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ON DIAMETER PRESERVING LINEAR MAPS

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Abstract. We study diameter preserving linear maps from $C(\mathrm{X})$ into $C(\mathrm{Y})$ where X and Y are compact Hausdorff spaces. By using the extreme points of $C(\mathrm{X})^*$ and $C(\mathrm{Y})^*$ and a linear condition on them, we obtain that there exists a diameter preserving linear map from $C(\mathrm{X})$ into $C(\mathrm{Y})$ if and only if X is homeomorphic to a subspace of Y. We also consider the case when X and Y are noncompact but locally compact spaces.

1. Introduction

A problem related to Banach-Stone theorem is the study of linear bijections between function spaces which preserve the diameter of the range. Such maps are called diameter preserving linear bijections.

The study of this problem begins with Györy and Molnár [6] and González and Uspenkij [5]. These authors in [5] and Cabello in [3] obtain the next result:

Let X be a compact Hausdorff space, then there exists a diameter preserving linear bijection $T : C(\mathrm{X}) \longrightarrow C(\mathrm{X})$ if and only if there exist a homeomorphism $\varphi : \mathrm{X} \longrightarrow \mathrm{X}$, a linear functional $\mu : C(\mathrm{X}) \longrightarrow \mathbb{K}$, where $\mathbb{K}$ is the scalar field, a scalar z with $|z| = 1$ and $\mu(1_X) + z \neq 0$ such that $Tf = zf \circ \varphi + \mu(f)1_X$ for each $f \in C(\mathrm{X})$.

Recently, there have been new results about diameter preserving linear bijections ([1], [2], [4]).

The diameter preserving linear bijections from a subspace of $C_0(\mathrm{X})$ onto a subspace of $C_0(\mathrm{Y})$ are studied in [4].

2. Main result

In this section X, Y are compact, Hausdorff topological spaces. We denote by $C(\mathrm{X})$ the space of scalar continuous functions endowed with the supremum norm. If $f \in C(\mathrm{X})$ we define the diameter of f by:

$$\rho(f) = \sup\{|f(x_1) - f(x_2)| : x_1, x_2 \in \mathrm{X}\}.$$

The next lemma is similar to the proposition 2 of [8].

Lemma 1. *Let $T : C(\mathrm{X}) \longrightarrow C(\mathrm{Y})$ be a linear map such that there exist:*

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(i) $Y_0 \subset Y$ *and a map* $\varphi : Y_0 \longrightarrow X$
(ii) *A linear map* $L : C(X) \longrightarrow \mathbb{K}$
(iii) $\alpha \in \mathbb{K}$, *with* $|\alpha| = 1$

such that for every $f \in C(X)$ *and* $y \in Y_0$ *we have* $Tf(y) = \alpha(f(\varphi(y))) + L(f)$.
Then φ *is a continuous map.*

Proof. If φ is not continuous there exists a net $(y_i)_{i\in A}$ in Y_0 such that $\lim_i y_i = y_0 \in Y_0$ but $\lim_i \varphi(y_i) = x_0 \in X$ where $x_0 \neq \varphi(y_0)$. Let $\mathcal{U}$ be an open neighborhood of x_0 such that $\varphi(y_0) \notin \mathcal{U}$. Let $f \in C(X)$ such that $f(x_0) = 1$ and $f(X \backslash \mathcal{U}) = 0$. There exists $i_0 \in I$ such that if $i \geq i_0$ is $|f(\varphi(y_i)) - f(y_0)| = |f(\varphi(y_i)) - 1| < \frac{1}{3}$ and we deduce that $|Tf(y_i) - Tf(y_0)| = |Tf(y_i)| > \frac{2}{3}$ if $i \geq i_0$ which contradicts the continuity of Tf. □

We will denote by C the subspace of constant maps of $C(X)$; $\rho([f])$ is a well-defined, complete norm in $C(X)/C$ and the extreme points of the unit ball of the dual space $(C(X)/C, \rho)^*$ are precisely $\alpha(\delta_{x_1} - \delta_{x_2})$, where $\alpha \in S_{\mathbb{K}}$ and $x_1, x_2 \in X$ $(x_1 \neq x_2)$. (see [1])

Let M be a subspace of $C(X)$. Then the extreme points of the unit ball of M^* are restriction to M of some extreme points of the unit ball of $(C(X)/C, \rho)^*$, so they are some $\alpha(\delta_{x_1} - \delta_{x_2})$ with $\alpha \in S_{\mathbb{K}}$ and $x_1, x_2 \in X$, $x_1 \neq x_2$.

In this paper we obtain a characterization of the diameter preserving linear injections $T : C(X) \longrightarrow C(Y)$ such that $\{\delta_y : y \in Y\}$ is linearly independent in M^*, where $M = T(C(X))$.

Theorem 1. *Let* X, Y *be compact, Hausdorff spaces. The next conditions are equivalent:*

(1) *There exists a diameter preserving linear map* $T : C(X) \longrightarrow C(Y)$ *such that* $\{\delta_y : y \in Y\}$ *is linearly independent in* M^* *where* $M = T(C(X))$.
(2) X *is homeomorphic to a subspace* $Y_0 \subset Y$ *and there exists a diameter preserving linear extension* $H : C(Y_0) \longrightarrow C(Y)$ *such that* $\{\delta_y : y \in Y\}$ *is linearly independent in* M_0^* *where* $M_0 = H(C(Y_0))$.

Proof. If $T : C(X) \longrightarrow C(Y)$ is a diameter preserving linear map such that $\{\delta_y : y \in Y\}$ is linearly independent in $T(C(X))$ we can define the map:

$$\widehat{T} : (C(X)/C, \rho) \longrightarrow (C(Y)/C, \rho)$$

by $\widehat{T}([f]) = [Tf]$. It will be an isometry if we consider the diameter norm.

We will denote by $\widehat{M}$ the subspace $\mathrm{Im}(\widehat{T})$ and we consider the dual map:

$$\widehat{T}^* : \widehat{M}^* \longrightarrow (C(X)/C, \rho)^*.$$

The extreme points of the unit ball of $\widehat{M}^*$ are some $\alpha(\delta_{y_1} - \delta_{y_2})$ where $y_1, y_2 \in Y$ and $\alpha \in S_{\mathbb{K}}$.

Given $x_1 \in X$ we have that, for each $x \in X \backslash \{x_1\}$ there exists $\alpha \in S_{\mathbb{K}}$ and $\{y, y_1\} \subset Y$, $y \neq y_1$ such that $\widehat{T}^{*^{-1}}(\delta_x - \delta_{x_1}) = \alpha(\delta_y - \delta_{y_1})$.

Consider $x' \notin \{x, x_1\}$, there exists $\beta \in S_{\mathbb{K}}$ and $\{y_2, y_3\} \subset \mathrm{Y}$, $y_2 \neq y_3$ such that $\widehat{T}^{*^{-1}}(\delta_{x'} - \delta_{x_1}) = \beta(\delta_{y_2} - \delta_{y_3})$.

Let us study the cardinal of the set $\{y, y_1\} \cap \{y_2, y_3\}$ (we will denote it by l).

If $l = 0$, then $\widehat{T}^{*^{-1}}(\delta_x - \delta_{x'}) = \alpha(\delta_y - \delta_{y_1}) - \beta(\delta_{y_2} - \delta_{y_3})$ and it must be an extreme point, so there exist $\gamma \in S_{\mathbb{K}}$ and $\{y_4, y_5\} \subset \mathrm{Y}$ such that $\alpha(\delta_y - \delta_{y_1}) - \beta(\delta_{y_2} - \delta_{y_3}) = \gamma(\delta_{y_4} - \delta_{y_5})$ which contradicts the fact that $\{\delta_y : y \in \mathrm{Y}\}$ is linearly independent in M^*.

If $l = 2$, then $\widehat{T}^{*^{-1}}(\delta_x - \delta_{x_1}) = \alpha(\delta_y - \delta_{y_1})$ and $\widehat{T}^{*^{-1}}(\delta_{x'} - \delta_{x_1}) = \gamma(\delta_y - \delta_{y_1})$ where $\gamma = \beta$ or $\gamma = -\beta$ and we have that: $\widehat{T}^*(\delta_y - \delta_{y_1}) = \dfrac{1}{\alpha}(\delta_x - \delta_{x_1})$ and $\widehat{T}^*(\delta_y - \delta_{y_1}) = \dfrac{1}{\gamma}(\delta_{x'} - \delta_{x_1})$ which is a contradiction because $\widehat{T}^*$ is a one-to-one map.

So that $l = 1$ and we have the next possibilities:

a) $y_1 = y_3$
b) $y_1 = y_2$
c) $y = y_3$
d) $y = y_2$

We will study a) (in the other cases we obtain a similar conclusion analogously).

In this case, $\widehat{T}^{*^{-1}}(\delta_x - \delta_{x'}) = \alpha(\delta_y - \delta_{y_1}) - \beta(\delta_{y_2} - \delta_{y_1})$ and we have that $\alpha = \beta$ because $\widehat{T}^{*^{-1}}(\delta_x - \delta_{x'})$ must be an extreme point. So we obtain that:

$$\widehat{T}^{*^{-1}}(\delta_x - \delta_{x'}) = \alpha(\delta_y - \delta_{y_2})$$

$$\widehat{T}^{*^{-1}}(\delta_x - \delta_{x_1}) = \alpha(\delta_y - \delta_{y_1})$$

$$\widehat{T}^{*^{-1}}(\delta_{x'} - \delta_{x_1}) = \alpha(\delta_{y_2} - \delta_{y_1}).$$

If $x'' \in \mathrm{X}$ and $x'' \notin \{x, x', x_1\}$ there exists $\gamma \in S_{\mathbb{K}}$ and $\{y_0, y_0'\} \subset \mathrm{Y}$ such that $\widehat{T}^{*^{-1}}(\delta_{x''} - \delta_{x_1}) = \gamma(\delta_{y_0} - \delta_{y_0'})$.

The sets $\{y_0, y_0'\} \cap \{y, y_1\}$, $\{y_0, y_0'\} \cap \{y_2, y_1\}$ are unitary.

If $y_0 = y$ and $y_0' = y_2$ we have that: $\widehat{T}^{*^{-1}}(\delta_x - \delta_{x''}) = \alpha(\delta_y - \delta_{y_1}) - \gamma(\delta_y - \delta_{y_2})$ so that $\alpha = \gamma$ and $\widehat{T}^{*^{-1}}(\delta_x - \delta_{x''}) = \alpha(\delta_{y_2} - \delta_{y_1}) = \widehat{T}^{*^{-1}}(\delta_{x'} - \delta_{x_1})$ which is a contradiction.

Analogously, if $y_0 = y_2$ and $y_0' = y$ we obtain a contradiction, so we deduce that $y_0 = y_1$ or $y_0' = y_1$.

Suppose then that $y_0' = y_1$ (If $y_0 = y_1$ we obtain a similar result).

In this case we have that $\widehat{T}^{*^{-1}}(\delta_{x''} - \delta_{x_1}) = \gamma(\delta_{y_0} - \delta_{y_1})$ and $\widehat{T}^{*^{-1}}(\delta_{x'} - \delta_{x''}) = \alpha(\delta_{y_2} - \delta_{y_1}) - \gamma(\delta_{y_0} - \delta_{y_1})$ so $\alpha = \beta$ and we have that $\widehat{T}^{*^{-1}}(\delta_{x''} - \delta_{x_1}) = \alpha(\delta_{y_0} - \delta_{y_1})$.

We deduce that given $x_1 \in \mathrm{X}$ we obtain $y_1 \in \mathrm{Y}$, and $\alpha \in S_{\mathbb{K}}$ such that if $x \in \mathrm{X} \backslash \{x_1\}$ there exists $y \in \mathrm{Y} \backslash \{y_1\}$ such that

$$\widehat{T}^{*^{-1}}(\delta_x - \delta_{x_1}) = \alpha(\delta_y - \delta_{y_1})$$

so we have a one-to-one map $\varphi_1 : X \longrightarrow Y$ defined by $\varphi_1(x_1) = y_1$ and $\varphi_1(x) = y$ for $x \neq x_1$ if and only if $\widehat{T}^{*^{-1}}(\delta_x - \delta_{x_1}) = \alpha(\delta_y - \delta_{y_1})$.

We will prove that the result does not depend on x_1.

Given $x_2 \in X$, there exist $y_2 \in Y$, $\beta \in S_{\mathbb{K}}$, and a one-to-one map $\varphi_2 : X \longrightarrow Y_2$ such that $\varphi_2(x_2) = y_2$ and $\varphi_2(x) = y$ for $x \neq x_2$ if and only if $\widehat{T}^{*^{-1}}(\delta_x - \delta_{x_2}) = \beta(\delta_y - \delta_{y_2})$.

We have that: $\widehat{T}^{*^{-1}}(\delta_{x_1} - \delta_{x_2}) = \beta(\delta_{\varphi_2(x_1)} - \delta_{y_2}) = -\alpha(\delta_{\varphi_1(x_2)} - \delta_{y_1}) = \alpha(\delta_{y_1} - \delta_{\varphi_1(x_2)})$.

We deduce that either $\beta = -\alpha$ or $\beta = \alpha$.

If $\beta = -\alpha$ we have that $\varphi_2(x_1) = \varphi_1(x_2)$ and $y_1 = y_2$, but if $x \notin \{x_1, x_2\}$ then

$$\widehat{T}^{*^{-1}}(\delta_x - \delta_{x_1}) = \alpha(\delta_{\varphi_1(x)} - \delta_{y_1})$$

$$\widehat{T}^{*^{-1}}(\delta_x - \delta_{x_2}) = -\alpha(\delta_{\varphi_2(x)} - \delta_{y_1}),$$

where $\varphi_1(x) \neq y_1$, $\varphi_2(x) \neq y_1$ and we obtain that:

$$\widehat{T}^{*^{-1}}(\delta_{x_2} - \delta_{x_1}) = \alpha(\delta_{\varphi_1(x)} + \delta_{\varphi_2} - 2\delta_{y_1})$$

which is a contradiction.

So that, necessarily $\beta = \alpha$ and $\varphi_2(x_1) = y_1 = \varphi_1(x_1)$, $\varphi_2(x_2) = y_2 = \varphi_2(x_1)$, and $\widehat{T}^{*^{-1}}(\delta_{x_2} - \delta_{x_1}) = \alpha(\delta_{\varphi_1(x)} - \delta_{y_1} - \delta_{\varphi_2(x)} + \delta_{y_2})$ and since $y_1 \neq y_2$ we obtain that $\varphi_1(x) = \varphi_2(x)$.

Then there exists a one-to-one map $\varphi : X \longrightarrow Y$. We denote $\varphi(X)$ by Y_0 and we consider $t = \varphi^{-1}$, $t : Y_0 \longrightarrow X$. We have that t is a bijection and if $y_1, y_2 \in Y_0$ then $\widehat{T}^*(\delta_{y_1} - \delta_{y_2}) = \beta(\delta_{t(y_1)} - \delta_{t(y_2)})$ where $\beta = \dfrac{1}{\alpha}$.

We will prove that if $f \in C(X)$ then $[Tf|_{Y_0}] = [g]$ where $g : Y_0 \longrightarrow \mathbb{K}$ is defined by $g(y) = \beta f(t(y))$.

If $[Tf|_{Y_0}] \neq [g]$ then $\rho(Tf|_{Y_0} - g) \neq 0$ so that, there exist $y_1, y_2 \in Y_0$ such that $Tf(y_1) - Tf(y_2) \neq \beta(f(t(y_1)) - f(t(y_2)))$ which is not possible. So, if $f \in C(X)$ there exists $z \in \mathbb{K}$ (we will denote it by $L(f)$) such that $Tf(y) = \beta f(t(y)) + L(f)$, if $y \in Y_0$, and we obtain a linear map $L : C(X) \longrightarrow \mathbb{K}$. L is continuous if and only if T is.

By lemma 1 we deduce that t is continuous, so that if we prove that Y_0 is a closed subspace of Y we will have that t is a homeomorphism.

Let $(y_r)_{r\in I}$ be a net in Y_0 such that $\lim_r y_r = y_0$, with $y_0 \in Y$. We consider the net $(x_r)_{r\in I}$ in X defined by $x_r = t(y_r)$ if $r \in I$. There exists a subnet $(x_p)_{p\in I}$ of $(x_r)_{r\in I}$ such that $\lim_p x_p = x_0$, so that, given $x_1 \in X$, $w^* \lim_p \widehat{T}^{*^{-1}}(\delta_{x_p} - \delta_{x_1}) = \widehat{T}^{*^{-1}}(\delta_{x_0} - \delta_{x_1})$ and we have that $w^* \lim_p \alpha(\delta_{y_p} - \delta_{\varphi(x_1)}) = \widehat{T}^{*^{-1}}(\delta_{x_0} - \delta_{x_1})$.

Since $\lim_r y_r = y$ we have that $w^* \lim_p(\delta_{y_p} - \delta_{\varphi(x_1)}) = \delta_{y_0} - \delta_{\varphi(x_1)}$ so that $\widehat{T}^{*^{-1}}(\delta_{x_0} - \delta_{x_1}) = \alpha(\delta_{y_0} - \delta_{\varphi(x_1)})$ and we obtain that $y_0 = \varphi(x_0)$, so that $y_0 \in Y_0$.

Consider now the map $l : C(\mathrm{Y}_0) \longrightarrow C(\mathrm{X})$ defined by $l(g)(x) = \alpha g(\varphi(x))$. We have that l is a diameter preserving surjective isometry.

Let $T_0 : C(\mathrm{X}) \longrightarrow C(\mathrm{Y})$ the map defined by $(T_0 f)(y) = Tf(y) - L(f)$. T_0 preserves the diameter.

Let $H : C(\mathrm{Y}_0) \longrightarrow C(\mathrm{Y})$ be defined by $H = T_0 l$. Obviously H preserves the diameter and if $g \in C(\mathrm{Y}_0)$, $y \in \mathrm{Y}_0$ we have: $(Hg)(y) = T_0 lg(y) = T(lg)(y) - L(lg) = \beta(\alpha g(\varphi(t(y)))) + L(lg) - L(lg) = g(y)$.

Let $M_0 = H(C(\mathrm{Y}_0))$, if we prove that $M = M_0$ we will have that $\{\delta_y : y \in \mathrm{Y}\}$ is linearly independent in M_0^*.

Let $f \in C(\mathrm{X})$, since $T(1_X)$ is a constant map we have that there exists $\beta \in \mathbb{K}$ such that $T(1_X)(y) = \beta + L(1_X)$ if $y \in \mathrm{Y}$, so that $T_0(1_X) = \beta$.

Consider $g \in C(\mathrm{Y}_0)$ such that $l(g) = f - \dfrac{L(f)}{\beta} 1_X$, we have that $H(g) = T_0(lg) = T_0(f) - \dfrac{L(f)}{\beta} T_0(1_X) = Tf$, so that $M \subset M_0$ and it is clear that $M_0 \subset M$ and we obtain that $M = M_0$.

Conversely, if there exists a homeomorphism $t : \mathrm{Y}_0 \longrightarrow \mathrm{X}$ where Y_0 is a subspace of Y and there exists a diameter preserving linear extension $H : C(\mathrm{Y}_0) \longrightarrow C(\mathrm{Y})$ such that $\{\delta_y : y \in \mathrm{Y}\}$ is linearly independent in M_0^* where $M_0 = H(C(\mathrm{Y}_0))$, we can consider $l : C(\mathrm{X}) \longrightarrow C(\mathrm{Y}_0)$ defined by $lf(y) = f(t(y))$.

Let $T : C(\mathrm{X}) \longrightarrow C(\mathrm{Y})$ be defined by $T = Hl$.

It is clear that T is a diameter preserving linear map and $M = T(C(X)) = H(C(\mathrm{Y}_0)) = M_0$, so that $\{\delta_y : y \in \mathrm{Y}\}$ is linearly independent in M^*. $\square$

Remark 1.

1) Let us consider $I = [0, 1]$ and let $\mathcal{C} \subset I$ be the Cantor set.

   It is well known that there exists a sequence $((a_n, b_n))_n$ of open and disjoint intervals such that $\mathcal{C} = I \setminus \bigcup_n (a_n, b_n)$.

   If $f \in C(\mathcal{C})$ we can define $\overline{f} : [0, 1] \longrightarrow \mathbb{R}$ by $\overline{f}(x) = x$ if $x \in \mathcal{C}$ and by $\overline{f}(x) = \lambda a_n + (1 - \lambda) b_n$ if $x \in (a_n, b_n)$ and $x = \lambda a_n + (1 - \lambda) b_n$ for $\lambda \in (0, 1)$.

   It is easily seen that the map $T : C(\mathcal{C}) \longrightarrow C([0, 1])$ defined by $Tf = \overline{f}$ is a diameter preserving and non surjective isometry; and if $M = T(C(\mathcal{C}))$ then $\{\delta_y : y \in [0, 1]\}$ is linearly independent in M^*.
2) Let us suppose that for $f \in C(\mathrm{X})$ we have that $f \neq 0$ but $Tf = 0$, then if $y \in \mathrm{Y}_0$, $\beta f(t(y)) = -L(f)$, so $f = -\dfrac{1}{\beta} L(f) 1_X$, so that, T is one-to-one if and only if the restriction of T to the constant maps is one-to-one.
3) In the last theorem we could consider in 1) that $\{\delta_y : y \in \mathrm{Y} \setminus \{y_0\}\}$ is linearly independent in M^* but $\delta_{y_0} = 0$. In this case, the extreme

points of the unit ball of M^* would be some $\alpha(\delta_{y_1} - \delta_{y_2})$ and $\alpha\delta_{y_1}$ (if $y_2 = y_0$).

We could use the same procedure but in 2) we would obtain that $\{\delta_y : y \in \mathrm{Y}\backslash\{y_0\}\}$ is linearly independent in M_0^* and $\delta_{y_0} = 0$, where $M_0^* = H(C(\mathrm{Y}_0))$.

3. Locally compact case

In this section X, Y are two locally compact, noncompact spaces and we will denote by $\gamma\mathrm{X}, \gamma\mathrm{Y}$ the corresponding Alexandroff compactifications of X and Y. We will study the diameter preserving linear maps in this situation.

Theorem 2. *Let* X, Y *be locally compact, noncompact Hausdorff topological spaces. The next conditions are equivalent:*

1) *There exists a diameter preserving linear map $T : C_0(\mathrm{X}) \longrightarrow C_0(Y)$ such that $\{\delta_y : y \in \mathrm{Y}\}$ is linearly independent in M^* where $M = T(C_0(\mathrm{X}))$.*
2) *There exists a homeomorphism $\varphi : \gamma\mathrm{X} \longrightarrow \mathrm{Y}_0$ where Y_0 is a subspace of* Y *satisfying that there exists a diameter preserving linear extension $H : A \longrightarrow C(\gamma\mathrm{Y})$ where $A = \{g \in C(Y_0) : g(y_0) = 0\}$ and $y_0 = \varphi(\infty)$ and $\{\delta_y : y \in \gamma\mathrm{Y}\backslash\{y_0\}\}$ is linearly independent in M_0^* where $M_0 = H(A)$.*

Proof. If $f \in C_0(\mathrm{X})$ we will denote by $\overline{f}$ the map of $C(\gamma\mathrm{X})$ defined by $\overline{f}(x) = f(x)$ if $x \in \mathrm{X}$ and $\overline{f}(\infty) = 0$.

Consider the map $q_1 : (C_0(\mathrm{X}), \rho) \longrightarrow (C(\gamma\mathrm{X})/C, \rho)$ defined by $q_1(f) = [\overline{f}]$. q_1 is a surjective linear isometry.

Analogously we can define $q_2 : (C_0(\mathrm{Y}), \rho) \longrightarrow (C(\gamma\mathrm{Y})/C, \rho)$ by $q_2(g) = [\overline{g}]$, where $\overline{g}(y) = g(y)$ if $y \in \mathrm{Y}$ and $\overline{g}(\infty) = 0$.

We define now $\widehat{T} : (C(\gamma\mathrm{X})/C, \rho) \longrightarrow (C(\gamma\mathrm{Y})/C, \rho)$ by $\widehat{T}[\overline{f}] = [\overline{Tf}]$, then $\widehat{T}(q_1 f) = q_2(Tf)$.

The subspace $\mathrm{Im}(\widehat{T})$ will be $Q_0 = \{[\overline{Tf}] : f \in C_0(\mathrm{X})\}$ and we can consider the dual map: $\widehat{T}^* : Q_0^* \longrightarrow (C(\gamma\mathrm{X})/C, \rho)^*$

The extreme points of the unit ball of Q_0^* are included in $\{\alpha(\delta_{y_1} - \delta_{y_2}) : \alpha \in S_{\mathbb{K}} \,;\, y_1, y_2 \in \gamma\mathrm{Y} \,,\, y_1 \neq y_2\}$.

We can now proceed analogously to the proof of the last theorem and we obtain that there exists a subspace $\mathrm{Y}_0 \subset \gamma\mathrm{Y}$, a homeomorphism $t : \mathrm{Y}_0 \longrightarrow \gamma\mathrm{X}$ and $\beta \in S_{\mathbb{K}}$ such that if $y_1, y_2 \in \mathrm{Y}_0$ then: $\overline{Tf}(y_1) - \overline{Tf}(y_2) = \beta(\overline{f}(t(y_1)) - \overline{f}(t(y_2)))$.

Let $y_0 \in \mathrm{Y}_0$ be such that $ty_0 = \infty$, then for each $y \in \mathrm{Y}_0\backslash\{y_0\}$ we have that $Tf(y) - Tf(y_0) = \beta f(t(y))$.

Let $A = \{g \in C(y_0) : g(y_0) = 0\}$.

Consider the map $Q : A \longrightarrow C_0(\mathrm{X})$ defined by $(Qg)(x) = \dfrac{1}{\beta} g(\varphi(x))$ where $\varphi = t^{-1}$. Q is well defined and preserves the diameter.

Consider now the map $T' : C_0(\mathrm{X}) \longrightarrow C(\gamma\mathrm{Y})$ defined by $T'h(y) = Th(y) - Th(y_0)$ if $y \in \mathrm{Y}$ and $T'h(\infty) = -Th(y_0)$. T is a diameter preserving linear map.

Let $H : A \longrightarrow C(\gamma\mathrm{Y})$ be defined by $H = T'Q$.

H is a diameter preserving linear map and if $g \in \mathrm{Y}$, $y \in \mathrm{Y}_0$, we have that: $(Hg)(y) = T'Qg(y) = TQg(y) - TQg(y_0) = \beta(\frac{1}{\beta}g(\varphi(t(y)))) + TQg(y_0) - TQg(y_0) = g(y_0)$, so that, Hg is an extension map of g.

We will prove now that $\{\delta_y : y \in \gamma\mathrm{Y}\backslash\{y_0\}\}$ is linearly independent in M_0^*, where $M_0 = H(A)$.

Let us suppose that there exist a set $\{y_1, \ldots, y_n, \infty\} \subset \gamma\mathrm{Y}\backslash\{y_0\}$ and a set $\{\alpha_1, \ldots, \alpha_{n+1}\} \subset \mathbb{K}$ such that $\sum_{i=1}^{n} \alpha_i\delta_{y_i} + \alpha_{n+1}\delta_\infty = 0$, then if $f \in C_0(\mathrm{X})$, we can consider $g \in C(\mathrm{Y}_0)$ defined by $g(y) = \beta(f(t(y)))$ if $y \in \mathrm{Y}_0\backslash\{y_0\}$ and $g(y_0) = 0$; it is clear that $g \in A$ and $Qg = f$. If $y \in \mathrm{Y}$ we have that $T'Qg(y) = Tf(y) - Tf(y_0)$ and $T'f(\infty) = -Tf(y_0)$, so we deduce that $\alpha_1(Tf(y_1) - Tf(y_0)) + \cdots + \alpha_n(Tf(y_n) - Tf(y_0)) + \alpha_{n-1}(-Tf(y_0)) = 0$ and we obtain that $\alpha_1 = \alpha_2 = \cdots = \alpha_{n+1} = 0$.

Conversely, let us suppose that we have a homeomorphism $t : \mathrm{Y}_0 \longrightarrow \gamma\mathrm{X}$ where $\mathrm{Y}_0 \subset \gamma\mathrm{Y}$ and let $y_0 \in \mathrm{Y}$ be such that $t(y_0) = \infty$ and let us suppose that if $A = \{g \in C(y_0) : g(y_0) = 0\}$ then there exists a diameter preserving linear extension $H : A \longrightarrow C(\gamma\mathrm{Y})$.

We can define:

$\varphi_1 : C_0(\mathrm{X}) \longrightarrow C(\gamma\mathrm{X})$ by $\varphi_1(f) = \overline{f}$, where $\overline{f}(x) = f(x)$ if $x \in \mathrm{X}$ and $\overline{f}(\infty) = 0$.

$\varphi_2 : C(\gamma\mathrm{X}) \longrightarrow C(\mathrm{Y}_0)$ by $\varphi_2 f(y) = f(t(y))$.

$\varphi_3 : C(\gamma\mathrm{Y}) \longrightarrow C_0(\mathrm{Y})$ by $\varphi_3 g(y) = g(y) - g(\infty)$.

We have that $\varphi_1, \varphi_2, \varphi_3$ are diameter preserving linear maps, then we can define $T : C_0(\mathrm{X}) \longrightarrow C_0(\mathrm{Y})$ by $T = \varphi_3 H\varphi_2\varphi_1$.

Let $M = T(C_0(\mathrm{X}))$, we will prove that $\{\delta_y : y \in \mathrm{Y}\}$ is linearly independent in M^*. Let $\{y_0, y_1, \ldots, y_n\} \subset \mathrm{Y}$ and $\{\alpha_0, \alpha_1, \ldots, \alpha_n\} \subset \mathbb{K}$ be such that $\sum_{i=0}^{n} \alpha_i\delta_{y_i} = 0$ in M^*.

Then if $f \in C_0(\mathrm{X})$ and $g = \varphi_2\varphi_1 f$ we have that $g(y_0) = 0$ and $g \in A$, so that $Tf = \varphi_3 Hg$, so that $\alpha_0(Hg(y_0) - Hg(\infty)) + \alpha_1(Hg(y_1) - Hg(\infty)) + \cdots + \alpha_n(Hg(y_n) - Hg(\infty)) = 0$ and then $\alpha_1 Hg(y_1) + \cdots + \alpha_n Hg(y_n) - (\alpha_0 + \cdots + \alpha_n)Hg(\infty) = 0$, but $\{\delta_y : y \in \gamma\mathrm{Y}\backslash\{y_0\}\}$ is linearly independent in M_0^* where $M_0 = H(C(y_0))$ so that $\alpha_0 = \alpha_1 = \cdots = \alpha_n = 0$. □

Holsztynski in [7] studied a version of Banach-Stone theorem for non-surjective linear isometries. Recently, a bilinear version of Holsztynski theorem has been obtained [9]. We study the case of diameter preserving linear maps and we think that it is more complicated than the non-surjective case of Banach-Stone problem.

To study this problem we could define a set similar to the Choquet boundary for a diameter preserving linear map:

If X is a compact topological space and M is a subspace of $(C(\mathrm{X})/C, \rho)$ we define the diametral boundary of M by $dch(M) = \{(x_1, x_2) : \delta_{x_1} - \delta_{x_2} \in ExB_{M^*}\}$ where ExB_{M^*} is the set of extreme points of the unit ball of M^*.

If $T : C(\mathrm{X}) \longrightarrow C(\mathrm{X})$ is a diameter preserving linear map, then there is some relations between the properties of T and the properties of $dch(M)$, where $M = \widehat{T}(C(\mathrm{X})/C)$.

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