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ON SOLVABILITY OF GENERALIZED NONLINEAR VARIATIONAL-LIKE INEQUALITIES

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ABSTRACT. In this paper, we introduce and study a new class of generalized nonlinear variational-like inequalities. By employing the auxiliary principle technique we suggest an iterative algorithm to compute approximate solutions of the generalized nonlinear variational-like inequalities. We discuss the convergence of the iterative sequences generated by the algorithm in Banach spaces and prove the existence of solutions and convergence of the algorithm for the generalized nonlinear variational-like inequalities in Hilbert spaces, respectively. Our results extend, improve and unify several known results due to Ding, Liu et al, and Zeng, and others.

1. Introduction

Variational inequality theory has become a rich source of inspiration in pure and applied mathematics, and variational inequalities have been used in a large variety of problems arising in elasticity, structural analysis, economics, optimization, physical and engineering sciences, etc. For details, we refer to [1-14] and the references therein. A useful and important generalization of variational inequalities is the variational-like inequalities. In [5, 7, 9-11, 14], the authors have studied several classes of variational-like inequalities. It is worth mentioning that the standard projection technique can no longer be applied to suggest the iterative algorithm for variational-like inequalities. Glowinski-Lions-Tremolieres [8] suggested another technique, which does not depend on the projection. This technique is called the auxiliary principle technique. Ansari-Yao [1] and Zeng [14] studied mixed variational-like inequalities in real Hilbert spaces, Ding-Yao [7] obtained the existence of solutions for mixed quasi-variational-like inclusions in reflexive Banach spaces, and Ding [5] proved the existence and uniqueness theorems of solutions for nonlinear mixed variational-like inequalities dealing with a nonlinear form $b(u, v)$ in reflexive

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Banach spaces. Recently, Ding-Tarafdar [6] and Liu-Ume-Kang [11] suggested the existence of solutions for general nonlinear variational inequalities and generalized nonlinear variational-like inequalities involving a bilinear functional $a(u,v)$, a nonlinear form $b(u,v)$ and some nonlinear monotone mappings in reflexive Banach spaces.

Motivated and inspired by the results in [1-14], we introduce and study a new class of generalized nonlinear variational-like inequalities, including as special cases of the variational inequalities and the variational-like inequalities due to Ansari-Yao [1], Ding [3-5], Ding-Tarafdar [6], Ding-Yao [7], Liu-Ume-Kang [11] and Zeng [14]. By using a new auxiliary variational-like inequality technique, firstly we give an existence and uniqueness theorem of solution for an auxiliary problem dealing with the generalized nonlinear variational-like inequalities, secondly we suggest an iterative algorithm to compute the approximate solutions of the generalized nonlinear variational-like inequalities, at last the convergence of the iterative sequences generated by the algorithm is also proved. The results presented here improve previously known results in this field.

2. Preliminaries

Let B be a reflexive Banach space with norm $||\cdot||$ and B^* be the topological dual space of B. Let $\langle u,v\rangle$ be the pairing between $u \in B^*$ and $v \in B$. In particular, if B is a Hilbert space, $\langle u,v\rangle$ denotes an inner product in it. Let D be a nonempty closed convex subset of B and let $a : D \times D \to (-\infty,+\infty)$ be a continuous linear functional in both arguments and there exist positive constants $\alpha > 0$ and $\beta > 0$ such that

$$a(x,x) \geq \alpha ||x||^2, \quad \forall x \in D; \tag{1a}$$

$$a(x,y) \leq \beta ||x|| ||y||, \quad \forall x,y \in D. \tag{1b}$$

Clearly, $\alpha \leq \beta$. Let the functional $b : D \times D \to (-\infty,+\infty)$ satisfy the following conditions:

(2a) b is linear in the first argument;

(2b) b is convex in the second argument;

there exists a constant $\gamma > 0$ satisfying

$$b(x,y) \leq \gamma ||x|| ||y||, \quad \forall x,y \in D; \tag{2c}$$

$$b(x,y) - b(x,z) \leq b(x,y-z), \quad \forall x,y,z \in D. \tag{2d}$$

It is easy to show that b is continuous in the second argument by (2c) and (2d).

Let $T, A : D \to B^*, N : B^* \times B^* \to B^*, \eta : D \times D \to B, g : B \to D$ be five nonlinear mappings. For given $w^* \in B^*$, we consider the following generalized nonlinear variational-like inequality problem (GNVLIP):

Find $u \in D$ such that

$$(3)\qquad \begin{aligned} &a(u, v-u) + b(gu, v) - b(gu, u) \\ &\geq \langle N(Tu, Au) - w^*, \eta(v, u)\rangle, \quad \forall v \in D, \end{aligned}$$

where a satisfies (1a) and (1b), b satisfies (2a)-(2d) and b is not necessarily differentiable.

Special cases

(A) If $w^* = 0$ and $gu = u$ for all $u \in D$, then GNVLIP (3) reduces to the following problem:

Find $u \in D$ such that

$$a(u, v-u) + b(u, v) - b(u, u) \geq \langle N(Tu, Au), \eta(v, u)\rangle, \quad \forall v \in D,$$

which was introduced and studied by Liu-Ume-Kang [11].

(B) If $N(u, v) = v$ for all $u, v \in B^*$, $gu = u$ for all $u \in D$ and $\eta(v, u) = fv - fu$ for all $v, u \in D$, where $f : D \to B$ is a given mapping and $w^* = 0$, then GNVLIP (3) is equivalent to finding $u \in D$ such that

$$a(u, v-u) + b(u, v) - b(u, u) \geq \langle Au, fv - fu\rangle, \quad \forall v \in D,$$

which was studied by Ding-Tarafdar [6].

(C) If B is a Hilbert space, $N(u, v) = v - u$ for all $u, v \in B$, $a = 0, b(u, v) = \phi(v)$ for all $u, v \in D$, where $\phi : D \to (-\infty, +\infty)$ is a real-valued function and $w^* = 0$, then GNVLIP (3) collapses to finding $u \in D$ such that

$$\langle Tu - Au, \eta(v, u)\rangle + \phi(v) - \phi(u) \geq 0, \quad \forall v \in D,$$

which was called *mixed variational-like inequality* and studied by Zeng [14].

(D) If B is a Hilbert space, $N(u, v) = 2w^*$ for all $u, v \in B$, $\eta(v, u) = v - u$ and $gu = u$ for all $u, v \in D$, then GNVLIP (3) reduces to finding $u \in D$ such that

$$a(u, v-u) + b(u, v) - b(u, u) \geq \langle w^*, v - u\rangle, \quad \forall v \in D,$$

which was introduced and studied by Siddiqi and Ansari [12].

For suitable and appropriate choices of the mappings T, A, N, η, g, a and b, one can obtain various new and previously known variational inequalities and variational-like inequalities as special cases of GNVLIP (3). In brief, GNVLIP (3) is a more general and unifying one, which is also one of the main motivations of this paper.

Definition 2.1 ([11]). Let D be a nonempty convex subset of a reflexive Banach space B with dual space B^*. Let $T, A : D \to B^*$, $N : B^* \times B^* \to B^*$ and $\eta : D \times D \to B$ be four mappings.

(1) T is said to be *(t, η)-relaxed Lipschitz* with respect to the first argument of N if there exists a constant $t > 0$ such that

$$(4)\qquad \begin{aligned} &\langle N(Tx, u), \eta(y, x)\rangle + \langle N(Ty, u), \eta(x, y)\rangle \\ &\geq t\|x - y\|^2, \quad \forall x, y \in D,\ u \in B^*; \end{aligned}$$

(2) A is said to be (t, η)-*pseudocontractive* with respect to the second argument of N if there exists a constant $t > 0$ such that

$$\begin{aligned} &\langle N(u, Ax), \eta(y, x)\rangle + \langle N(u, Ay), \eta(x, y)\rangle \\ &\geq -t\|x-y\|^2, \quad \forall x, y \in D,\ u \in B^*; \end{aligned} \tag{5}$$

(3) If $t = 0$ in (4) (resp. (5)), then T is called η-*antimonotone* with respect to the first argument of N (resp. η-*antimonotone* with respect to the second argument of N);

(4) T is said to be t-*Lipschitz continuous* if there exists a constant $t > 0$ such that

$$\|Tx - Ty\| \leq t\|x - y\|, \quad \forall x, y \in D;$$

(5) T is said to be η-*strongly monotone* if there exists a constant $t > 0$ such that

$$\langle Tx - Ty, \eta(x, y)\rangle \geq t\|x - y\|^2, \quad \forall x, y \in D;$$

(6) T is said to be t-*relaxed Lipschitz* with respect to the first argument of N if there exists a constant $t > 0$ such that

$$\langle N(Tx, u) - N(Ty, u), x - y\rangle \leq -t\|x - y\|^2, \quad \forall x, y \in D,\ u \in B^*;$$

(7) η is said to be t-*Lipschitz continuous* if there exists a constant $t > 0$ such that

$$\|\eta(x, y)\| \leq t\|x - y\|, \quad \forall x, y \in D;$$

(8) η is said to be t-*strongly monotone* if there exists a constant $t > 0$ such that

$$\langle x - y, \eta(x, y)\rangle \geq t\|x - y\|^2, \quad \forall x, y \in D;$$

(9) N is said to be t-*Lipschitz continuous* in the first argument if there exists a constant $t > 0$ such that

$$\|N(x, u) - N(y, u)\| \leq t\|x - y\|, \quad \forall x, y, u \in B^*.$$

In a similar way, we can define the Lipschitz continuity of the mapping $N(\cdot, \cdot)$ in the second argument.

Definition 2.2 ([1]). A differential function $h : D \to (-\infty, +\infty)$ on a convex set D is called:

(1) η-*convex* if

$$h(y) - h(x) \geq \langle h'(x), \eta(y, x)\rangle, \quad \forall x, y \in D,$$

where $h'(x)$ is the Fréchet derivative of h at x;

(2) η-*strongly convex* if there exists a constant $\mu > 0$ such that

$$h(y) - h(x) - \langle h'(x), \eta(y, x)\rangle \geq (\mu/2)\|x - y\|^2, \quad \forall x, y \in D.$$

It is easy to prove the following result.

Proposition 2.1. *Let h be a differentiable and η-strongly convex functional on a convex subset D of B, and let $\eta : D \times D \to B$ be a mapping such that $\eta(x, y) + \eta(y, x) = 0, \forall x, y \in D$. Then h' is η-strongly monotone.*

Proof. Since h is a differentiable and η-strongly convex functional on a convex subset D of B,

$$h(y) - h(x) - \langle h'(x), \eta(y,x)\rangle \geq (\mu/2)\|x-y\|^2, \quad \forall x,y \in D; \tag{6}$$

$$h(x) - h(y) - \langle h'(y), \eta(x,y)\rangle \geq (\mu/2)\|x-y\|^2, \quad \forall x,y \in D. \tag{7}$$

Adding (6) and (7), we have

$$\langle h'(y) - h'(x), \eta(y,x)\rangle \geq \mu\|x-y\|^2,$$

that is, h' is η-strongly monotone. This completes the proof. □

Lemma 2.1 ([2]). *Let E be a topological vector space, X a nonempty convex subset of E and $f, g : X \times X \to [-\infty, +\infty]$ such that*

(a) *for each $x, y \in X$, $f(x,y) > 0$ implies $g(x,y) > 0$;*

(b) *for each fixed $x \in X$, $y \mapsto f(x,y)$ is lower semicontinuous on any nonempty compact subset of X;*

(c) *for each $A \in \Im(X)$ and for each $y \in co(A)$, $min_{x\in A} g(x,y) \leq 0$, where $\Im(X)$ denotes the set of all nonempty finite subset of X;*

(d) *there exist a nonempty closed and compact subset K of X and $x_0 \in K$ such that $g(x_0, y) > 0$ for all $y \in X \backslash K$.*

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq 0$ for all $x \in X$.

In order to obtain our results, we need the following assumption.

Assumption 2.1. *The mapping $\eta : D \times D \to B$ satisfies the following conditions:*

(a) $\eta(x,y) + \eta(y,x) = 0$, $\forall x, y \in D$;

(b) *for given $u, v, w^* \in D$, mapping $w \to \langle N(Tu, Au) - w^*, \eta(v,w)\rangle$ is convex and lower semicontinuous.*

Remark 2.1. It is easy to see that $\eta(x,x) = 0$ and $w \to \langle N(Tu, Au) - w^*, \eta(w,v)\rangle$ is concave.

3. Existence and uniqueness theorem of solution for the auxiliary problem

Now we extend the auxiliary principle technique to study the existence and uniqueness of solution for GNVLIP (3).

Let $h : D \to (-\infty, +\infty]$ be a given Fréchet differentiable convex functional and $\rho > 0$ be a constant. For given $u \in D$ and $w^* \in B^*$, we consider the following auxiliary problem: Find $w \in D$ such that

$$\begin{aligned}&\langle h'(w), v - w\rangle \\ &\geq \langle h'(u), v-w\rangle + \rho\langle N(Tu, Au) - w^*, \eta(v,w)\rangle \\ &\quad - \rho b(gu, v) + \rho b(gu, w) - \rho a(u, v-w), \quad \forall v \in D.\end{aligned} \tag{8}$$

Firstly, we prove the existence and uniqueness of solution for the auxiliary problem (8).

Theorem 3.1. *Let D be a nonempty closed convex subset of a reflexive Banach space B with dual space B^*. Assume that $a : D \times D \to (-\infty, +\infty)$ is a continuous linear function in both arguments and satisfies* (1a) *and* (1b), *and $b : D \times D \to (-\infty, +\infty)$ satisfies conditions* (2b), (2c) *and* (2d). *Let $\eta : D \times D \to B$ be a mapping and $h : D \to (-\infty, +\infty]$ be a differentiable functional such that*

(a) *η is $\xi > 0$-Lipschitz continuous and Assumption 2.1 holds*;

(b) *h is strongly convex with constant $\mu > 0$ and h' is continuous.*

Then for given $w^* \in B^*$ and $u \in D$, the auxiliary problem (8) *has a unique solution.*

Proof. For given $w^* \in B^*$ and $u \in D$, define $\varphi, \psi : D \times D \to (-\infty, +\infty)$ by

$$\begin{aligned} \varphi(v,w) = {} & \langle h'(u) - h'(v), v - w\rangle + \rho\langle N(Tu, Au) - w^*, \eta(v,w)\rangle \\ & - \rho b(gu, v) + \rho b(gu, w) - \rho a(u, v - w), \quad \forall v, w \in D, \end{aligned} \tag{9}$$

$$\begin{aligned} \psi(v,w) = {} & \langle h'(u) - h'(w), v - w\rangle + \rho\langle N(Tu, Au) - w^*, \eta(v,w)\rangle \\ & - \rho b(gu, v) + \rho b(gu, w) - \rho a(u, v - w), \quad \forall v, w \in D. \end{aligned} \tag{10}$$

We verify that the mappings φ, ψ satisfy all conditions of Lemma 2.1. It follows from the strong convexity of h and Proposition 2.1 that h' is strongly monotone with constant $\mu > 0$ and

$$\begin{aligned} \psi(v,w) - \varphi(v,w) &= \langle h'(v) - h'(w), v - w\rangle \\ &\geq \mu\|v - w\|^2, \quad \forall v, w \in D, \end{aligned}$$

which yields that $\varphi(v,w) > 0$ implies that $\psi(v,w) > 0$. The continuity of a and b in the second argument and Assumption 2.1 ensure that $w \to \varphi(v,w)$ is lower semicontinuous on D. We claim that condition (c) of Lemma 2.1 holds. If it were false, then there exist a finite set $\{v_1, \ldots, v_n\} \subset D$ and $w = \sum_{i=1}^n \lambda_i v_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$ such that

$$\begin{aligned} \psi(v_i,w) = {} & \langle h'(u) - h'(w), v_i - w\rangle + \rho\langle N(Tu, Au) - w^*, \eta(v_i,w)\rangle \\ & - \rho b(gu, v_i) + \rho b(gu, w) - \rho a(u, v_i - w) > 0, \quad 1 \leq i \leq n. \end{aligned}$$

Note that a is linear in both arguments, b is convex in the second argument and $w \to \langle N(Tu, Au) - w^*, \eta(v,w)\rangle$ is convex. It follows that

$$\begin{aligned} 0 &< \sum_{i=1}^n \lambda_i \psi(v_i, w) \\ &= \sum_{i=1}^n \lambda_i \langle h'(u) - h'(w), v_i - w\rangle + \rho \sum_{i=1}^n \lambda_i \langle N(Tu, Au) - w^*, \eta(v_i, w)\rangle \\ &\quad - \rho \sum_{i=1}^n \lambda_i b(gu, v_i) + \rho \sum_{i=1}^n \lambda_i b(gu, w) - \rho \sum_{i=1}^n \lambda_i a(u, v_i - w) \end{aligned}$$

$$= \rho \sum_{i=1}^{n} \lambda_i \langle N(Tu, Au) - w^*, \eta(v_i, w) \rangle - \rho \sum_{i=1}^{n} \lambda_i b(gu, v_i) + \rho b(gu, w)$$
$$\leq \rho \langle N(Tu, Au) - w^*, \eta(w, w) \rangle = 0,$$

which is a contradiction. Hence, condition (c) of Lemma 2.1 is satisfied. Let v^* be an arbitrary element in D. Put $R = \frac{1}{\mu}(\|h'(u) - h'(v^*)\| + \rho\xi\|N(Tu, Au) - w^*\| + \rho\gamma\|gu\| + \rho\beta\|u\|)$ and $K = \{w \in D : \|w - v^*\| \leq R\}$. Then K is a weakly compact convex subset of D.

For any $w \in D \backslash K$, we know that

$$\begin{aligned}
\psi(v^*, w) &= \langle h'(u) - h'(w), v^* - w \rangle + \rho \langle N(Tu, Au) - w^*, \eta(v^*, w) \\
&\quad - \rho b(gu, v^*) + \rho b(gu, w) - \rho a(u, v^* - w) \\
&\geq \langle h'(u) - h'(v^*), v^* - w \rangle + \langle h'(v^*) - h'(w), v^* - w \rangle \\
&\quad + \rho \langle N(Tu, Au) - w^*, \eta(v^*, w) \rangle - \rho b(gu, v^* - w) \\
&\quad - \rho\beta\|u\|\|v^* - w\| \\
&\geq \mu\|v^* - w\|^2 - \|h'(u) - h'(v^*)\|\|v^* - w\| \\
&\quad - \rho\xi\|N(Tu, Au) - w^*\|\|v^* - w\| - \rho\gamma\|gu\|\|v^* - w\| \\
&\quad - \rho\beta\|u\|\|v^* - w\| \\
&= \mu\|v^* - w\|[\|v^* - w\| - \frac{1}{\mu}(\|h'(u) - h'(v^*)\| \\
&\quad + \rho\xi\|N(Tu, Au) - w^*\| + \rho\gamma\|gu\| + \rho\beta\|u\|)] \\
&> 0.
\end{aligned}$$

Therefore, the condition (d) of Lemma 2.1 holds. Consequently, Lemma 2.1 ensures that there exists a $\bar{w} \in K$ such that $\varphi(v, \bar{w}) \leq 0$ for all $v \in D$, that is,

$$\begin{aligned}
&\langle h'(v), v - \bar{w} \rangle \\
&\geq \langle h'(u), v - \bar{w} \rangle + \rho \langle N(Tu, Au) - w^*, \eta(v, \bar{w}) \rangle \\
&\quad - \rho b(gu, v) + \rho b(gu, \bar{w}) - \rho a(u, v - \bar{w}), \quad \forall v \in D.
\end{aligned} \tag{11}$$

For each $t \in (0, 1]$ and $v \in D$, let $x_t = tv + (1 - t)\bar{w}$. Replacing v by x_t in (11), we get that

$$\begin{aligned}
&t\langle h'(x_t), v - \bar{w} \rangle \\
&\geq t\langle h'(u), v - \bar{w} \rangle + \rho \langle N(Tu, Au) - w^*, \eta(tv + (1 - t)\bar{w}, \bar{w}) \rangle \\
&\quad - \rho b(gu, tv + (1 - t)\bar{w}) + \rho b(gu, \bar{w}) - \rho t a(u, v - \bar{w}) \\
&\geq \rho t \langle N(Tu, Au) - w^*, \eta(v, \bar{w}) \rangle - \rho t b(gu, v) \\
&\quad + \rho t b(gu, \bar{w}) - \rho t a(u, v - \bar{w}),
\end{aligned}$$

that is,

$$\begin{aligned}
\langle h'(x_t), v - \bar{w} \rangle &\geq \langle h'(u), v - \bar{w} \rangle + \rho \langle N(Tu, Au) - w^*, \eta(v, \bar{w}) \rangle \\
&\quad - \rho b(gu, v) + \rho b(gu, \bar{w}) - \rho a(u, v - \bar{w}).
\end{aligned}$$

Letting $t \to 0^+$ in the above inequality, we have

$$\begin{aligned}\langle h'(\bar{w}), v-\bar{w}\rangle \ge \langle h'(u), v-\bar{w}\rangle + \rho\langle N(Tu, Au) - w^*, \eta(v, \bar{w})\rangle \\ - \rho b(gu, v) + \rho b(gu, \bar{w}) - \rho a(u, v-\bar{w}).\end{aligned}$$

That is, $\bar{w}$ is a solution of the auxiliary problem (8).

Next we prove the uniqueness of the solution of the auxiliary problem (8). For given $w^* \in B^*$ and $u \in D$, suppose that w_1 and w_2 are two different solutions of the auxiliary problem (8). It is clear that

$$\begin{aligned}&\langle h'(w_1), v-w_1\rangle \\ &\ge \langle h'(u), v-w_1\rangle + \rho\langle N(Tu, Au) - w^*, \eta(v, w_1)\rangle \\ &\quad - \rho b(gu, v) + \rho b(gu, w_1) - \rho a(u, v-w_1), \quad \forall v \in D,\end{aligned} \tag{12}$$

$$\begin{aligned}&\langle h'(w_2), v-w_2\rangle \\ &\ge \langle h'(u), v-w_2\rangle + \rho\langle N(Tu, Au) - w^*, \eta(v, w_2)\rangle \\ &\quad - \rho b(gu, v) + \rho b(gu, w_2) - \rho a(u, v-w_2), \quad \forall v \in D.\end{aligned} \tag{13}$$

Taking $v = w_2$ in (12) and $v = w_1$ in (13), respectively, and adding them, we obtain that

$$\langle h'(w_1) - h'(w_2), w_2 - w_1\rangle \ge 0,$$

which is contradiction with the strong monotonicity of h'. Hence the auxiliary problem (8) has a unique solution. This completes the proof. □

Remark 3.1. It is not necessary that b is linear in the first argument in Theorem 3.1.

It follows from Theorem 3.1 that the auxiliary problem (8) yields a mapping $F : D \to D$ defined by $F(u) = w$ for each $u \in D$, where w satisfies (8). Based on Theorem 3.1, we suggest the following algorithm for GNVLIP (3).

Algorithm 3.1. *For given $w^* \in B^*$ and $u_0 \in D$, compute $\{u_n\}_{n\ge 0} \subseteq D$ by solving the auxiliary problem* (8) *with $u = u_n$:*

$$\begin{aligned}&\langle h'(w), v-w\rangle \\ &\ge \langle h'(u_n), v-w\rangle + \rho\langle N(Tu_n, Au_n) - w^*, \eta(v, w)\rangle \\ &\quad - \rho b(gu_n, v) + \rho b(gu_n, w) - \rho a(u_n, v-w), \quad \forall v \in D.\end{aligned}$$

Let u_{n+1} denote the solution of the above problem. That is,

$$\begin{aligned}&\langle h'(u_{n+1}), v-u_{n+1}\rangle \\ &\ge \langle h'(u_n), v-u_{n+1}\rangle + \rho\langle N(Tu_n, Au_n) - w^*, \eta(v, u_{n+1})\rangle \\ &\quad - \rho b(gu_n, v) + \rho b(gu_n, u_{n+1}) - \rho a(u_n, v-u_{n+1}), \quad \forall v \in D.\end{aligned} \tag{14}$$

4. Convergence of iterative sequence in reflexive Banach spaces

Now we prove that the sequence $\{u_n\}_{n\geq 0}$ generated by Algorithm 3.1 converges strongly to a solution of GNVLIP (3).

Theorem 4.1. *Let D be a nonempty closed convex subset of a reflexive Banach space B with dual space B^*. Let $\eta : D \times D \to B$ be a mapping, $a : D \times D \to R$ be a continuous linear function in both arguments and satisfy* (1a), (1b) *and* $b : D \times D \to R$ *satisfy* (2a)-(2d). *Assume that $T, A : D \to B^*, N : B^* \times B^* \to B^*, \eta : D\times D \to B$ are four mappings and $h : D \to (-\infty, +\infty]$ is a differentiable functional such that*

(a) *N is σ_1 and σ_2-Lipschitz continuous in the first and the second arguments, respectively;*

(b) *T is (t, η)-relaxed Lipschitz with respect to the first argument of N and is τ-Lipschitz continuous;*

(c) *A is (s, η)-pseudocontractive with respect to the second argument of N and is δ-Lipschitz continuous;*

(d) *g is θ-Lipschitz continuous;*

(e) *η is $\xi > 0$-Lipschitz continuous,* (b) *of Assumption 2.1 holds and $\eta(x,y) = \eta(x,z)+\eta(z,y), \forall x,y,z \in D$;*

(f) *h is strongly convex with constant $\mu > 0$ and h' is continuous;*

(g) *there exists a constant $\rho > 0$ satisfying*

$$0 < \rho < \frac{2\mu(t + s - \gamma\theta + \alpha)}{(\xi\tau\sigma_1 + \xi\delta\sigma_2 + \gamma\theta + \beta)^2}.$$

Suppose that the solution set of GNVLIP (3) *is nonempty. Then the sequence $\{u_n\}_{n\geq 0}$ defined by* (14) *converges strongly to a solution $u \in D$ of GNVLIP* (3).

Proof. Let u be a solution of GNVLIP (3). Define $C : D \to (-\infty, +\infty]$ by

$$C(y) = h(u) - h(y) - \langle h'(y), u - y\rangle, \quad \forall y \in D.$$

Since h' is μ-strongly monotone, it follows that

$$C(y) \geq \frac{\mu}{2}\|y - u\|^2, \quad \forall y \in D. \tag{15}$$

Taking $v = u_{n+1}$ in (3) and $v = u$ in (14), by (1a), (1b), (2a)-(2d) and conditions (a)-(f), we get that

$$\begin{aligned}
&C(u_n) - C(u_{n+1}) \\
&= h(u_{n+1}) - h(u_n) - \langle h'(u_n), u_{n+1} - u_n\rangle + \langle h'(u_n), u_{n+1} - u_n\rangle \\
&\quad - \langle h'(u_n), u - u_n\rangle + \langle h'(u_{n+1}), u - u_{n+1}\rangle
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\mu}{2}\|u_{n+1}-u_n\|^2 - \langle h'(u_n), u-u_{n+1}\rangle + \langle h'(u_{n+1}), u-u_{n+1}\rangle \\
&\geq \frac{\mu}{2}\|u_{n+1}-u_n\|^2 + \rho\langle N(Tu_n, Au_n) - w^*, \eta(u, u_{n+1})\rangle \\
&\quad - \rho b(gu_n, u) + \rho b(gu_n, u_{n+1}) - \rho a(u_n, u-u_{n+1}) \\
&\geq \frac{\mu}{2}\|u_{n+1}-u_n\|^2 + \rho\langle N(Tu_n, Au_n) - w^*, \eta(u, u_{n+1})\rangle \\
&\quad - \rho\langle N(Tu, Au) - w^*, \eta(u, u_{n+1})\rangle \\
&\quad + \rho\langle N(Tu, Au) - w^*, \eta(u, u_{n+1})\rangle + \rho b(gu, u_{n+1}) - \rho b(gu, u) \\
&\quad + \rho a(u, u_{n+1}-u) \\
&\quad - \rho b(gu, u_{n+1}) + \rho b(gu, u) - \rho b(gu_n, u) + \rho b(gu_n, u_{n+1}) \\
&\quad - \rho a(u, u_{n+1}-u) - \rho a(u_n, u-u_{n+1}) \\
&\geq \frac{\mu}{2}\|u_{n+1}-u_n\|^2 + \rho\langle N(Tu_n, Au_n) - N(Tu, Au), \eta(u, u_{n+1})\rangle \\
&\quad - \rho b(gu-gu_n, u_{n+1}) + \rho b(gu-gu_n, u) \\
&\quad + \rho a(u_n-u, u_{n+1}-u) \\
&= \frac{\mu}{2}\|u_{n+1}-u_n\|^2 + \rho\langle N(Tu_n, Au_n) - N(Tu, Au_n), \eta(u, u_{n+1})\rangle \\
&\quad + \rho\langle N(Tu, Au_n) - N(Tu, Au), \eta(u, u_{n+1})\rangle + \rho[b(gu-gu_n, u) \\
&\quad - b(gu-gu_n, u_n) + b(gu-gu_n, u_n) - b(gu-gu_n, u_{n+1})] \\
&\quad + \rho[a(u_n-u, u_{n+1}-u_n) + a(u_n-u, u_n-u)] \\
&\geq \frac{\mu}{2}\|u_{n+1}-u_n\|^2 + \rho\langle N(Tu_n, Au_n) - N(Tu, Au_n), \eta(u, u_n)\rangle \\
&\quad + \rho\langle N(Tu_n, Au_n) - N(Tu, Au_n), \eta(u_n, u_{n+1})\rangle \\
&\quad + \rho\langle N(Tu, Au_n) - N(Tu, Au), \eta(u, u_n)\rangle \\
&\quad + \rho\langle N(Tu, Au_n) - N(Tu, Au), \eta(u_n, u_{n+1})\rangle \\
&\quad - \rho\gamma(\|gu-gu_n\|\|u_n-u\| + \|gu-gu_n\|\|u_n-u_{n+1}\|) \\
&\quad - \rho\beta\|u_n-u\|\|u_{n+1}-u_n\| + \rho\alpha\|u_n-u\|^2 \\
&\geq \frac{\mu}{2}\|u_{n+1}-u_n\|^2 + \rho t\|u_n-u\|^2 - \rho\xi\tau\sigma_1\|u_n-u\|\|u_{n+1}-u_n\| \\
&\quad - \rho s\|u_n-u\|^2 - \rho\xi\delta\sigma_2\|u_n-u\|\|u_{n+1}-u_n\| \\
&\quad - \rho\gamma\theta(\|u_n-u\|^2 + \|u_n-u\|\|u_n-u_{n+1}\|) \\
&\quad - \rho\beta\|u_n-u\|\|u_{n+1}-u_n\| + \rho\alpha\|u_n-u\|^2 \\
&= \frac{\mu}{2}\|u_{n+1}-u_n\|^2 + \rho(t-s-\gamma\theta+\alpha)\|u_n-u\|^2 \\
&\quad - \rho(\xi\tau\sigma_1 + \xi\delta\sigma_2 + \gamma\theta + \beta)\|u_n-u\|\|u_{n+1}-u_n\| \\
&\geq \rho\left[(t-s-\gamma\theta+\alpha) - \frac{\rho(\xi\tau\sigma_1 + \xi\delta\sigma_2 + \gamma\theta + \beta)^2}{2\mu}\right]\|u_n-u\|^2,
\end{aligned}
$$

that is,

$$\begin{aligned}&C(u_n) - C(u_{n+1})\\ (16)\qquad &\geq \rho\left[(t - s - \gamma\theta + \alpha) - \frac{\rho(\xi\tau\sigma_1 + \xi\delta\sigma_2 + \gamma\theta + \beta)^2}{2\mu}\right]\|u_n - u\|^2\end{aligned}$$

for all $n \geq 0$. It follows from (g) that

$$(t - s - \gamma\theta + \alpha) - \frac{\rho(\xi\tau\sigma_1 + \xi\delta\sigma_2 + \gamma\theta + \beta)^2}{2\mu} > 0.$$

Thus, (15) and (16) ensure that the nonnegative sequence $\{C(u_n)\}_{n\geq 0}$ is non-increasing, hence it is convergent. Letting $n \to \infty$ in (16), we conclude that $\lim_{n\to\infty} \|u_n - u\| = 0$. This completes the proof. □

5. Existence of solutions for GNVLIP (3) in Hilbert spaces

In this section, we study the existence of solutions for GNVLIP (3) and convergence of the iterative sequence generated by Algorithm 3.1 in Hilbert spaces.

Theorem 5.1. *Let D be a nonempty closed convex subset of a Hilbert space B. Let $\eta : D\times D \to B$ be a mapping, $a : D\times D \to (-\infty, +\infty)$ be a continuous linear function in both arguments and satisfy* (1a), (1b) *and $b : D \times D \to (-\infty, +\infty)$ satisfy* (2a)-(2d). *Assume that $T, A : D \to B, N : B \times B \to B, \eta : D \times D \to B$ are four mappings and $h : D \to (-\infty, +\infty]$ is a differentiable functional such that*

(a) *N is σ_1 and σ_2-Lipschitz continuous in the first and the second arguments, respectively;*

(b) *A, T and g are s, t and δ-Lipschitz continuous, respectively, and T is ε-relaxed Lipschitz with respect to the first argument of N;*

(c) *η is $\xi > 0$-Lipschitz continuous and ς-strongly monotone, and Assumption 2.1 holds ;*

(d) *h is strongly convex with constant $\mu > 0$ and h' is τ-Lipschitz continuous;*

(e) *$e = \sqrt{\tau^2 - 2\mu + 1}$, $k = \sigma_1 t\sqrt{1 - 2\varsigma + \xi^2} + \sigma_2 s\xi + \gamma\delta + \beta$, there exists a constant $\rho > 0$ satisfying*

$$\rho < \frac{\mu - e}{k},$$

and

$$\sigma_1 t > k, \quad [\varepsilon - k(\mu - e)]^2 > [1 - (\mu - e)^2](\sigma_1^2 t^2 - k^2),$$

$$\left|\rho - \frac{\varepsilon - k(\mu - e)}{\sigma_1^2 t^2 - k^2}\right| < \frac{\sqrt{[(\mu - e)^2 - 1](\sigma_1^2 t^2 - k^2) + [\varepsilon - k(\mu - e)]^2}}{\sigma_1^2 t^2 - k^2};$$

or

$$\sigma_1 t < k,$$

$$\left|\rho - \frac{\varepsilon - k(\mu - e)}{\sigma_1^2 t^2 - k^2}\right| > \frac{\sqrt{[(\mu - e)^2 - 1](\sigma_1^2 t^2 - k^2) + [\varepsilon - k(\mu - e)]^2}}{k^2 - \sigma_1^2 t^2}.$$

Then the sequence $\{u_n\}_{n\geq 0}$ *defined by* (14) *converges strongly to a solution* $u \in D$ *of* GNVLIP (3).

Proof. In order to show that GNVLIP (3) has a solution $u \in D$, it is enough to prove that the mapping $F : D \to D$ defined by (8) has a unique fixed point $u \in D$. Let x, y be arbitrary elements in D. It follows from Theorem 3.1 that

$$\begin{aligned} &\langle h'(Fx), v - Fx\rangle \\ &\geq \langle h'(x), v - Fx\rangle + \rho\langle N(Tx, Ax) - w^*, \eta(v, Fx)\rangle \\ &\quad - \rho b(gx, v) + \rho b(gx, Fx) - \rho a(x, v - Fx), \quad \forall v \in D, \end{aligned} \tag{17}$$

$$\begin{aligned} &\langle h'(Fy), v - Fy\rangle \\ &\geq \langle h'(y), v - Fy\rangle + \rho\langle N(Ty, Ay) - w^*, \eta(v, Fy)\rangle \\ &\quad - \rho b(gy, v) + \rho b(gy, Fy) - \rho a(y, v - Fy), \quad \forall v \in D. \end{aligned} \tag{18}$$

Taking $v = Fy$ in (17) and $v = Fx$ in (18), respectively, and adding the inequalities, we obtain that

$$\begin{aligned} &\langle h'(Fx) - h'(Fy), Fx - Fy\rangle \\ &\leq \langle h'(x) - h'(y), Fx - Fy\rangle \\ &\quad + \rho\langle N(Tx, Ax) - N(Ty, Ay), \eta(Fx, Fy)\rangle \\ &\quad + \rho b(gx - gy, Fy - Fx) + \rho a(x - y, Fy - Fx). \end{aligned}$$

From condition (d), we have

$$\begin{aligned} &\mu\|Fx - Fy\|^2 \\ &\leq \langle h'(x) - h'(y) - (x - y), Fx - Fy\rangle \\ &\quad + \langle x - y - \rho[N(Ty, Ax) - N(Tx, Ax)], Fx - Fy\rangle \\ &\quad + \rho\langle N(Ty, Ax) - N(Tx, Ax), Fx - Fy - \eta(Fx, Fy)\rangle \\ &\quad + \rho\langle N(Ty, Ax) - N(Ty, Ay), \eta(Fx, Fy)\rangle \\ &\quad + \rho b(gx - gy, Fy - Fx) + \rho a(x - y, Fy - Fx) \\ &\leq \|h'(x) - h'(y) - (x - y)\|\|Fx - Fy\| \\ &\quad + \|x - y - \rho[N(Ty, Ax) - N(Tx, Ax)]\|\|Fx - Fy\| \\ &\quad + \rho\|N(Ty, Ax) - N(Tx, Ax)\|\|Fx - Fy - \eta(Fx, Fy)\| \\ &\quad + \rho\|N(Ty, Ax) - N(Ty, Ay)\|\|\eta(Fx, Fy)\| \\ &\quad + \rho\gamma\|gx - gy\|\|Fx - Fy\| + \rho\beta\|x - y\|\|Fx - Fy\|. \end{aligned} \tag{19}$$

It follows from conditions (a)-(d) that

$$\begin{aligned} &\|h'(x) - h'(y) - (x - y)\|^2 \\ &= \|h'(x) - h'(y)\|^2 - 2\langle h'(x) - h'(y), x - y\rangle + \|x - y\|^2 \\ &\leq (\tau^2 - 2\mu + 1)\|x - y\|^2, \end{aligned} \tag{20}$$

$$\begin{aligned}
&\|x - y - \rho[N(Ty, Ax) - N(Tx, Ax)]\|^2 \\
&= \|x - y\|^2 - 2\rho\langle N(Ty, Ax) - N(Tx, Ax), x - y\rangle \\
&\quad + \rho^2\|N(Ty, Ax) - N(Tx, Ax)\|^2 \\
&\le (1 - 2\varepsilon\rho + \sigma_1^2 t^2 \rho^2)\|x - y\|^2,
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\|Fx - Fy - \eta(Fx, Fy)\|^2 \\
&= \|Fx - Fy\|^2 - 2\langle Fx - Fy, \eta(Fx, Fy)\rangle + \|\eta(Fx, Fy)\|^2 \\
&\le (1 - 2\varsigma + \xi^2)\|Fx - Fy\|^2.
\end{aligned} \tag{22}$$

Using (20)-(22) in (19) and from conditions (a)-(e), we get that

$$\|Fx - Fy\|^2 \le \frac{1}{\mu}\Big[e + \sqrt{1 - 2\varepsilon\rho + \sigma_1^2 t^2 \rho^2} + k\rho\Big]\|x - y\|\|Fx - Fy\|,$$

that is,

$$\|Fx - Fy\| \le \theta\|x - y\|,$$

where

$$\theta = \frac{1}{\mu}\Big[e + \sqrt{1 - 2\varepsilon\rho + \sigma_1^2 t^2 \rho^2} + k\rho\Big].$$

It follows from (e) that $0 < \theta < 1$ and F is a contraction mapping. Hence F has a unique fixed point $u \in D$, which is a solution of GNVLIP (3).

Now we show that $\lim_{n\to\infty} u_n = u$. Since u is a solution of GNVLIP (3), it follows that for given $w^* \in B^*$,

$$\begin{aligned}
\langle h'(u), v - u\rangle \ge{}& \langle h'(u), v - u\rangle + \rho\langle N(Tu, Au) - w^*, \eta(v, u)\rangle \\
&- \rho b(gu, v) + \rho b(gu, u) - \rho a(u, v - u), \quad \forall v \in D.
\end{aligned} \tag{23}$$

Taking $v = u$ in (14) and $v = u_{n+1}$ in (23) and adding them, we can easily obtain that $\|u_{n+1} - u\| \le \theta\|u_n - u\| \le \theta^{n+1}\|u_0 - u\|$ and $u_n \to u$ as $n \to \infty$. This completes the proof. □

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References

[1] Q. H. Ansari and J. C. Yao, *Iterative schemes for solving mixed variational-like inequalities*, J. Optim. Theory Appl. **108** (2001), no. 3, 527–541.

[2] M. S. R. Chowdhury and K. K. Tan, *Generalized variational inequalities for quasi-monotone operators and applications*, Bull. Polish Acad. Sci. Math. **45** (1997), no. 1, 25–54.

[3] X. P. Ding, *General algorithm of solutions for nonlinear variational inequalities in Banach space*, Comput. Math. Appl. **34** (1997), no. 9, 131–137.

[4] ———, *Generalized quasi-variational-like inclusions with nonconvex functionals*, Appl. Math. Comput. **122** (2001), no. 3, 267–282.

[5] ———, *Existence and algorithm of solutions for nonlinear mixed variational-like inequalities in Banach spaces*, J. Comput. Appl. Math. **157** (2003), no. 2, 419–434.

[6] X. P. Ding and E. Tarafdar, *Existence and uniqueness of solutions for a general nonlinear variational inequality*, Appl. Math. Lett. **8** (1995), no. 1, 31–36.
[7] X. P. Ding and J. C. Yao, *Existence and algorithm of solutions for mixed quasi-variational-like inclusions in Banach spaces*, Comput. Math. Appl. **49** (2005), no. 5-6, 857–869.
[8] R. Glowinski, J. Lions, and R. Trémolières, *Numerical Analysis of variational inequalities*, North-Holland, Amsterdam, 1987.
[9] Z. Liu, Z. An, S. M. Kang, and J. S. Ume, *Convergence and stability of the three-step iterative schemes for a class of general quasivariational-like inequalities*, Int. J. Math. Math. Sci. **70** (2004), no. 69-72, 3849–3857.
[10] Z. Liu, J. H. Sun, S. H. Shim, and S. M. Kang, *On solvability of general nonlinear variational-like inequalities in reflexive Banach spaces*, Int. J. Math. Math. Sci. (2005), no. 9, 1415–1420.
[11] Z. Liu, J. S. Ume, and S. M. Kang, *Generalized nonlinear variational-like inequalities in reflexive Banach spaces*, J. Optim. Theory Appl. **126** (2005), no. 1, 157–174.
[12] A. H. Siddiqi and Q. H. Ansani, *An algorithm for a class of quasivariational inequalities*, J. Math. Anal. Appl. **145** (1990), no. 2, 413–418.
[13] Q. H. Wu, Y. Hao, Z. Liu, and S. M. Kang, *Sensitivity analysis of solutions for parametric general quasivariational-like inequalities*, Int. J. Pure Appl. Math. **21** (2005), no. 1, 121–130.
[14] L. C. Zeng, *Iterative algorithm for finding approximate solutions of a class of mixed variational-like inequalities*, Acta Math. Appl. Sin. Engl. Ser. **20** (2004), no. 3, 477–486.

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