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HERMITE AND HERMITE-FEJÉR INTERPOLATION OF HIGHER ORDER AND ASSOCIATED PRODUCT INTEGRATION FOR ERDŐS WEIGHTS

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ABSTRACT. Using the results on the coefficients of Hermite-Fejér interpolations in [5], we investigate convergence of Hermite and Hermite-Fejér interpolation of order m, $m = 1, 2, \ldots$ in $L_p(0 < p < \infty)$ and associated product quadrature rules for a class of fast decaying even Erdős weights on the real line.

1. Introduction

Let $X := \{x_{k,n} : 1 \le k \le n, n \ge 1\} \subset \mathbb{R}$, satisfying

$$-\infty < x_{n,n} < x_{n-1,n} < \cdots < x_{2,n} < x_{1,n} < \infty, \ n = 1, 2, \ldots,$$

be a set of pairwise distinct nodes. For any real-valued function $f(x)$ on $\mathbb{R}$ and an integer $m \ge 1$, we let $H_{n,m}(f, X, x)$ be the Hermite-Fejér interpolation polynomial of order m for f on X, which is the unique polynomial of degree $\le nm - 1$ satisfying

$$H^{(t)}_{n,m}(f, X, x_{k,n}) = \delta_{0,t} f(x_{k,n}), \quad 1 \le k \le n, \ 0 \le t \le m - 1.$$

On the other hand, if $f \in C^{m-1}(\mathbb{R})$, then the Hermite interpolation polynomial $\hat{H}_{n,m}(f, X, x)$ of order m for f on X is defined to be the unique polynomial of degree $\le nm - 1$ satisfying

$$\hat{H}^{(t)}_{n,m}(f, X, x_{k,n}) = f^{(t)}(x_{k,n}), \quad 1 \le k \le n, \ 0 \le t \le m - 1. \tag{1.1}$$

In this paper, we deal with Hermite-Fejér and Hermite interpolations with respect to X whose elements are the zeros of a sequence of certain orthogonal polynomials described below. We consider $w(x) := \exp(-Q(x))$, where $Q : I \to \mathbb{R}$ is even, continuous, and decays faster than any polynomial at infinity. Then X consists of the zeros $\{x_{j,n}(w^2) : 1 \le j \le n, n \ge 1\}$ of the orthonormal polynomial $\{p_n(w^2, x)\}_{n=0}^{\infty}$, where

$$p_n(w^2, x) = \gamma_n(w^2)x^n + \text{ lower degree terms } (\gamma_n(d\alpha) > 0)$$

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with respect to $w^2(x)$, defined by the condition

$$\int_I p_n(w^2,x)p_m(w^2,x)w^2(x)dx = \delta_{m,n}, \quad m,n=0,1,2,\ldots.$$

Then all $\{x_{j,n}(w^2)\}_{j=1}^n$ belongs to $\mathbb{R}$, which we arrange as

$$x_{n,n}(w^2) < x_{n-1,n}(w^2) < \cdots < x_{2,n}(w^2) < x_{1,n}(w^2).$$

Let $H_{n,m}[w^2;\cdot]$ and $\hat{H}_{n,m}[w^2;\cdot]$ be the Hermite-Fejér and Hermite interpolation operators of order m with respect to the zeros $\{x_{j,n}(w^2)\}_{j=1}^n$ of $p_n(w^2;x)$. Then $H_{n,m}[w^2;f]$ can be written as

$$H_{n,m}[w^2;f](x) = \sum_{k=1}^{n} f(x_{k,n})h_k(x), \quad n=1,2,\ldots, \tag{1.2}$$

where for $1 \le k \le n$

$$h_k(x) := h_{k,n,m}(x) = l_{k,n}^m(x)\sum_{i=0}^{m-1} e_{i,k}(x-x_{k,n})^i \tag{1.3}$$

is the unique polynomial of degree $nm-1$ satisfying

$$h_k^{(t)}(x_{l,n}) = \delta_{0,t}\delta_{l,k}, \quad 1 \le k,l \le n, 0 \le t \le m-1$$

and $l_{k,n}(x)$ is the fundamental Lagrange interpolation polynomial ([3, p. 23]), given by

$$l_{k,n}(w^2;x) := \frac{p_n(w^2;x)}{p_n'(w^2;x_{k,n})(x-x_{k,n})}, \quad k=1,2,\ldots,n.$$

Also, we can express $\hat{H}_{n,m}[w^2;f]$ as

$$\hat{H}_{n,m}[w^2;f](x) = \sum_{t=0}^{m-1}\sum_{k=1}^{n} f^{(t)}(x_{k,n})h_{t,k}(x), \; m=1,2,\ldots, \tag{1.4}$$

where for $0 \le t \le m-1$

$$h_{t,k}(x) := h_{t,k,n,m}(x) = l_{k,n}^m(x)\frac{(x-x_{k,n})^t}{t!}\sum_{i=0}^{m-1-t} e_{t,i,k}(x-x_{k,n})^i \tag{1.5}$$

is the unique polynomial of degree $nm-1$ satisfying

$$h_{t,k}^{(i)}(x_{j,n}) = \delta_{t,i}\delta_{k,j}, \quad 0 \le i, \; t \le m-1, \; 1 \le j, \; k \le n.$$

In (1.3) and (1.5), $e_{i,k}$ and $e_{t,i,k}$ depend also on n and m. We also have $h_{0,k}(x) = h_k(x)$. Then we have for $f \in C^{m-1}(\mathbb{R})$,

$$\hat{H}_{n,m}[w^2;f](x) = H_{n,m}[w^2;f](x) + \sum_{t=1}^{m-1}\sum_{k=1}^{n} f^{(t)}(x_{k,n})h_{t,k}(x)$$

and specially for any polynomial P of degree $\le nm-1$, $P(x) = \hat{H}_{n,m}[w^2;P](x)$ from (1.1).

Our main concern is the following problem: Under what conditions on weight functions $w_1(x)$ and $w_2(x)$ will the relation for $0 < p < \infty$

$$\lim_{n\to\infty} \left\| \left(f(x) - H_{n,m}[w^2; f](x)\right) w_1(x) \right\|_{L_p(\mathbb{R})} = 0$$

hold for all continuous functions f satisfying $\lim_{|x|\to\infty} |fw_2(x)| = 0$? Specially, we will consider this problem for $m = 1, 2, \ldots$, in case

$$w_1(x) = w^m(x)(1 + Q(x))^{-\Delta}, \quad w_2(x) = w^m(x)\tilde{Q}(x),$$

where $w(x)$ is an Erdős weight and $\tilde{Q}(x)$ is defined as in (2.1) below.

Once we have the convergence of Hermite-Fejér and Hermite interpolations of order m, we can consider the convergence of associated *product quadrature rules*, involving approximation of

$$I[k; f] := \int_{\mathbb{R}} k(x)f(x)dx$$

by quadrature rules

$$I_{n,m}[k; f] := \sum_{j=1}^{n} w_{j,n,m}(k) f(x_{j,n}),$$

where the weights $\{w_{j,n,m}(k)\}$ are usually determined by integration of some approximation to f. Here, the kernel $k(x)$ is typically the difficult component of the integrand $k(x)f(x)$, with known types of singularities or oscillatory behavior and usually f has smooth behavior. The product quadrature treated in this paper, is to approximate $I[k; f]$ by

$$I_{n,m}[k; f] := \int_{\mathbb{R}} k(x)H_{n,m}[w^2; f]dx = \sum_{j=1}^{n} f(x_{j,n}) \left(\int_{\mathbb{R}} k(x)h_j(x)dx \right).$$

Analogous rule generated by $\hat{H}_{n,m}$ is

$$\hat{I}_{n,m}[k; f] := \int_{\mathbb{R}} k(x)\hat{H}_{n,m}[w^2; f]dx.$$

We shall prove these product quadratures converge to $I[k; f]$ under mild conditions on $f(x)$ and $k(x)$.

This paper is organized as follows. In section 2, we introduce a subclass of weights from [6] and state main results. In section 3, we estimate some quadrature sum for the main result and prove Theorem 2.2 and Theorem 2.3 in section 4. Finally, section 5 is an appendix containing various estimates on $p_n(w^2)$ and its zeros which are taken from [6]. They are largely independent of the paper and should be used only for following the technical details of the proofs.

2. Main results

We first introduce some notations, which we use in the following. For any two sequences $\{b_n\}_n$ and $\{c_n\}_n$ of non-zero real numbers(or functions), we write $b_n \lesssim c_n$ if there exists a constant $C > 0$ independent of n(or x) such that $b_n \leq Cc_n$ for n large enough and write $b_n \sim c_n$ if $b_n \lesssim c_n$ and $c_n \lesssim b_n$. We denote by $\mathcal{P}_n$ the space of polynomials of degree at most n.

The main feature of our weight $w(x) := \exp(-Q(x))$ is that $Q(x)$ is even and of faster than smooth polynomial decay at infinity. For example, we will consider the following examples below, see [6]:

$$w_{k,\alpha}(x) := \exp\big(-Q_{k,\alpha}(x)\big) \quad \text{and} \quad w_{A,B}(x) := \exp\big(-Q_{A,B}(x)\big),$$

where for $k \geq 1, \alpha > 1$ and for $B > 1$

$$Q_{k,\alpha}(x) := \exp_k(|x|^\alpha), \quad \text{and} \quad Q_{A,B}(x) = \exp\big(\log(A + x^2)\big)^B.$$

Here, $\exp_k(;) = \exp\big(\cdots(\exp(;))\big)$ denotes the kth iterated exponential and A is a large enough but fixed constant. More precisely, we shall be interested in the following subclass of weights from [6] of which the weights above are natural examples.

Definition 2.1. Let $w := \exp(-Q)$, where $Q : \mathbb{R} \to \mathbb{R}$ is even, continuous, $Q^{(j)} \geq 0$ in $(0, \infty)$ for $j = 0, 1, 2$, and the function

$$T(x) := 1 + xQ''(x)/Q'(x)$$

is increasing in $(0, \infty)$ with

$$\lim_{x\to\infty} T(x) = \infty, \quad T(0+) := \lim_{x\to 0+} T(x) > 1.$$

Assume that for some constants $C_1, C_2, C_3 > 0$,

$$C_1 \leq T(x)/\left(\frac{xQ'(x)}{Q(x)}\right) \leq C_2, x \geq C_3,$$

and for every $\epsilon > 0$ and for some positive constant C independent of x

$$T(x) \leq C(Q(x))^\varepsilon, \; x \to \infty.$$

Then we write $w \in \mathcal{E}_1$.

Now, we state our main result in the following.

Theorem 2.2. *Let $w \in \mathcal{E}_1$, $0 < p < \infty$, $\Delta \in \mathbb{R}$ and $\kappa > 0$. Let m be a positive integer and*

$$\tilde{Q}(x) := Q^{\max\{\frac{m}{6}-\frac{1}{3},0\}}(x)T^{\frac{m-1}{2}}(x)\log^{1+\kappa}(1+|x|). \tag{2.1}$$

Assume that

$$\Delta > \max\left[0, \frac{2}{3}\left\{\frac{m}{4} - \min\left(\frac{1}{p}, 1\right)\right\}\right]. \tag{2.2}$$

Then

$$\lim_{n\to\infty} \|(f(x) - H_{n,m}[f](x))w^m(x)(1+Q(x))^{-\Delta}\|_{L_p(\mathbb{R})} = 0$$

for every continuous function $f : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$\lim_{|x|\to\infty} |f(x)|w^m(x)\tilde{Q}(x) = 0. \tag{2.3}$$

Moreover, for $m \geq 2$

$$\lim_{n\to\infty} \|\big(f(x) - \hat{H}_{n,m}[w^2; f](x)\big)w^m(x)(1+Q(x))^{-\Delta}\|_{L_p(\mathbb{R})} = 0$$

for every $f \in C^{m-1}(\mathbb{R})$ *satisfying* (2.3) *and*

$$\sup_{x\in\mathbb{R}} \left| f^{(t)}(x)w^m(x)\tilde{Q}(x)\right| < \infty,\ t = 1, 2, \ldots, m-1. \tag{2.4}$$

The following is a result for associated product quadrature derived from Theorem 2.2.

Theorem 2.3. *Assume that hypotheses of Theorem 2.2. Let* $k : \mathbb{R} \to \mathbb{R}$ *be measurable and assume that for some* $1 < p < \infty$,

$$\left\|k(x)w^{-m}(x)(1+Q(x))^{\Delta}\right\|_{L_p(\mathbb{R})} < \infty.$$

If

$$\Delta > \max\left\{0, \frac{2}{3}\left(\frac{m}{4} + \frac{1}{p} - 1\right)\right\},$$

then for every continuous function $f : \mathbb{R} \to \mathbb{R}$ *satisfying* (2.3),

$$\lim_{n\to\infty} I_{n,m}[k; f] = I[k; f]$$

and for every $f \in C^{m-1}(\mathbb{R})$ *satisfying* (2.3) *and* (2.4),

$$\lim_{n\to\infty} \hat{I}_{n,m}[k; f] = I[k; f].$$

3. Technical lemmas

In this section, we give coefficients $e_{r,k}$ and $e_{t,r,k}$ of Hermite-Fejér and Hermite interpolation polynomials and some quadrature sum estimates to prove our main results. Throughout for convenience, we set for $n \geq 1$

$$x_{0,n} := x_{1,n} + L\delta_n a_n \quad \text{and} \quad x_{n+1,n} := x_{n+1,n} - L\delta_n a_n.$$

First, we recall the results for $e_{r,k}$ and $e_{t,r,k}$ from [5].

Lemma 3.1 ([5]). *For any integer* $m \geq 1$, *we have uniformly for* $1 \leq k \leq n$,

(3.1)

$$|e_{r,k}| \lesssim \left(\frac{n}{a_n}T^{\frac{1}{2}}(x_{k,n})\right)^r \quad \textit{and} \quad |e_{t,r,k}| \lesssim \left(\frac{n}{a_n}T^{\frac{1}{2}}(x_{k,n})\right)^r, \quad 0 \leq r, t \leq m-1.$$

Proof. It is proved in [5, Theorem 2.8]. □

Next, we give some quadrature sum estimates.

Lemma 3.2. *For $\beta \in (0,1/4)$ and $r = 0,1,2,\ldots,m-1$, let*

$$\sum\nolimits_r := \left(\frac{n}{a_n}T^{\frac{1}{2}}(a_n)\right)^r \sum_{|x_{k,n}|\geq a_{\beta n}} \left(|l_{k,n}(x)|w^{-1}(x_{k,n})\right)^m |x-x_{k,n}|^r. \tag{3.2}$$

Then we have for some constant $L > 0$

$$w^m(x)\sum\nolimits_r \lesssim \begin{cases} A_n, & |x| \leq a_{\beta n/2},\ |x| \geq a_{2n}, \\ B_n, & a_{\beta n/2} \leq |x| \leq (1-L\delta_n)a_n, \\ C_n, & (1-L\delta_n)a_n \leq |x| \leq (1+L\delta_n)a_n, \\ D_n, & (1+L\delta_n)a_n \leq |x| \leq a_{2n}, \end{cases}$$

where

$$\begin{aligned} A_n &:= \left(\frac{T^{\frac{1}{2}}(a_n)}{n}\right)^{m-r-1} T^{-\max\{\frac{m}{12}-\frac{1}{6},\frac{1-m}{4}\}}(a_n)n^{\max\{\frac{m}{6}-\frac{1}{3},0\}}, \\ B_n &:= (T(a_n)(1-|x|/a_n))^{\frac{r}{2}} + (T(a_n)(1-|x|/a_n))^{-\frac{r}{2}} \\ &\quad + T^{-\max\{\frac{m}{3}-\frac{1}{6},\frac{1}{4}\}}(a_n)n^{\max\{\frac{m}{6}-\frac{1}{3},0\}}\left|a_n^{1/2}p_n w(x)\right|^m \log n, \\ C_n &:= \left(\frac{n}{a_n}T^{\frac{1}{2}}(a_n)\right)^r \left(||x|-(1+2L\delta_n)a_n|^r + ||x|-(1-2L\delta_n)a_n|^r\right) \\ &\quad + T^{-\max\{\frac{m}{3}-\frac{1}{6},\frac{1}{4}\}}(a_n)n^{\max\{\frac{m}{6}-\frac{1}{3},0\}}\left|a_n^{1/2}p_n w(x)\right|^m \log n, \\ D_n &:= T^{-\max\{\frac{m}{3}-\frac{1}{6},\frac{1}{4}\}}(a_n)n^{\max\{\frac{m}{6}-\frac{1}{3},0\}}\left|a_n^{1/2}p_n w(x)\right|^m \log n. \end{aligned}$$

Proof. Choose $l = l(x)$ such that $x \in [x_{l+1,n}, x_{ln}]$ and split

$$\sum\nolimits_r(x) := \sum\nolimits_{r1}(x) + \sum\nolimits_{r2}(x),$$

where $\sum_{r1}$ sums over those k in $\sum_r$ for which $k \in [l-3, l+3]$ and $\sum_{r2}$ contains the rest. For $|x_{k,n}| \geq a_{\beta n}$ by (5.9) and (5.4)

$$\begin{aligned} &\left|l_{k,n}(x)w^{-1}(x_{k,n})w(x)\right|^m |x-x_{k,n}|^r \\ \sim\ &\left(\tfrac{a_n}{n}\right)^{m-1}\left|a_n^{1/2}p_n w(x)\right|^m \frac{\Psi_n^{m-1}(x_{k,n})(1-|x_{k,n}|/a_n+L\delta_n)^{\frac{m}{4}}(x_{k,n}-x_{k+1,n})}{|x-x_{k,n}|^{m-r}}. \end{aligned}$$

Here, since for $|x_{k,n}| \geq a_{\beta n}$ by (5.2), (5.3), (5.12), and (5.15)

$$\Psi_n(x_{k,n}) \lesssim \frac{(1-|x_{k,n}|/a_n+L\delta_n)^{-1/2}}{T(a_n)}, \quad (1-|x_{k,n}|/a_n+L\delta_n)^{-1} \lesssim \delta_n^{-1}$$

and

$$(1-|x_{k,n}|/a_n+L\delta_n)^{1/4} \lesssim T^{-\frac{1}{4}}(a_n), \tag{3.3}$$

we obtain for $m \geq 2$

$$\begin{aligned} &\left|l_{k,n}(x)w^{-1}(x_{k,n})w(x)\right|^m |x-x_{k,n}|^r \\ \lesssim\ &\left(\frac{a_n}{n}T^{-\frac{1}{2}}(a_n)\right)^{m-1}\left|a_n^{1/2}p_n w(x)\right|^m T^{-\frac{m}{3}+\frac{1}{6}}(a_n)n^{\frac{m}{6}-\frac{1}{3}}\frac{(x_{k,n}-x_{k+1,n})}{|x-x_{k,n}|^{m-r}} \end{aligned}$$

and for $m = 1$ since $r = 0$, by (3.3)

$$\left|l_{k,n}(x)w^{-1}(x_{k,n})w(x)\right|^m |x - x_{k,n}|^r \lesssim \frac{\left|a_n^{1/2}p_n w(x)\right|}{T^{\frac{1}{4}}(a_n)} \frac{(x_{k,n} - x_{k+1,n})}{|x - x_{k,n}|}.$$

Therefore, we have

$$\begin{aligned} &w^m \sum\nolimits_{r2}(x) \\ \lesssim\ & \left(\frac{a_n}{n} T^{-\frac{1}{2}}(a_n)\right)^{m-r-1} T^{-\max\{\frac{m}{3}-\frac{1}{6},\frac{1}{4}\}}(a_n) n^{\max\{\frac{m}{6}-\frac{1}{3},0\}} \left|a_n^{1/2} p_n w(x)\right|^m \\ (3.4) \qquad & \times \sum_{|x_{k,n}| \ge a_{\beta n}} \frac{(x_{k,n} - x_{k+1,n})}{|x - x_{k,n}|^{m-r}}. \end{aligned}$$

Case 1. $|x| \le a_{\beta n/2}$, $|x| \ge a_{2n}$: For $|x_{k,n}| \ge a_{\beta n}$, $\sum_{r1} = 0$. Since we have by (5.5) and (5.15)

$$\left|a_n^{1/2} p_n w(x)\right| \lesssim |1 - |x|/a_n|^{-1/4} \lesssim T^{1/4}(a_n),$$

by (5.13)

$$|x - x_{k,n}| \gtrsim \begin{cases} a_n/T(a_n) & \text{if } |x| \le a_{\beta n/2}, \\ |x| & \text{if } |x| \ge a_{2n} \end{cases} \gtrsim \frac{a_n}{T(a_n)},$$

and by (5.12) and (5.13)

$$\sum_{|x_{k,n}| \ge a_{\beta n}} (x_{k,n} - x_{k+1,n}) \sim \frac{a_n}{T(a_n)},$$

we have by (3.4)

$$w^m \sum\nolimits_{r}(x) \lesssim \left(\frac{T^{\frac{1}{2}}(a_n)}{n}\right)^{m-r-1} T^{-\max\{\frac{m}{12}-\frac{1}{6},\frac{1-m}{4}\}}(a_n) n^{\max\{\frac{m}{6}-\frac{1}{3},0\}}.$$

Case 2. $a_{\beta n/2} \le |x| \le a_{2n}$: For $\sum_{r2}$, if we let $A := [a_{\beta n}, x_{0,n}] \setminus [x_{l+2,n}, x_{l-2,n}]$, then since for $a_{\beta n/2} \le |x| \le a_{2n}$ by (5.2)

$$\Psi_n^{-1}(x) \lesssim T(a_n)\sqrt{1 - |x|/a_n + L\delta_n} \lesssim T^{\frac{1}{2}}(a_n)$$

and by (5.4) and (5.12)

$$\sum_{|x_{k,n}| \ge a_{\beta n}} \frac{(x_{k,n} - x_{k+1,n})}{|x - x_{k,n}|^{m-r}} \lesssim \int_A \frac{dt}{|x - t|^{m-r}} \lesssim \left(\frac{n}{a_n} T^{\frac{1}{2}}(a_n)\right)^{m-r-1} \log n$$

we obtain by (3.4)

$$w^m \sum\nolimits_{r2}(x) \lesssim T^{-\max\{\frac{m}{3}-\frac{1}{6},\frac{1}{4}\}}(a_n) n^{\max\{\frac{m}{6}-\frac{1}{3},0\}} \left|a_n^{1/2} p_n w(x)\right|^m \log n.$$

For $\sum_{r1}$, we split this case into the following three cases.

Case 2-1. $a_{\beta n/2} \le |x| \le (1 - L\delta_n)a_n$: Since by (5.10), (5.6), and (5.4) for $-3 \le i \le 3$,

$$|x - x_{l+i,n}|^r \left|l_{l+i,n}(x)w^{-1}(x_{l+i,n})w(x)\right|^m \lesssim |x_{l-3,n} - x_{l+3,n}|^r \lesssim \left(\frac{n}{a_n}\Psi_n(x)\right)^r,$$

we have by (5.2)

$$\begin{aligned} w^m \sum\nolimits_{r1}(x) &\lesssim \left(\frac{n}{a_n}T^{\frac{1}{2}}(a_n)\right)^r \Big(|x - x_{l+3,n}|^r \left(l_{l+3,n}(x)w^{-1}(x_{l+3,n})w(x)\right)^m \\ &\qquad + \cdots + |x - x_{l-3,n}|^r \left(l_{l-3,n}(x)w^{-1}(x_{l-3,n})w(x)\right)^m\Big) \\ &\lesssim \left(T^{\frac{1}{2}}(a_n)\Psi_n(x)\right)^r \\ &\lesssim (T(a_n)(1 - |x|/a_n))^{\frac{r}{2}} + (T(a_n)(1 - |x|/a_n))^{-\frac{r}{2}}. \end{aligned}$$

Case 2-2. $(1 - L\delta_n)a_n \le |x| \le (1 + L\delta_n)a_n$: By (5.10), (5.2), (5.3), and (5.4),

$$w^m \sum\nolimits_{r1}(x) \lesssim \left(\frac{n}{a_n}T^{\frac{1}{2}}(a_n)\right)^r \left(||x| - (1 + 2L\delta_n)a_n|^r + ||x| - (1 - 2L\delta_n)a_n|^r\right).$$

Case 2-3. $(1 + L\delta_n)a_n \le |x| \le a_{2n}$: From (5.3) $w^m \sum_{r1}(x) = 0$. Therefore, Lemma 3.2 is proved completely. □

4. Proofs of Theorem 2.2 and Theorem 2.3.

In this section, we prove our main results. To prove Theorem 2.2, we split our functions to be approximated into pieces that vanishes inside or outside $[-a_{n/9}, a_{n/9}]$. For simplicity, we use the following notation: For $r = 0, \ldots, m-1$, let

$$H_{n,m,r}[f](x) := \sum_{k=1}^{n} e_{r,k} l_{k,n}^m(x)(x - x_{k,n})^r f(x_{k,n})$$

and

$$\phi(x) := \log^{-(1+\kappa)}(2 + x^2).$$

Then

$$H_{n,m}[w^2; f](x) = \sum_{r=0}^{m-1} H_{n,m,r}[f](x). \tag{4.1}$$

Throughout this section, we assume (2.2).

Lemma 4.1. *Let $1 < p < \infty$. Assume that $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f_n(x) = 0$ for $|x| < a_{n/9}$ and*

$$|f_n(x)w^m(x)| \lesssim \tilde{Q}^{-1}(x) \quad x \in \mathbb{R} \text{ and } n \ge 1. \tag{4.2}$$

Then for $r = 0, 1, \ldots, m-1$, we have

$$\lim_{n\to\infty} ||H_{n,m,r}[f_n](x)w^m(x)(1 + Q(x))^{-\Delta}||_{L_p(\mathbb{R})} = 0.$$

Proof. Since $\tilde{Q}(x)$ is increasing, we have by (3.1), (4.2), and the definition of $\sum_r$ in Lemma 3.2,

$$\begin{aligned} & |H_{n,m,r}[f_n](x)w^m(x)| \\ \lesssim & \sum_{|x_{k,n}|\geq a_{n/9}} |e_{r,k}| \left|l_{k,n}(x)w^{-1}(x_{k,n})w(x)\right|^m |x-x_{k,n}|^r \tilde{Q}^{-1}(x_{k,n}) \\ \lesssim & \tilde{Q}^{-1}(a_{n/9}) \left|w^m(x)\sum\nolimits_r(x)\right|, \end{aligned}$$

where by (5.11) and (5.12), $\tilde{Q}^{-1}(a_{n/9})$ is estimated as

$$\tilde{Q}^{-1}(a_{n/9}) \sim (nT^{-\frac{1}{2}}(a_n))^{-\max\{\frac{m}{6}-\frac{1}{3},0\}} T^{-\frac{m-1}{2}}(a_n)\phi(a_{n/9}). \tag{4.3}$$

In order to use Lemma 3.2, we let

$$\begin{aligned} S_1 &:= \{\, x \mid |x| \leq a_{n/18},\ |x| \geq a_{2n} \,\}, \\ S_2 &:= \{\, x \mid a_{n/18} \leq |x| \leq (1-L\delta_n)a_n \,\}, \\ S_3 &:= \{\, x \mid (1-L\delta_n)a_n \leq |x| \leq (1+L\delta_n)a_n \,\}, \\ S_4 &:= \{\, x \mid (1+L\delta_n)a_n \leq |x| \leq a_{2n} \,\}. \end{aligned}$$

Then we have

$$||H_{n,m,r}[f_n](x)w^m(x)(1+Q(x))^{-\Delta}||_{L_p(\mathbb{R})} \lesssim \tau_n^{(1)} + \tau_n^{(2)} + \tau_n^{(3)} + \tau_n^{(4)},$$

where for $1 \leq i \leq 4$

$$\tau_n^{(i)} := \tilde{Q}^{-1}(a_{n/9})||w^m(x)\sum\nolimits_r(x)(1+Q(x))^{-\Delta}||_{L_p(S_i)}.$$

For $\tau_n^{(1)}$, since Q grows faster than any power of x from (a) of Proposition 5.3, we have by Lemma 3.2 and (4.3)

$$\begin{aligned} \lim_{n\to\infty} \tau_n^{(1)} &\lesssim \lim_{n\to\infty} \tilde{Q}^{-1}(a_{n/9})A_n||(1+Q(x))^{-\Delta}||_{L_p(S_1)} \\ &\lesssim \lim_{n\to\infty} T^{-\frac{m-1}{2}}(a_n)\phi(a_{n/9}) = 0. \end{aligned}$$

Next, for $i = 2,3,4$, we know by (5.8)

$$\left\|\left(a_n^{1/2}p_n w(x)\right)^m\right\|_{L_p(S_i)} \lesssim a_n^{\frac{1}{p}} \left(nT(a_n)\right)^{\frac{2}{3}\max\{\frac{m}{4}-\frac{1}{p},0\}} (\log n)^{\frac{m}{4}}. \tag{4.4}$$

Then for $\tau_n^{(2)}$

$$\tau_n^{(2)} \lesssim \tilde{Q}^{-1}(a_{n/9})||B_n(1+Q(x))^{-\Delta}||_{L_p(S_2)}.$$

Since

$$||(1-|x|/a_n)^{\frac{r}{2}}||_{L_p(S_2)} \lesssim a_n^{\frac{1}{p}} T^{-(\frac{r}{2}+\frac{1}{p})}(a_n)$$

and

$$||(1-|x|/a_n)^{-\frac{r}{2}}||_{L_p(S_2)} \lesssim a_n^{\frac{1}{p}} (nT(a_n))^{\max\{\frac{2}{3}(\frac{r}{2}-\frac{1}{p}),0\}} (\log n)^{\frac{1}{p}},$$

we have by the monotonicity of Q, (4.3), (5.11) and (4.4),

$$\tau_n^{(2)} \lesssim n^{-\Delta+\max\{\frac{2}{3}(\frac{m}{4}-\frac{1}{p}),0\}}(a_n, T(a_n), \log n).$$

Here, we set for some constants C_1, C_2, and C_3, independent of n,

$$(a_n, T(a_n), \log n) := a_n^{C_1} T(a_n)^{C_2} \log n^{C_3},$$

where the constants C_1, C_2, C_3, need not be the same in different occurrences. Hence, we obtain by (2.2) and (5.14)

$$\lim_{n\to\infty} \tau_n^{(2)} = 0.$$

Similarly, for $\tau_n^{(3)}$

$$\tau_n^{(3)} \lesssim \tilde{Q}^{-1}(a_{n/9})||C_n(1+Q(x))^{-\Delta}||_{L_p(S_2)}.$$

Since

$$||\left(|x| - (1 \pm 2C\delta_n)a_n\right)^r||_{L_p(S_3)} \lesssim (nT(a_n))^{-\frac{2}{3}(r+\frac{1}{p})} a_n^{r+\frac{1}{p}}$$

we have by (4.3), (5.11), and (4.4),

$$\tau_n^{(3)} \lesssim \left(n^{-\Delta-\max\{\frac{m}{6}-\frac{1}{3},0\}+\frac{2}{3}(\frac{r}{2}-\frac{1}{p})} + n^{-\Delta+\max\{\frac{2}{3}(\frac{m}{4}-\frac{1}{p}),0\}}\right)(a_n, T(a_n), \log n)$$

and then by (2.2) and (5.14)

$$\lim_{n\to\infty} \tau_n^{(3)} = 0.$$

Finally, for $\tau_n^{(4)}$

$$\begin{aligned}\tau_n^{(4)} &\lesssim \tilde{Q}^{-1}(a_{n/9})||D_n(1+Q(x))^{-\Delta}||_{L_p(S_2)} \\ &\lesssim n^{-\Delta+\max\{\frac{2}{3}(\frac{m}{4}-\frac{1}{p}),0\}}(a_n, T(a_n), \log n)\end{aligned}$$

and we have by (2.2) and (5.14)

$$\lim_{n\to\infty} \tau_n^{(4)} \lesssim \epsilon.$$

Therefore,we have for $r = 0, 1, \ldots, m-1$,

$$\lim_{n\to\infty} ||H_{n,m,r}[f_n](x)w^m(x)(1+Q(x))^{-\Delta}||_{L_p(\mathbb{R})} = 0.$$

□

Lemma 4.2. *Let $1 < p < \infty$. Assume that $\{g_n\}_{n=1}^{\infty}$ is a sequence of measurable functions from $\mathbb{R}$ to $\mathbb{R}$ such that $g_n(x) = 0$ for $|x| \geq a_{n/9}$ and*

$$|g_n(x)w^m(x)| \lesssim \tilde{Q}^{-1}(x).$$

Then for $r = 0, 1, \ldots, m-1$, we have

$$\lim_{n\to\infty} ||H_{n,m,r}[g_n](x)w^m(x)(1+Q(x))^{-\Delta}||_{L_p(|x|\geq a_{n/8})} = 0.$$

Proof. For $|x| \geq a_{n/8}$ and $|x_{k,n}| \leq a_{n/9}$, we estimate some factors. Since $\tilde{Q}$ and Q are increasing,

$$Q^{-\Delta}(x) \lesssim Q^{-\Delta}(a_n), \quad \tilde{Q}^{-1}(x_{k,n}) \lesssim 1,$$

and we have by (5.15) and (5.2)

$$\frac{1}{|x - x_{k,n}|} \lesssim \frac{1}{a_{\frac{n}{8}} - |x_{k,n}|} \lesssim \frac{T(x_{k,n})}{a_{\frac{n}{8}}},$$

$$(1 - |x_{k,n}|/a_n + L\delta_n) \lesssim 1, \quad \text{and} \quad \Psi_n(x_{k,n}) \lesssim 1. \tag{4.5}$$

By (5.4), (5.9), (5.12), and the above estimations,

$$\begin{aligned} &\left|l_{k,n}(x)w^{-1}(x_{k,n})w(x)\right|^m |x - x_{k,n}|^r \tilde{Q}^{-1}(x_{k,n}) \\ \lesssim\ &\left(\frac{a_n}{n}\right)^{m-1} \left|a_n^{1/2}p_n(x)w(x)\right|^m \left(\frac{T(a_n)}{a_n}\right)^{m-r-1} (x_{k,n} - x_{k+1,n}). \end{aligned} \tag{4.6}$$

Then we have for $r = 0, 1, \ldots, m-1$ and $|x| \geq a_{n/8}$ by (4.6), (5.9), and (3.1),

$$\begin{aligned} &|H_{n,m,r}[g_n](x)w^m(x)(1 + Q(x))^{-\Delta}| \\ \lesssim\ &Q^{-\Delta}(a_n) \sum_{|x_{k,n}| \leq a_{n/9}} |e_{r,k}| \left|l_{k,n}(x)w^{-1}(x_{k,n})w(x)\right|^m |x - x_{k,n}|^r \tilde{Q}^{-1}(x_{k,n}) \\ \lesssim\ &n^{-\Delta-m+1+r} \left|a_n^{1/2}p_n(x)w(x)\right|^m (a_n, T(a_n), \log n) \quad \text{by (5.11)} \\ \lesssim\ &n^{-\Delta} \left|a_n^{1/2}p_n(x)w(x)\right|^m (a_n, T(a_n), \log n). \end{aligned}$$

Therefore, since by (4.4)

$$\begin{aligned} &||H_{n,m,r}[g_n](x)w^m(x)(1 + Q(x))^{-\Delta})||_{L_p(|x| \geq a_{n/8})} \\ \lesssim\ &n^{-\Delta + \max\{\frac{2}{3}(\frac{m}{4} - \frac{1}{p}), 0\}}(a_n, T(a_n), \log n), \end{aligned}$$

we have by (2.2) and (5.14),

$$\lim_{n\to\infty} ||H_{n,m,r}[g_n](x)w^m(x)(1 + Q(x))^{-\Delta}||_{L_p(|x| \geq a_{n/8})} = 0.$$

□

Lemma 4.3. *Let $1 < p < \infty$ and $\epsilon > 0$. Assume that $\tilde{\Delta} > \max\{\frac{2}{3}(\frac{1}{4} - \frac{1}{p}), 0\}$ and $\{g_n\}_{n=1}^{\infty}$ is a sequence of measurable functions from $\mathbb{R}$ to $\mathbb{R}$ such that $g_n(x) = 0$ for $|x| \geq a_{n/9}$ and*

$$|g_n(x_{k,n})w(x_{k,n})| \lesssim \epsilon\phi(x_{k,n}) \quad 1 \leq k \leq n.$$

Then

$$\limsup_{n\to\infty} ||L_n[g_n](x)w(x)(1 + Q(x))^{-\tilde{\Delta}}||_{L_p(|x| \leq a_{n/8})} \lesssim \epsilon.$$

Proof. It is proved in [1, Lemma 4.4]. □

Lemma 4.4. *Let* $1 < p < \infty$, $\tilde{\Delta} > 0$ *and* $\epsilon > 0$. *Assume that* $\{g_n\}_{n=1}^{\infty}$ *is a sequence of measurable functions from* $\mathbb{R}$ *to* $\mathbb{R}$ *such that* $g_n(x) = 0$ *for* $|x| \geq a_{n/9}$ *and*

$$\left|g_n(x)w^2(x)\right| \lesssim \epsilon\phi(x) \quad x \in \mathbb{R}.$$

Then

$$\limsup_{n\to\infty} \|\sum_{k=1}^{n} l_{k,n}^2(x)w^2(x)g_n(x_{k,n})(1+Q(x))^{-\tilde{\Delta}}\|_{L_p(|x|\leq a_{n/8})} \lesssim \epsilon.$$

Proof. It follows easily from Lemma 3.4 (b) in [4]. □

Lemma 4.5. *Let* $1 < p < \infty$. *Assume that* $\{g_n\}_{n=1}^{\infty}$ *is a sequence of measurable functions from* $\mathbb{R}$ *to* $\mathbb{R}$ *such that* $g_n(x) = 0$ *for* $|x| \geq a_{n/9}$ *and*

$$|g_n(x)w^m(x)| \lesssim \epsilon\tilde{Q}^{-1}(x) \quad x \in \mathbb{R}. \tag{4.7}$$

Then for $r = 0, 1, \ldots, m-1$, *we have*

$$\limsup_{n\to\infty} \|H_{n,m,r}[g_n](x)w^m(x)(1+Q(x))^{-\Delta}\|_{L_p(|x|\leq a_{n/8})} \lesssim \epsilon.$$

Proof. Note that for $|x| \leq a_{n/8}$ by (5.5) and (5.15)

$$\left|a_n^{1/2}p_n(x)w(x)\right| \lesssim (1-|x|/a_n)^{-1/4} \lesssim T^{\frac{1}{4}}(x). \tag{4.8}$$

First, we estimate $H_{n,m,m-1}[g_n](x)$. For $|x| \leq a_{n/8}$ and $|x_{k,n}| \leq a_{n/9}$

$$\begin{aligned}
&|w^m(x)H_{n,m,m-1}[g_n](x)(1+Q(x))^{-\Delta}| \\
\lesssim\;& T^{\frac{m-1}{4}}(x)\left|\sum_{k=1}^{k} e_{m-1,k}l_{k,n}(x)w(x)\left(a_n^{1/2}p_n'(x_{k,n})\right)^{-(m-1)}\right. \\
&\qquad \left.\times g_n(x_{k,n})(1+Q(x))^{-\Delta}\right|
\end{aligned}$$

by (4.8). If we let

$$\tilde{g}_n(x) := g_n(x)\alpha_n(x), \quad x \in \mathbb{R} \text{ and } n \geq 1,$$

where $\alpha_n(x)$ is a function satisfying

$$\alpha_n(x_{k,n}) = e_{m-1,k}\left(a_n^{1/2}p_n'(x_{k,n})\right)^{-(m-1)}, \quad k = 1, 2, \ldots, n$$

then we have

$$\begin{aligned}
&|w^m(x)H_{n,m,m-1}[g_n](x)(1+Q(x))^{-\Delta}| \\
\lesssim\;& T^{\frac{m-1}{4}}(x)\left|\sum_{k=1}^{n} l_{k,n}(x)w(x)\tilde{g}_n(x_{k,n})(1+Q(x))^{-\Delta}\right| \\
=\;& T^{\frac{m-1}{4}}(x)\left|L_n[\tilde{g}_n](x)w(x)(1+Q(x))^{-\Delta}\right|.
\end{aligned}$$

By (2.2), choose $\hat{\Delta}$ satisfying

$$\Delta > \hat{\Delta} > \max\left\{\frac{2}{3}\left(\frac{m}{4}-\frac{1}{p}\right), 0\right\} \geq \max\left\{\frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right), 0\right\}.$$

Then

$$\limsup_{n\to\infty} \|w^m(x)H_{n,m,m-1}[g_n](x)(1+Q(x))^{-\Delta}\|_{L_p(|x|\le a_{n/8})}$$

$$(4.9) \qquad \lesssim \limsup_{n\to\infty} \|L_n[\tilde{g}_n](x)w(x)(1+Q(x))^{-\hat{\Delta}}\|_{L_p(|x|\le a_{n/8})}$$

since $|T^{\frac{m-1}{4}}(x)Q^{-(\Delta-\hat{\Delta})}(x)|$ is bounded by (5.11) and (5.14). Here, the definition of $\tilde{g}_n(x)$ implies that $\tilde{g}_n(x) = 0$ on $|x| \ge a_{n/9}$ and since for $|x_{k,n}| \le a_{n/9}$ by (4.5) and (5.7)

$$(4.10) \qquad \left|\frac{1}{p_n'(x_{k,n})}\right| \lesssim \frac{a_n^{3/2}}{n} w(x_{k,n}),$$

for $|x_{k,n}| < a_{n/9}$, by (2.1), (3.1), and the monotonicity of Q,

$$|\tilde{g}_n(x_{k,n})w(x_{k,n})| \lesssim T^{\frac{m-1}{2}}(x_{k,n})|g_n(x_{k,n})|w^m(x_{k,n}) \lesssim \epsilon\phi_n(x_{k,n}).$$

Therefore, we have by Lemma 4.3 and (4.9),

$$\limsup_{n\to\infty} \|H_{n,m,m-1}[g_n](x)w^m(x)(1+Q(x))^{-\Delta}\|_{L_p(|x|\le a_{n/8})} \lesssim \epsilon.$$

Specially, if $m = 1$, then this lemma is proved. In fact, for the case of $m = 1$ Lemma 4.5 means Lemma 4.3 directly. Next, we assume $m \ge 2$ and estimate $H_{n,m,r}[g_n](x)$ for $0 \le r < m-1$. For $0 \le r < m-1$ we have by (3.1), (4.8), and (4.10),

$$(4.11) \qquad \begin{aligned} & \left|e_{r,k} l_{k,n}^m(x) w^m(x)(x-x_{k,n})^r g_n(x_{k,n})\right| \\ \lesssim \;& T^{\frac{r}{4}}(x) l_{k,n}^2(x) w^2(x) |g_n(x_{k,n})| w^{m-2}(x_{k,n}) T^{\frac{m-2}{2}}(x_{k,n}). \end{aligned}$$

If we set

$$\hat{g}_n(x) := g_n(x)w^{m-2}(x)T^{\frac{m-2}{2}}(x), \quad x \in \mathbb{R} \text{ and } n \ge 1,$$

then $\hat{g}_n(x) = 0$ on $|x| \ge a_{n/9}$ and by (2.1) and (4.7)

$$\left|\hat{g}_n(x)w^2(x)\right| = \left|g_n(x)w^m(x)T^{\frac{m-2}{2}}(x)\right| \lesssim \epsilon\phi(x).$$

Therefore, we have for $r = 0, 1, \ldots, m-1$ by (4.11) and Lemma 4.4

$$\begin{aligned} & \limsup_{n\to\infty} \|H_{n,m,r}[g_n](x)w^m(x)(1+Q(x))^{-\Delta}\|_{L_p(|x|\le a_{n/8})} \\ \lesssim \;& \limsup_{n\to\infty} \left\|\sum_{k=1}^n e_{r,k} l_{k,n}^m(x) w^m(x)(x-x_{k,n})^r g_n(x_{k,n})(1+Q(x))^{-\Delta}\right\|_{L_p(|x|\le a_{n/8})} \\ \lesssim \;& \limsup_{n\to\infty} \left\|\sum_{k=1}^n l_{k,n}^2(x) w^2(x) \hat{g}_n(x_{k,n})(1+Q(x))^{-\frac{\Delta}{2}}\right\|_{L_p(|x|\le a_{n/8})} \lesssim \epsilon \end{aligned}$$

since $T^{\frac{r}{4}}(x)Q^{-\frac{\Delta}{2}}(x) < \infty$ by (5.11) and (5.14). □

Let

$$\tilde{H}_{n,m,r}[f](x) := \sum_{k=1}^{n} \left(\frac{n}{a_n} T^{\frac{1}{2}}(x_{k,n})\right)^r l_{k,n}^m(x)(x - x_{k,n})^r f(x_{k,n}).$$

Since $|e_{r,k}| \lesssim \left(\frac{n}{a_n} T^{\frac{1}{2}}(x_{k,n})\right)^r \lesssim \left(\frac{n}{a_n} T^{\frac{1}{2}}(a_n)\right)^r$, if we inspect the proofs of Lemma 4.1, Lemma 4.2, and Lemma 4.5, we can check the following corollaries easily.

Corollary 4.6. (a) *Under the same conditions as in Lemma 4.1, we have for* $0 \leq r \leq m-1$,

$$\limsup_{n\to\infty} ||\tilde{H}_{n,m,r}[f_n](x) w^m(x)(1+Q(x))^{-\Delta}||_{L_p(\mathbb{R})} = 0.$$

(b) *Under the same conditions as in Lemma 4.2, we have for* $0 \leq r \leq m-1$,

$$\limsup_{n\to\infty} ||\tilde{H}_{n,m,r}[g_n](x) w^m(x)(1+Q(x))^{-\Delta}||_{L_p(|x|\geq a_{n/8})} = 0$$

(c) *Under the same conditions as in Lemma 4.5, we have for* $0 \leq r \leq m-1$,

$$\limsup_{n\to\infty} ||\tilde{H}_{n,m,r}[g_n](x) w^m(x)(1+Q(x))^{-\Delta}||_{L_p(|x|\leq a_{n/8})} = 0.$$

Lemma 4.7. *Let* $1 < p < \infty$. *Then for any fixed polynomial* $R(x)$,

$$\limsup_{n\to\infty} ||(H_{n,m}[w^2; R](x) - R(x)) w^m(x)(1+Q(x))^{-\Delta}||_{L_p(\mathbb{R})} = 0.$$

Proof. For any fixed polynomial $R(x)$ since $\left|w^{m-1}(x)\tilde{Q}(x)\right|$ is bounded, we have by (5.1)

$$|R^{(t)}(x) w^m(x) \tilde{Q}(x)| \leq M, \quad x \in \mathbb{R},\ t = 0, 1, \ldots, m-1,$$

where M is a constant independent of x and t. Then for $n \geq \deg R$, by (1.2) and (1.4),

$$R(x) - H_{n,m}[w^2; R](x) = \begin{cases} \sum_{t=1}^{m-1} \sum_{k=1}^{n} R^{(t)}(x_{k,n}) h_{t,k}(x), & m \geq 2 \\ 0, & m = 1. \end{cases} \tag{4.12}$$

So, this lemma is proved for $m = 1$. Now, we assume $m \geq 2$. For $1 \leq t \leq m-1$

$$h_{t,k}(x) = \frac{1}{t!} \sum_{i=0}^{m-1-t} \frac{e_{t,i,k}}{\left(\frac{n}{a_n} T^{\frac{1}{2}}(x_{k,n})\right)^{t+i}} \left(\frac{n}{a_n} T^{\frac{1}{2}}(x_{k,n})\right)^{t+i} l_{k,n}^m(x)(x - x_{k,n})^{t+i}. \tag{4.13}$$

If we set

$$R_n^{[t,i]}(x) := R^{(t)}(x) r_n^{[i]}(x),$$

where $r_n^{[i]}(x)$ is a function satisfying

$$r_n^{[i]}(x_{k,n}) = \frac{e_{t,i,k}}{\left(\frac{n}{a_n} T^{\frac{1}{2}}(x_{k,n})\right)^{t+i}} \quad k = 1, 2, \ldots, n, \tag{4.14}$$

then for sufficiently large n by (3.1) and (5.14)

$$|R_n^{[t,i]}(x_{k,n})w^m(x_{k,n})\tilde{Q}(x_{k,n})| \lesssim \left(\frac{n}{a_n}T^{\frac{1}{2}}(x_{k,n})\right)^{-t} \leq \epsilon.$$

Then by (4.12) and (4.13)

$$R(x) - H_{n,m}[w^2; R](x) = \sum_{t=1}^{m-1}\sum_{i=0}^{m-1-t}\frac{1}{t!}\tilde{H}_{n,m,t+i}[R_n^{[t,i]}]$$

and

$$\begin{aligned}&||(H_{n,m}[w^2; R](x) - R(x))w^m(x)(1 + Q(x))^{-\Delta}||_{L_p(\mathbb{R})}\\ =& \sum_{t=1}^{m-1}\sum_{i=0}^{m-1-t}\frac{1}{t!}||\tilde{H}_{n,m,t+i}[R_n^{[t,i]}]w^m(x)(1 + Q(x))^{-\Delta}||_{L_p(\mathbb{R})}.\end{aligned}$$

Let χ_n be the characteristic function of $[-a_{n/9}, a_{n/9}]$ and

$$R_n^{[t,i]}(x) = \chi_n R_n^{[t,i]}(x) + (1 - \chi_n)R_n^{[t,i]}(x) := g_n + f_n.$$

Then by Corollary 4.6

$$\limsup_{n\to\infty} ||(H_{n,m}[w^2; R](x) - R(x))w^m(x)(1 + Q(x))^{-\Delta}||_{L_p(\mathbb{R})} = 0.$$

□

Proof of Theorem 2.2. First we assume that $1 < p < \infty$. For any $\epsilon > 0$, we can find a polynomial $P(x)$ such that

$$|f(x) - P(x)|w^m(x)\tilde{Q}(x) \leq \epsilon, \quad x \in \mathbb{R}$$

(cf. [2, p. 180]). Then for sufficiently large n,

$$\begin{aligned}&||(f(x) - H_{n,m}[w^2; f](x))w^m(x)(1 + Q(x))^{-\Delta}||_{L_p(\mathbb{R})}\\ \leq\ & ||(f(x) - P(x))w^m(x)(1 + Q(x))^{-\Delta}||_{L_p(\mathbb{R})}\\ &+||(P(x) - H_{n,m}[w^2; P](x))w^m(x)(1 + Q(x))^{-\Delta}||_{L_p(\mathbb{R})}\\ &+||H_{n,m}[w^2; P - f](x)w^m(x)(1 + Q(x))^{-\Delta}||_{L_p(\mathbb{R})}.\end{aligned}$$

For the first term, since $Q(x)$ grows faster than any power of x and $\tilde{Q}(x)$ is increasing,

$$||(f(x) - P(x))w^m(x)(1+Q(x))^{-\Delta}||_{L_p(\mathbb{R})} \leq \epsilon||\tilde{Q}^{-1}(x)(1+Q(x))^{-\Delta}||_{L_p(\mathbb{R})} \lesssim \epsilon$$

and for the second term by Lemma 4.7,

$$\limsup_{n\to\infty} ||(P(x) - H_{n,m}[w^2; P](x))w^m(x)(1 + Q(x))^{-\Delta}||_{L_p(\mathbb{R})} \lesssim \epsilon.$$

For the third term we can apply Lemma 4.1, Lemma 4.2, and Lemma 4.5. Let χ_n be the characteristic function of $[-a_{n/9}, a_{n/9}]$ and write

$$P - f = (P - f)\chi_n + (P - f)(1 - \chi_n) := g_n + f_n.$$

Then we deduce that from (4.1)

$$\begin{aligned}&\limsup_{n\to\infty}||H_{n,m}[w^2;P-f](x)w^m(x)(1+Q(x))^{-\Delta}||_{L_p(\mathbb{R})}\\ \le\ &\sum_{r=0}^{m-1}\limsup_{n\to\infty}||H_{n,m,r}[P-f](x)w^m(x)(1+Q(x))^{-\Delta}||_{L_p(\mathbb{R})}\lesssim\epsilon.\end{aligned}$$

Therefore, we have for $1<p<\infty$

$$\limsup_{n\to\infty}||(f(x)-H_{n,m}[w^2;f](x))w^m(x)(1+Q(x))^{-\Delta}||_{L_p(\mathbb{R})}\lesssim\epsilon.$$

Moreover, for $m\ge 2$ if we set

$$f_n^{[t,i]}(x):=f^{(t)}(x)r_n^{[i]}(x),$$

then by (1.4) and (4.13), similarly to Lemma 4.7

$$\hat{H}_{n,m}[w^2;f](x)=H_{n,m}[w^2;f](x)+\sum_{t=1}^{m-1}\sum_{i=0}^{m-1-t}\frac{1}{t!}\tilde{H}_{n,m,t+i}[f_n^{[t,i]}](x),$$

where $r_n^{[i]}(x)$ is defined in (4.14). Since for sufficiently large n by (2.4), (3.1), and (5.14),

$$|f_n^{[t,i]}(x_{k,n})w^m(x)\tilde{Q}(x_{k,n})|=\frac{|e_{t,i,k}|}{\left(\frac{n}{a_n}T^{\frac{1}{2}}(x_{k,n})\right)^{t+i}}f^{(t)}(x_{k,n})w^m(x_{k,n})\tilde{Q}(x_{k,n})\lesssim\epsilon$$

and

$$\begin{aligned}&||(\hat{H}_{n,m}[w^2;f](x)-f(x))w^m(x)(1+Q(x))^{-\Delta}||_{L_p(\mathbb{R})}\\ \le\ &||(H_{n,m}[w^2;f](x)-f(x))w^m(x)(1+Q(x))^{-\Delta}||_{L_p(\mathbb{R})}\\ &+\sum_{t=1}^{m-1}\sum_{i=0}^{m-1-t}\frac{1}{t!}||\tilde{H}_{n,m,t+i}[f_n^{[t,i]}](x)w^m(x)(1+Q(x))^{-\Delta}||_{L_p(\mathbb{R})},\end{aligned}$$

we also have by the same method as above

$$\limsup_{n\to\infty}||(f(x)-\hat{H}_{n,m}[w^2;f](x))w^m(x)(1+Q(x))^{-\Delta}||_{L_p(\mathbb{R})}\lesssim\epsilon.$$

Hence, Theorem 2.2 is proved for $1<p<\infty$, since $\epsilon>0$ is arbitrary.

Now, we assume that $0<p\le 1$. By $\Delta>\max\left\{\frac{2}{3}\left(\frac{m}{4}-1\right),0\right\}$, we know that $\Delta>0$ and $1>\frac{m}{4}-\frac{3}{2}\Delta$. So, we can choose q satisfying $pq>1$ and $1>\frac{1}{pq}>\frac{m}{4}-\frac{3}{2}\Delta$ so that $\Delta>\max\left\{\frac{2}{3}\left(\frac{m}{4}-\frac{1}{pq}\right),0\right\}$. Then, choose Δ_1 satisfying

$$\Delta>\Delta_1>\max\left\{\frac{2}{3}\left(\frac{m}{4}-\frac{1}{pq}\right),0\right\} \tag{4.15}$$

and let q' be the conjugate of q, that is, $\frac{1}{q}+\frac{1}{q'}=1$. Since by Hölder's inequality,

$$\begin{aligned} &||(f(x)-H_{n,m}[w^2;f](x))w^m(x)(1+Q(x))^{-\Delta}||^p_{L_p(\mathbb{R})} \\ \leq\ & \left(\int |(f(x)-H_{n,m}[w^2;f](x))w^m(x)(1+Q(x))^{-\Delta_1}|^{pq}dx\right)^{1/q} \\ & \times\left(\int (1+Q(x))^{-(\Delta-\Delta_1)pq'}dx\right)^{1/q'}, \end{aligned}$$

we have from the result of the case $1<p<\infty$

$$\lim_{n\to\infty}\int |(f(x)-H_{n,m}[w^2;f](x))w^m(x)(1+Q(x))^{-\Delta_1}|^{pq}dx=0$$

by (4.15) and $pq>1$ and since $\Delta-\Delta_1>0$ and $Q(x)$ grows faster that any power of x, we have

$$\int (1+Q(x))^{-(\Delta-\Delta_1)pq'}dx<\infty.$$

Therefore, for $0<p\leq 1$

$$\lim_{n\to\infty}||(f(x)-H_{n,m}[w^2;f](x))w^m(x)(1+Q(x))^{-\Delta}||_{L_p(\mathbb{R})}=0.$$

By the same method as above, we also have for $0<p\leq 1$

$$\lim_{n\to\infty}||(f(x)-\hat{H}_{n,m}[w^2;f](x))w^m(x)(1+Q(x))^{-\Delta}||_{L_p(\mathbb{R})}=0.$$

This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3. Since the proof for $\hat{I}_{n,m}[k;f]$ is the same as the case of $I_{n,m}[k;f]$, we prove only the case of $I_{n,m}[k;f]$. For $p>1$, we have by Hölder inequality and Theorem 2.2

$$\begin{aligned} |I_{n,m}[k;f]-I[k;f]| &= \left|\int_{\mathbb{R}} k(x)\left(H_{n,m}[w^2;f](x)-f(x)\right)dx\right| \\ &\leq \left\|k(x)w^{-m}(x)(1+Q(x))^{\Delta}\right\|_{L_p(\mathbb{R})} \\ &\quad\times\left\|(f-H_{n,m}[w^2;f])w^m(x)(1+Q(x))^{-\Delta}\right\|_{L_q(\mathbb{R})} \\ &\longrightarrow 0 \quad \text{as } n\to\infty, \end{aligned}$$

where q is the conjugate of p, that is, $\frac{1}{q}=1-\frac{1}{p}$. □

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5. Appendix

We let a_u, for $u > 0$, be the u-th Mhaskar-Rakhmanov-Saff number, which is the unique positive root of the equation

$$u = \frac{2}{\pi} \int_0^1 a_u t Q(a_u t) dt / \sqrt{1-t^2}.$$

Then, a_u is increasing with u. The importance of a_n lies in the identity

$$\|Pw\|_{L_\infty[I]} = \|Pw\|_{L_\infty[-a_n, a_n]} \tag{5.1}$$

for any polynomial $P \in \mathcal{P}_n$ (cf. [8]). We define some auxiliary quantities which we will need in the sequel, see [6]. Set:

$$\delta_n := (nT(a_n))^{-\frac{2}{3}}, n \geq 1.$$

We shall also need the sequence of functions

$$\Psi_n(x) := \begin{cases} \max\left\{\sqrt{1 - \frac{|x|}{a_n} + L\delta_n}, \frac{1}{T(a_n)\sqrt{1-\frac{|x|}{a_n}+L\delta_n}}\right\}, & |x| \leq a_n, \\ \Psi_n(a_n), & |x| \geq a_n. \end{cases} \tag{5.2}$$

We begin by recalling a number of estimates from [6]. Throughout the sections, we assume that $w \in \mathcal{E}_1$.

Proposition 5.1. (a) *For $n \geq 1$, and for some constant L*

$$|x_{1,n}/a_n - 1| \leq \frac{L}{2}\delta_n, \tag{5.3}$$

and uniformly for $n \geq 2$ and $1 \leq j \leq n-1$,

$$x_{j,n} - x_{j+1,n} \sim \frac{a_n}{n}\Psi_n(x_{j,n}). \tag{5.4}$$

(b) *For $n \geq 1$,*

$$\sup_{x\in\mathbb{R}} |p_n(x)|w(x)|1 - |x|/a_n|^{1/4} \sim a_n^{-1/2}. \tag{5.5}$$

(c) *Uniformly for $n \geq 2$ and $1 \leq j \leq n-1$,*

$$1-|x_{j,n}|/a_n+L\delta_n \sim 1-|x_{j+1,n}|/a_n+L\delta_n \quad \textit{and} \quad \Psi_n(x_{j,n}) \sim \Psi_n(x_{j+1,n}). \tag{5.6}$$

(d) *Uniformly for $n \geq 2$ and $2 \leq j \leq n-1$,*

$$\frac{a_n^{3/2}}{n}\Psi_n(x_{j,n})(1 - |x_{j,n}|/a_n + L\delta_n)^{1/2}|p_n'w|(x_{j,n}) \sim a_n^{1/2}|p_{n-1}w|(x_{j,n}) \tag{5.7}$$

$$\sim (1 - |x_{j,n}|/a_n + L\delta_n)^{1/4}.$$

Proof. (a) This is part of Corollary 1.3 in [6, p. 205].
(b) This is Corollary 1.4(a) in [6, p. 205].
(c) These are in [6, p. 265].
(d) This is Corollary 1.4(b) in [6, p. 205]. □

Next, we recall some results from [7], involving mostly the fundamental polynomials of Lagrange interpolation.

Proposition 5.2. (a) *Given* $0 < p < \infty$, *we have for* $n \geq 2$,

$$(5.8)\qquad \|p_n w\|_{L_p(\mathbb{R})} \sim a_n^{\frac{1}{p}-\frac{1}{2}} \begin{cases} 1, & p < 4, \\ (\log n)^{1/4}, & p = 4, \\ (nT(a_n))^{\frac{2}{3}(\frac{1}{4}-\frac{1}{p})}, & p > 4. \end{cases}$$

(b) *Uniformly for* $n \geq 1$, $1 \leq j \leq n$, *and* $x \in \mathbb{R}$,

$$(5.9)\qquad |l_{j,n}(x)| \sim \frac{a_n^{3/2}}{n}(\Psi_n w)(x_{j,n})(1 - |x_{j,n}|/a_n + L\delta_n)^{1/4}\left|\frac{p_n(x)}{x - x_{j,n}}\right|.$$

(c) *Uniformly for* $n \geq 1$, $1 \leq j \leq n$, *and* $x \in \mathbb{R}$,

$$(5.10)\qquad |l_{j,n}(x)| w^{-1}(x_{j,n}) w(x) \lesssim 1.$$

Proof. (a) This is Theorem 1.1 in [7].
(b),(c) These are Theorem 1.2 in [7]. □

Finally, some technical estimates on growth of a_n, $Q(a_n)$, $T(a_n)$, etc.

Proposition 5.3. (a) *Given* $r > 0$, *there exists* x_0 *such that for* $x \geq x_0$ *and* $j = 0, 1, 2$, $Q^{(j)}(x)/x^r$ *is increasing in* $[x_0, \infty)$.

(b) *Uniformly for* $u \geq C$ *and* $j = 0, 1, 2$,

$$(5.11)\qquad a_u^j Q^{(j)}(a_u) \sim uT(a_u)^{j-1/2}.$$

(c) *Let* $0 < \alpha < \beta$. *Then uniformly for* $u \geq C$ *and* $j = 0, 1, 2$,

$$(5.12)\qquad a_{\alpha u} \sim a_{\beta u}, \quad T(a_{\alpha u}) \sim T(a_{\beta u}), \quad \text{and} \quad Q^{(j)}(a_{\alpha u}) \sim Q^{(j)}(a_{\beta u}).$$

(d) *Uniformly for* $u \in (C, \infty)$, *and* $v \in [\frac{u}{2}, 2u]$, *we have*

$$(5.13)\qquad \left|\frac{a_u}{a_v} - 1\right| \sim \left|\frac{u}{v} - 1\right| \frac{1}{T(a_u)}.$$

(e) *For any* $\epsilon > 0$

$$(5.14)\qquad a_n \leq Cn^{\epsilon} \quad \text{and} \quad T(a_n) \leq Cn^{\epsilon},\ n \geq 1.$$

(f) *Let* $0 < \eta < 1$. *Uniformly for* $n \geq 1$, $0 < |x| \leq a_{\eta n}$ *and* $|x| = a_s$, *we have*

$$(5.15)\qquad C_1 \leq T(x)\left(1 - \frac{|x|}{a_n}\right) \leq C_2 \log\frac{n}{s}.$$

Proof. (a) This is Lemma 2.1(iii) in [6, p. 207].
(b)-(d) These are part of Lemma 2.2 in [6, p.208-209].
(e)-(f) These are part of Lemma 2.4 in [1, p.720]. □

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