J. Korean Math. Soc. **45** (2008), No. 1, pp. 137–150

ON NON-ISOMORPHIC GROUPS WITH THE SAME SET OF ORDER COMPONENTS

Mohammad Reza Darafsheh

Abstract. In this paper we will prove that the simple groups $B_p(3)$ and $C_p(3)$, p an odd prime number, are 2-recognizable by the set of their order components. More precisely we will prove that if G is a finite group and $OC(G)$ denotes the set of order components of G, then $OC(G) = OC(B_p(3))$ if and only if $G \cong B_p(3)$ or $C_p(3)$.

1. Introduction

For a positive integer n, let $\pi(n)$ be the set of all prime divisors of n. If G is a finite group, we set $\pi(G) = \pi(|G|)$. The Gruenberg-Kegel graph of G, or the prime graph of G, is denoted by $GK(G)$ and is defined as follows. The vertex set of $GK(G)$ is the set $\pi(G)$ and two distinct primes p and q are joined by an edge if and only if G contains an element of order pq. We denote the connected components of $GK(G)$ by $\pi_1, \pi_2, \ldots, \pi_{s(G)}$, where $s(G)$ denotes the number of connected components of $GK(G)$. If the order of G is even, the notation is chosen so that $2 \in \pi_1$. It is clear that the order of G can be expressed as the product of the numbers $m_1, m_2, \ldots, m_{s(G)}$, where $\pi(m_i) = \pi_i$, $1 \leq i \leq s(G)$. If the order of G is even and $s(G) \geq 2$, according to our notation $m_2, \ldots, m_{s(G)}$ are odd numbers. The positive integers $m_1, m_2, \ldots, m_{s(G)}$ are called the order components of G and $OC(G) = \{m_1, m_2, \ldots, m_{s(G)}\}$ is called the set of order components of G. It is a natural question to ask: If the finite groups G and H have the same order components does it follow G is isomorphic to H? For many simple groups H with the number of order components $s(H)$ at least 2, the answer to the above question is affirmative. However if $s(H) = 1$ the answer is negative. The simple groups $B_n(q)$ and $C_n(q)$ where $n = 2^m \geq 4$ and q is odd, have the same order components but they are not isomorphic. Hence it is natural to adopt the following definition.

Definition 1. Let G be a finite group. The number of non-isomorphic finite groups with the same order components as G is denoted by $h(G)$ and is called the h-function of G. For any natural number k we say the finite group G is

Received May 30, 2006; Revised November 3, 2006.
2000 *Mathematics Subject Classification.* 20D06, 20D60.
Key words and phrases. prime graph, order component, linear group.

k-recognizable by its set of order components if $h(G) = k$. If $h(G) = 1$ we say that G is characterizable by its set of order components or briefly G is a characterizable group. In this case G is uniquely determined by the set of its order components.

Obviously for any finite groups G we have $h(G) \geq 1$. The components of the Gruenberg-Kegel graph $GK(P)$ of any non-abelian finite simple group P with $GK(P)$ disconnected are found in [16] from which we can deduce the component orders of P. These information which will be used in proving our main result are listed in Tables 1, 2 and 3.

Table 1. The order components of finite simple groups P with $s(P) = 2$ (p an odd prime)

P	Restrictions on P	m_1	m_2
$\mathbb{A}_n$	$6 < n = p, p+1, p+2$; one of $n, n-2$ is not a prime	$\frac{n!}{2p}$	p
$A_{p-1}(q)$	$(p,q) \neq (3,2),(3,4)$	$q^{\binom{p}{2}}\prod_{i=1}^{p-1}(q^i-1)$	$\frac{(q^p-1)}{(q-1)(p,q-1)}$
$A_p(q)$	$(q-1) \mid (p+1)$	$q^{\binom{p+1}{2}}(q^{p+1}-1)\prod_{i=2}^{p-1}(q^i-1)$	$\frac{(q^p-1)}{(q-1)}$
$^2A_{p-1}(q)$		$q^{\binom{p}{2}}\prod_{i=1}^{p-1}(q^i-(-1)^i)$	$\frac{(q^p+1)}{(q+1)(p,q+1)}$
$^2A_p(q)$	$(q+1) \mid (p+1)$, $(p,q) \neq (3,3),(5,2)$	$q^{\binom{p+1}{2}}(q^{p+1}-1)\prod_{i=1}^{p-1}(q^i-(-1)^i)$	$\frac{(q^p+1)}{(q+1)}$
$^2A_3(2)$		$2^6.3^4$	5
$B_n(q)$	$n = 2^m \geq 4$, q odd	$q^{n^2}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{(q^n+1)}{2}$
$B_p(3)$		$3^{p^2}(3^p+1)\prod_{i=1}^{p-1}(3^{2i}-1)$	$\frac{(3^p-1)}{2}$
$C_n(q)$	$n = 2^m \geq 2$	$q^{n^2}(q^n-1)\prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{(q^n+1)}{(2,q-1)}$
$C_p(q)$	$q = 2,3$	$q^{p^2}(q^p+1)\prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{(q^p-1)}{(2,q-1)}$
$D_p(q)$	$p \geq 5$, $q = 2,3,5$	$q^{p(p-1)}\prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{(q^p-1)}{(q-1)}$
$D_{p+1}(q)$	$q = 2,3$	$\frac{1}{(2,q-1)}q^{p(p+1)}(q^p+1)$ $(q^{p+1}-1)\prod_{i=1}^{p-1}(q^{2i}-1)$	$\frac{(q^p-1)}{(2,q-1)}$
$^2D_n(q)$	$n = 2^m \geq 4$	$q^{n(n-1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$\frac{(q^n+1)}{(2,q+1)}$
$^2D_n(2)$	$n = 2^m+1 \geq 5$	$2^{n(n-1)}(2^n+1)(2^{n-1}-1)$ $\prod_{i=1}^{n-2}(2^{2i}-1)$	$2^{n-1}+1$
$^2D_p(3)$	$5 \leq p \neq 2^m+1$	$3^{p(p-1)}\prod_{i=1}^{p-1}(3^{2i}-1)$	$\frac{(3^p+1)}{4}$
$^2D_n(3)$	$9 \leq n = 2^m+1 \neq p$	$\frac{1}{2}3^{n(n-1)}(3^n+1)(3^{n-1}-1)$ $\prod_{i=1}^{n-2}(3^{2i}-1)$	$\frac{(3^{n-1}+1)}{2}$
$G_2(q)$	$2 < q \equiv \epsilon (\mathrm{mod}\, 3)$, $\epsilon = \pm 1$	$q^6(q^3-\epsilon)(q^2-1)(q+\epsilon)$	$q^2-\epsilon q+1$
$^3D_4(q)$		$q^{12}(q^6-1)(q^2-1)$ (q^4+q^2+1)	q^4-q^2+1
$F_4(q)$	q odd	$q^{24}(q^8-1)(q^6-1)^2$ (q^4-1)	q^4-q^2+1
$^2F_4(2)'$		$2^{11}.3^3.5^2$	13
$E_6(q)$		$q^{36}(q^{12}-1)(q^8-1)(q^6-1)$ $(q^5-1)(q^3-1)(q^2-1)$	$\frac{(q^6+q^3+1)}{(3,q-1)}$

Table 1. (Continued)

$^2E_6(q)$	$q > 2$	$q^{36}(q^{12}-1)(q^8-1)(q^6-1)$ $(q^5+1)(q^3+1)(q^2-1)$	$\frac{(q^6-q^3+1)}{(3,q+1)}$
M_{12}		$2^6.3^3.5$	11
J_2		$2^7.3^3.5^2$	7
Ru		$2^{14}.3^3.5^3.7.13$	29
He		$2^{10}.3^3.5^2.7^3$	17
McL		$2^7.3^6.5^3.7$	11
Co_1		$2^{21}.3^9.5^4.7^2.11.13$	23
Co_3		$2^{10}.3^7.5^3.7.11$	23
Fi_{22}		$2^{17}.3^9.5^2.7.11$	13
HN		$2^{14}.3^6.5^6.7.11$	19

Table 2. The order components of finite simple groups P with $s(P) = 3$ (p an odd prime)

P	Restrictions on P	m_1	m_2	m_3
$\mathbb{A}_n$	$n > 6$, $n = p$, $p-2$ are primes	$\frac{n!}{2n(n-2)}$	p	$p-2$
$A_1(q)$	$3 < q \equiv \epsilon (\bmod 4)$, $\epsilon = \pm 1$	$q - \epsilon$	q	$\frac{(q+\epsilon)}{2}$
$A_1(q)$	$q > 2$, q even	q	$q-1$	$q+1$
$A_2(2)$		8	3	7
$^2A_5(2)$		$2^{15}.3^6.5$	7	11
$^2D_p(3)$	$p = 2^m + 1 \geq 5$	$2.3^{p(p-1)}(3^{p-1}-1)$ $\prod_{i=1}^{p-2}(3^{2i}-1)$	$\frac{(3^{p-1}+1)}{2}$	$\frac{(3^p+1)}{4}$
$^2D_{p+1}(2)$	$p = 2^n - 1$, $n \geq 2$	$2^{p(p+1)}(2^p-1)$ $\prod_{i=1}^{p-1}(2^{2i}-1)$	2^p+1	$2^{p+1}+1$
$G_2(q)$	$q \equiv 0 (\bmod 3)$	$q^6(q^2-1)^3$	q^2-q+1	q^2+q+1
$^2G_2(q)$	$q = 3^{2m+1} > 3$	$q^3(q^2-1)$	$q-\sqrt{3q}+1$	$q+\sqrt{3q}+1$
$F_4(q)$	q even	$q^{24}(q^6-1)^2(q^4-1)^2$	q^4+1	q^4-q^2+1
$^2F_4(q)$	$q = 2^{2m+1} > 2$	$q^{12}(q^4-1)(q^3+1)$	$q^2-\sqrt{2q^3}+$ $q-\sqrt{2q}+1$	$q^2+\sqrt{2q^3}+$ $q+\sqrt{2q}+1$
$E_7(2)$		$2^{63}.3^{11}.5^2.7^3.11.13.$ $17.19.31.43$	73	127
$E_7(3)$		$2^{23}.3^{63}.5^2.7^3.11^2.13^2.$ $19.37.41.61.73.547$	757	1093
M_{11}		$2^4.3^2$	5	11
M_{23}		$2^7.3^2.5.7$	11	23
M_{24}		$2^{10}.3^3.5.7$	11	23
J_3		$2^7.3^5.5$	17	19
HiS		$2^9.3^2.5^3$	7	11
P	Restrictions on P	m_1	m_2	m_3
Suz		$2^{13}.3^7.5^2.7$	11	13
Co_2		$2^{18}.3^6.5^3.7$	11	23
Fi_{23}		$2^{18}.3^{13}.5^2.7.11.13$	17	23
F_3		$2^{15}.3^{10}.5^3.7^2.13$	19	31

Table 2. (Continued)

F_2		$2^{41}.3^{13}.5^6.7^2.11.13.$ $17.19.23$	31	47

Table 3. The order components of finite simple groups P with $s(P) > 3$

P	Restrictions on P	m_1	m_2	m_3	m_4	m_5	m_6
$A_2(4)$		2^6	3	5	7		
${}^2B_2(q)$	$q = 2^{2m+1} > 2$	q^2	$q-1$	$q-\sqrt{2q}$ $+1$	$q+\sqrt{2q}$ $+1$		
${}^2E_6(2)$		$2^{36}.3^9.5^2.7^2.11$	13	17	19		
$E_8(q)$	$q \equiv 2,3$ $(\text{mod } 5)$	$q^{120}(q^{20}-1)(q^{18}-1)$ $(q^{14}-1)(q^{12}-1)$ $(q^{10}-1)(q^8-1)$ $(q^4+1)(q^4+q^2+1)$	$\frac{q^{10}-q^5+1}{q^2-q+1}$	$\frac{q^{10}+q^5+1}{q^2+q+1}$	q^8-q^4 $+1$		
M_{22}		$2^7.3^2$	5	7	11		
J_1		$2^3.3.5$	7	11	19		
$O'N$		$2^9.3^4.5.7^3$	11	19	31		
LyS		$2^8.3^7.5^6.7.11$	31	37	67		
Fi'_{24}		$2^{21}.3^{16}.5^2.7^3.11.13$	17	23	29		
F_1		$2^{46}.3^{20}.5^9.7^6.11^2.13^3.$ $17.19.23.29.31.47$	41	59	71		
$E_8(q)$	$q \equiv 0,1,4$ $(\text{mod } 5)$	$q^{120}(q^{18}-1)(q^{14}-1)$ $(q^{12}-1)^2(q^{10}-1)^2$ $(q^8-1)^2(q^4+q^2+1)$	$\frac{q^{10}-q^5+1}{q^2-q+1}$	$\frac{q^{10}+q^5+1}{q^2+q+1}$	q^8-q^4 $+1$	$\frac{q^{10}+1}{q^2+1}$	
J_4		$2^{21}.3^3.5.7.11^3$	23	29	31	37	43

In [15] and [14] it is proved that if $n = 2^m \geq 4$, then $h(B_n(q)) = h(C_n(q)) = 2$ for q odd and $h(B_n(q)) = h(C_n(q)) = 1$ for q even. Apart from the families $B_n(q)$ and $C_n(q)$, $n = 2^m \geq 4$, q odd, the following groups have been proved to be characterizable by their order components by various authors. All the sporadic simple groups [2], $PSL_2(q)$, ${}^2D_n(3)$ where $9 \leq n = 2^m + 1$ is not a prime, ${}^2D_{p+1}(2)$ in [3], [6] and [17], respectively. Some projective special linear (unitary) groups have been characterized in a series of articles in [9], [10], [11] and [12]. A few of the alternating or symmetric groups are proved to be characterizable by their order components in [1] and [13]. Based on these results we put forward the following conjecture.

Conjecture 1. Let P be a non-abelian finite simple group with $s(P) \geq 2$. If G is a finite group and $OC(G) = OC(P)$, then either $G \cong P$ or $G \cong B_n(q)$, $C_n(q)$ where $n = 2^m \geq 4$ and q is an odd number or $G \cong B_p(3)$, $C_p(3)$, where p is an odd prime number.

In this paper we consider the simple groups $B_p(3)$ and $C_p(3)$, where p is an odd prime, and prove that these groups are 2-recognizable by the set of their order components. Another names for the group $B_p(3)$ are $O_{2p+1}(3)$ or $\Omega_{2p+1}(3)$, and for $C_p(3)$ is $PSP_{2p}(3)$. More precisely we will prove:

Main Theorem. *If a finite group G has the same set of order components as $B_p(3)$, p an odd prime, then $G \cong B_p(3)$ or $C_p(3)$.*

2. Preliminary results

The structure of finite groups with disconnected Gruenberg-Kegel graph follows from Theorem A of [18] which will be stated below:

Lemma 1. *Let G be a simple group with $s(G) \geq 2$. Then one of the following hold:*

(1) *G is either a Frobenius or 2-Frobenius group.*

(2) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H is a nilpotent π_1-group, $\frac{K}{H}$ is a non-abelian simple group, $\frac{G}{K}$ is a π_1-group, $\left|\frac{G}{K}\right|$ divides $\left|\mathrm{Out}(\frac{K}{H})\right|$ and any odd order component of G is equal to one of the odd order components of $\frac{K}{H}$.*

To deal with the first case in the above Lemma we need the following results which are taken from [5] and [2], respectively.

Lemma 2. (a) *Let G be a Frobenius group of even order with kernel and complements K and H, respectively. Then $s(G) = 2$ and the prime graph components of G are $\pi(H)$ and $\pi(K)$.*

(b) *Let G be a 2-Frobenius group of even order. Then $s(G) = 2$ and G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that $\left|\frac{K}{H}\right| = m_2$, $|H|\left|\frac{G}{K}\right| = m_1$ and $\left|\frac{G}{K}\right|$ divides $\left|\frac{K}{H}\right| - 1$ and H is a nilpotent π_1-group.*

Lemma 3. *Let G be a finite group with $s(G) \geq 2$. If $H \trianglelefteq G$ is a π_i-group, then $(\prod_{j=1,\ j\neq i}^{s(G)} m_j) \mid (|H| - 1)$.*

The following result of Zsigmondy [19] is important in some number theoretical considerations.

Lemma 4. *Let n and a be integers greater than 1. There exists a prime divisor p of $a^n - 1$ such that p does not divide $a^i - 1$ for all $1 \leq i < n$, except in the following cases.*

(1) $n = 2$, $a = 2^k - 1$, *where* $k \geq 2$,

(2) $n = 6$, $a = 2$.

The prime p in Lemma 4 is called a Zsigmondy prime for $a^n - 1$.

Remark 1. If p is a Zsigmondy prime for $a^n - 1$, then $p > n$. Because if $p \leq n$, then $n = kp + r$, $0 \leq r < p$, and we can write $a^n - 1 = a^r(a^{kp} - a^k) + a^{k+r} - 1$. Since $(p, a) = 1$ we have $a^p \equiv a \pmod p$, hence $a^{kp} \equiv a^k \pmod p$, therefore $p \mid a^{k+r} - 1$. By assumption about p we must have $k + r \geq n$ which implies $k \geq kp$, hence $k = 0$. Therefore $n = r < p$ contradicting $p \leq n$.

Next we consider the group $B_n(q)$ whose order according to [7] is $|B_n(q)| = \frac{1}{(2,q-1)} q^{n^2} \prod_{i=1}^{n} (q^{2i} - 1)$. The order of the outer automorphism group of $B_n(q)$ is $(2, q-1)f$ if q is odd or $n > 2$, and it is $2f$ if q is even and $n = 2$, where $q = p^f$ is a power of a prime p. If $q = 2^m$ is a power of 2 it is known that for any natural number n we have $B_n(2^m) \cong C_n(2^m)$ where $C_n(2^m)$ is the projective special symplectic group. In general the groups $B_n(q)$ and $C_n(q)$ have the same order, but the groups $B_n(q)$ and $C_n(q)$ with q odd are not isomorphic. By [16] the group $B_n(q)$ has a disconnected Gruenberg-Kegel graph if and only if $n = 2^m \geq 4$ with q an odd number, or $n = p$ is an odd prime number and $q = 3$. It is proved in [14] that if $n = 2^m \geq 4$ and q is an odd prime power then $h(B_n(q)) = 2$, that is to say there are two non-isomorphic finite groups with the same set of order components as $B_n(q)$ if $n = 2^m \geq 4$ with q odd. Therefore it remains to consider the characterizability of the group $B_p(3)$. By [16] the prime graph of the group $B_p(3)$ has two components with $m_1 = 3^{p^2}(3^p+1)\prod_{i=1}^{p-1}(3^{2i}-1)$ and $m_2 = \frac{3^p-1}{2}$. The components of the prime graph of $B_p(3)$ are $\pi_1 = \pi(3(3^p+1)\prod_{i=1}^{p-1}(3^{2i}-1))$ and $\pi_2 = \pi(\frac{3^p-1}{2})$. Also, as can be seen from Table 1, the group $C_p(3)$ has the same order components as $B_p(3)$. Our main result asserts that $h(B_p(3)) = h(C_p(3)) = 2$.

3. Proof of the main theorem

Let G be a finite group with $OC(G) = \{m_1, m_2\}$, where m_1 and m_2 are the order components of the group $B_p(3)$ or $C_p(3)$. In order to use Lemma 1, first we will prove the following Lemma.

Lemma 5. *If G is a finite group with $OC(G) = \{m_1, m_2\}$, then G is neither a Frobenius nor a 2-Frobenius group.*

Proof. First we assume G is a Frobenius group and derive a contradiction. If H and K denote the complement and the kernel of the Frobenius group G, respectively, then by Lemma 2 we have $OC(G) = \{|H|, |K|\}$. Since $|H| \mid |K|-1$ we deduce $|H| < |K|$, from which it follows that $|K| = m_1$ and $|H| = m_2$. Let r be a Zsigmondy prime for 3^p+1. Since $r \neq 2$ we obtain $r \mid \frac{3^p+1}{4}$. Since $|G| = m_1 m_2$ and $(m_1, m_2) = 1$ and $|K| = m_1$, we observe that the order of a Sylow r-subgroup S of G, and hence of K, is a divisor of $\frac{3^p+1}{4}$. Since K is a nilpotent normal subgroup of G, we obtain $S \trianglelefteq G$ and $m_2 \mid |S|-1$ by Lemma 3. But $m_2 = \frac{3^p-1}{2}$ and hence $|S|-1 \geq m_2 = \frac{3^p-1}{2}$ implying $|S| \geq \frac{3^p+1}{2}$ which contradicts $|S| \mid \frac{3^p+1}{4}$.

Next we will assume G is a 2-Frobenius group and obtain a contradiction. In this case by Lemma 2(b) there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ for G such that H is a nilpotent π_1-group, $\left|\frac{K}{H}\right| = m_2$ and $\left|\frac{G}{K}\right| \mid (\left|\frac{K}{H}\right| - 1) = \frac{3(3^{p-1}-1)}{2}$. Arguing as above if r is a Zsigmondy prime for 3^p+1, then $r \nmid \left|\frac{G}{K}\right|$, hence by Lemma 2, $r \mid |H|$. If S is a Sylow $r-$subgroup of H then by Lemma 3, $m_2 \mid |S|-1$, therefore $|S| \geq m_2 + 1 = \frac{3^p+1}{2}$ contradicting $|S| \leq 3^p+1$. The Lemma is proved. □

Proof of the main theorem. By the Lemmas 1 and 4, if G is a finite group with $OC(G) = \{m_1, m_2\}$, then there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ for G

such that $\frac{K}{H}$ is a non-abelian simple group, H and $\frac{G}{K}$ are π_1-group and H is nilpotent. Moreover $\left|\frac{G}{K}\right|$ divides $\left|\mathrm{Out}(\frac{K}{H})\right|$ and the odd order component of G is one of the odd order components of $\frac{K}{H}$ and $s(\frac{K}{H}) \geq 2$.

Since $P = \frac{K}{H}$ is a non-abelian simple group with disconnected Gruenberg-Kegel graph, by the classification of finite simple groups we have one of the possibilities in Tables 1, 2 or 3 for P. In the following we deal with these groups.

Case 1. $P \cong {}^2A_3(2)$, ${}^2F_4(2)'$, $A_2(2)$, $A_2(4)$, ${}^2A_5(2)$, $E_7(2)$, $E_7(3)$, ${}^2E_6(2)$, or one of the 26 sporadic simple groups listed in Tables 1, 2 or 3.

The odd order component of G is a number of the form $m_2 = \frac{3^p-1}{2}$ and must be equal to one of the odd order components of the groups listed above. By inspecting Tables 1, 2 and 3, the largest odd order component of the above groups is 1093. But for $p = 3$, 5 and 7 the respected values of m_2 are 13, 121 and 1093, hence we have the following possibilities for P. If $p = 7$, then $m_2 = 1093$ corresponds to $P \cong E_7(3)$, and if $p = 3$, then $m_2 = 13$ corresponds to the simple groups ${}^2F_4'(2)$, Fi_{22}, Suz or ${}^2E_6(2)$. But examination of each case gives a contradiction compare with $|P| \mid |G|$. Therefore the above possibilities are ruled out.

Case 2. $P \cong \mathbb{A}_n$ and either $n = p'$, $p'+1$, $p'+2$, one of n or $n-2$ is not a prime, or $n = p'$, $p'-2$ are both prime where $p' > 6$ is a prime number.

By Tables 1 and 2, the odd order components of $\mathbb{A}_n$ are p' or(and) $p'-2$. If $p'-2 = m_2 = \frac{3^p-1}{2}$, then $p' = \frac{3^p+3}{2}$ is not a prime number. Therefore we may assume $p' = \frac{3^p-1}{2}$ is a prime number. In this case we have $P \cong \mathbb{A}_{p'}$ and the order of P must divide the order of G, in particular $|\mathbb{A}_{p'}|_3 \mid 3^{p^2}$. But the largest power of 3 dividing $p'!$ is $[\frac{p'}{3}] + [\frac{p'}{9}] + \cdots = \frac{3^p-2p-1}{4}$. If $p > 3$, then clearly $\frac{3^p-2p-1}{4} > p^2$ and hence $|\mathbb{A}_{p'}|_3$ does not divide $|G|_3$. However if $p = 3$, then $p' = 13$ and $|G| = |B_3(3)| = 2^9.3^9.5.7.13$, $|\mathbb{A}_{p'}| = 2^{10}.3^4.5^2.7.11.13$ and clearly $|P| \nmid |G|$. This final contradiction rules out the possibility of P being isomorphic to an alternating group.

Case 3. $P \cong {}^2E_6(q)$; $E_6(q)$, $q > 2$.

In these cases we have $\frac{q^6 \mp q^3+1}{(3,q\pm1)} = \frac{3^p-1}{2}$, respectively. We will deal with $P \cong E_6(q)$, because the other case is similar. Hence $\frac{q^6+q^3+1}{(3,q-1)} = \frac{3^p-1}{2}$. If $(3, q-1) = 1$, then $2q^3(q^3+1) = 3^p - 3$. Clearly q cannot be a power of 3, hence $q \equiv 2 \pmod 3$. Therefore $q^3 \equiv 8 \pmod 9$ from which it follows that $2q^3(q^3+1) \equiv 0 \pmod 9$. Since $3^p - 3 \equiv -3 \pmod 9$ a contradiction is obtained.

Next we assume $(3, q-1) = 3$, from which it follows $\frac{q^6+q^3+1}{3} = \frac{3^p-1}{2}$ and consequently

$$(*) \qquad 2q^3(q^3+1) = 3^{p+1} - 5.$$

From $q \equiv 1 \pmod 3$ we obtain $q^3 \equiv 1 \pmod 9$. Therefore we may write $q^3 = 1+9k$ for some $k \in \mathbb{N}$. Since $k \equiv 0, 1$ or $2 \pmod 3$, we will obtain $q^3 \equiv 1, 10$ or $19 \pmod{27}$. Now in any case we deduce $2q^3(q^3+1) \equiv 4 \pmod{27}$. But clearly $3^{p+1} - 5 \equiv 22 \pmod{27}$ which is against $(*)$. This contradiction rules out the possibility of $P \cong E_6(q)$. The case $P \cong {}^2E_6(q)$ is treated similarly.

Case 4. $P \cong F_4(q)$, q odd.

By Table 1 the odd order components of $F_4(q)$ is $q^4 - q^2 + 1$, and hence $q^4 - q^2 + 1 = \frac{3^p-1}{2}$ from which it follows that $q^2(q^2-1) = \frac{3(3^{p-1}-1)}{2}$. If the order components of $F_4(q)$ are denoted by M_1 and M_2, with M_1 even then clearly $M_1 = q^{24}(q^2-1)^4(q^2+1)^2(q^4+q^2+1)$ and from $|P| = M_1M_2 \mid |G| = m_1m_2$ we obtain $M_1 \mid m_1$. Since M_2 is a multiple of $(q^2(q^2-1))^3$, from $M_1 \mid m_1$ it follows that $3^{p-1} - 1 \mid (3^p+1)(3^{p-1}+1)\prod_{i=1, i\neq (p-1)/2}^{p-2}(3^{2i}-1)$. Now let r be a Zsigmondy prime for $3^{p-1}-1$. Since $r \neq 2$, we must have $r \mid \prod_{i=1, i\neq(p-1)/2}^{p-2}(3^{2i}-1)$. Let $r \mid 3^{2j}-1$ for some j, $1 \leq j \leq p-2$, $j \neq \frac{p-1}{2}$. By assumption about r we must have $2j \geq p-1$. But then from $r \mid 3^{p-1}-1$ and $r \mid 3^{2j}-1$ we deduce $r \mid 3^{2j}-3^{p-1}$, hence $r \mid 3^{2j-(p-1)}-1$. Since $2j-(p-1) \neq 0$ we must have $2j-(p-1) \geq p-1$ which implies $j \geq p-1$, a contradiction. Hence this case is ruled out.

Case 5. $P \cong {}^3D_4(q)$ or $P \cong G_2(q)$, $2 < q \equiv \epsilon \pmod 3$, $\epsilon = \pm 1$.

The odd order components of ${}^3D_4(q)$ and $G_2(q)$ are q^4-q^2+1 and $q^2-\epsilon q+1$, respectively. Equating these numbers with $m_2 = \frac{3^p-1}{2}$ will yield

$$(*) \qquad \begin{aligned} q^4 - q^2 &= 3(3^{p-1}-1)/2, \\ q^2 - \epsilon q &= 3(3^{p-1}-1)/2 \end{aligned}$$

clearly from the first equation of $(*)$ it follows that $3 \nmid q$. Therefore in both cases we have $3 \nmid q$. Since from here on the arguments for ${}^3D_4(q)$ and $G_2(q)$ are the same, we present the details for ${}^3D_4(q)$.

Since $3 \nmid q$ we have $q \equiv 1$ or $2 \pmod 3$, and since $9 \nmid q^4 - q^2$ we obtain $q \equiv 2, 4, 5, 7 \pmod 9$. Hence $q \not\equiv \pm 1 \pmod 3$. Let $q = r^f$, where r is a prime number. Clearly $3 \nmid r$, hence $r \equiv \pm 1 \pmod 3$ from which it follows that $r^3 \equiv \pm 1 \pmod 9$. If f is a multiple of 3, the we set $f = 3\alpha$ for some $\alpha \in \mathbb{N}$, from which it follows that $q = r^f = r^{3\alpha} \equiv (\pm 1)^\alpha \pmod 9$. Hence $q \equiv \pm 1 \pmod 9$ depending on whether α is even or odd, which contradicts $q \not\equiv \pm 1 \pmod 9$. obtained earlier. Therefor f cannot be a multiple of 3.

Now we set $\left|\frac{G}{K}\right| = t$ and obtain $t\,|H|\,|P| = |G|$, where t divides $|\mathrm{Out}(P)|$. But since $q^3 = r^{3f}$ is a power of the prime r, by [7] we have $|\mathrm{Out}(P)| = 3f$. Since we proved f is not a multiple of 3, t is either prime to 3 or $t = 3$. Now using Table 1 and substituting orders of P and G in $t\,|H|\,|P| = |G|$ we will obtain:

$$(**) \qquad t\,|H|\,q^{12}(q^2-1)^2(q^4+q^2+1)^2 = 3^{p^2}(3^p+1)\prod_{i=1}^{p-1}(3^{2i}-1).$$

Since $q \equiv 2, 4, 5, 7 \pmod 9$ we obtain $(q^4+q^2+1)_3 = 3$, and so a Sylow 3-subgroup S of H has orders 3^{p^2-2} or 3^{p^2-3} according to $(t,3) = 1$ or $t = 3$. But H is nilpotent, hence $S \trianglelefteq G$ and by Lemma 3 it follows that $m_2 \mid |S| - 1$, i.e., $3^p - 1 \mid 3^{p^2-2} - 1$ or $3^{p-1} - 1 \mid 3^{p^2-3} - 1$ which are impossible. This final contradiction shows that $P \cong {}^3D_4(q)$ is impossible. As we indicated at the beginning of the proof, the impossibility of $P \cong G_2(q)$, $2 < q \equiv \epsilon \pmod 3$, $\epsilon = \pm 1$, is proved similarly.

Case 6. $P \cong {}^2D_n(3)$, $9 \leq n = 2^m + 1 \neq p'$; ${}^2D_{p'}(3)$, $5 \leq p' \neq 2^m + 1$; ${}^2D_n(2)$, $n = 2^m + 1 \geq 5$.

If we equate the odd order components of the above groups with m_2 we obtain the following equations respectively: $3^{n-1} = 3^p - 2$, $3^{p'} = 2.3^p - 3$, $2^n = 3^p - 3$. Obviously all the above equations are impossible.

Case 7. $P \cong {}^2D_n(q)$, $n = 2^m \geq 4$; $P \cong C_n(q)$, $n = 2^m \geq 2$ or $P \cong B_n(q)$, $n = 2^m \geq 4$, q odd.

In the above cases the odd order components are $\frac{q^n+1}{(2,q+1)}$, $\frac{q^n+1}{(2,q-1)}$ or $\frac{q^n+1}{2}$, respectively. If q is even, then $q^n + 1 = \frac{3^p-1}{2}$ leads to $2q^n = 3^p - 3$ which is obviously impossible by [8]. Therefore we assume q is odd and in all the above cases the odd order component of each of the above group is $\frac{q^n+1}{2}$ and $\frac{q^n+1}{2} = \frac{3^p-1}{2}$ leads to $q^n = 3^p - 2$. Obviously from this equation we deduce $n < p$ and $(q,3) = 1$. Since $q^n - 1 = 3^p - 3$ is not a multiple of 9 and n is a power of 2, we obtain $q \pm 1 \mid q^n - 1 = 3^p - 3$, hence $q \equiv \pm 1 \pmod 3$. Therefore $q \equiv 2,\ 4,\ 5$ or $7 \pmod 9$, implying that $q \not\equiv -1 \pmod 9$. Now we assume $q = r^f$ is a power of the prime number r. Since $(q,3) = 1$, we have $r \neq 3$. Hence $r \equiv \pm 1 \pmod 3$ implying $r^3 \equiv \pm 1 \pmod 9$. If $f = 3\alpha$ is a multiple of 3, then

$$q = r^f = (r^3)^\alpha \equiv (\pm 1)^\alpha \pmod 9 \equiv \pm 1 \pmod 9.$$

Therefore f is not a multiple of 3.

Now if we set $\left|\frac{G}{K}\right| = t$, then $t \mid |\mathrm{Out}(P)|$, where by [7], $\left|\mathrm{Out}({}^2D_n(q))\right| = (4, q^n+1)f$ if $q^2 = r^f$, $|\mathrm{Out}(B_n(q))| = |\mathrm{Out}(C_n(q))| = (2, q-1)f$ if $q = r^f$. As usual we may write $t\,|H|\,|P| = |G|$, from which it follows that

$$(*) \qquad t\,|H|\,M_1 = 3^{p^2}(3^p + 1)\prod_{i=1}^{p-1}(3^{2i} - 1),$$

where $M_1 = q^{n(n-1)}\prod_{i=1}^{n-1}(q^{2i} - 1)$ for the group ${}^2D_n(q)$ and $M_1 = q^{n^2}(q^n - 1)\prod_{i=1}^{n-1}(q^{2i} - 1)$ for both groups $C_n(q)$ and $B_n(q)$. Since $t \mid |\mathrm{Out}(P)|$, we see that $t \mid 4f$ and since we proved f is not a multiple of 3, we deduce that t is not a multiple of 3. We consider that 3-part of the expression on the left hand side of $(*)$. First we calculate the 3-part of the expression $\prod_{i=1}^{n-1}(q^{2i} - 1)$. If the 3-part of $\prod_{i=1}^{n-1}(q^{2i} - 1)$ is 3^k, then since one of $q \pm 1$ is a multiple of 3 but not a multiple of 9 we obtain $k = \frac{n}{2} + [\frac{n-1}{6}] + [\frac{n-1}{18}] + \cdots$.

If we consider the equation (*) for the group ${}^2D_n(q)$, then the order of a Sylow 3-subgroup S of H would be $|S| = 3^{p^2-k}$. By Lemma 3 we must have

$3^p - 1 \mid 3^{p^2-k} - 1$ and this implies that $k = lp$ is a multiple of p. But

$$k = \frac{n}{2} + [\frac{n-1}{6}] + [\frac{n-1}{18}] + \cdots \leq \frac{n}{2} + \frac{n-1}{6} + \frac{n-1}{18} + \cdots \leq \frac{3n-1}{4}$$

which implies $\frac{3n-1}{4} \geq lp$. Therefore $3n \geq 4lp + 1$, hence $n \geq lp + \frac{lp+1}{3} \geq p$ contradicting $n < p$. This contradiction rules out the possibility $P \cong {}^2D_n(q)$.

If $P \cong C_n(q)$ or $B_n(q)$, then the 3-part of M_1 is $(q^n - 1)_3 . 3^k = 3^{k+1}$ because n is a power of 2. Therefore the order of a Sylow 3-subgroup of S of G in these cases is $3^{p^2-(k+1)}$. Now similar calculations as above give a contradiction which rules out the possibilities of P being isomorphic to $C_n(q)$ or $B_n(q)$. This final contradiction settles Case 7.

Case 8. $P \cong D_{p'+1}(q),\ q = 2,\ 3$ or $P \cong D_{p'}(q),\ p' \geq 5,\ q = 2,\ 3,\ 5$.

First suppose $P \cong D_{p'+1}(q),\ q = 2,\ 3$. If $q = 2$, then $2^{p'} - 1 = \frac{3^p-1}{2}$, hence $2^{p'+1} - 3^p = 1$ which is impossible by [8]. If $q = 3$, then $\frac{3^{p'}-1}{2} = \frac{3^p-1}{2}$ implying $p' = p$. But then the 3-part of $|D_{p+1}(3)|$ is $3^{p(p+1)}$ which does not divide the 3-part of G which is 3^{p^2}.

Next we assume $P \cong D_{p'}(q), p' \geq 5,\ q = 2,\ 3,\ 5$. If $q = 2$, then $2^{p'} - 1 = \frac{3^p-1}{2}$ implying $2^{p'+1} - 3^p = 1$ which is impossible by [8]. If $\frac{3^{p'}-1}{2} = \frac{3^p-1}{2}$, then $p' = p$. By setting $|\frac{G}{K}| = t \mid |\mathrm{Out}(P)|$ we observe that $t \mid 8$ and $t\,|H|\,|P| = |G|$ which implies $t\,|H| = 3^p(3^p + 1)$. Clearly $(3^p + 1)_2 = 4$, hence $t = 1,\ 2,\ 4$. Therefore $|H| = 3^p(3^p + 1),\ 3^p(3^p + 1)/2$ or $3^p(3^p + 1)/4$. Now let r be a prime divisor of $3^p + 1$ and consider a Sylow r-subgroup of H. Using Lemma 3 a contradiction is obtained.

Finally if $q = 5$, then from $\frac{5^{p'}-1}{4} = \frac{3^p-1}{2}$ we obtain $5^{p'} + 1 = 2.3^p$. But p' is a prime and either $p' \equiv 1 \pmod 6$ or $p' \equiv 5 \pmod 6$. If $p' \equiv 1 \pmod 6$, then $5^{p'} \equiv 5 \pmod 9$ and if $p' \equiv 5 \pmod 6$, then $5^{p'} \equiv 2 \pmod 9$. Therefore in any case $5^{p'} + 1 \not\equiv 0 \pmod 9$ contradicting the equation $5^{p'} + 1 = 2.3^p$. This final contradiction rules out the groups in Case 8.

Case 9. $P \cong C_{p'}(q),\ q = 2,\ 3$; or $B_{p'}(3)$.

If $P \cong C_{p'}(q),\ q = 2,\ 3$ then from $2^{p'-1} = \frac{3^p-1}{2}$ we will obtain $2^{p'+1} - 3^p = 1$ which by [8] is impossible. If $q = 3$, then from $\frac{3^{p'}-1}{2} = \frac{3^p-1}{2}$ we will obtain $p' = p$ and $P \cong C_p(3)$. Again by setting $|\frac{G}{K}| = t$ we obtain $t\,|H|\,|P| = |G|$ and since $|G| = |P|$, this implies $t\,|H| = 1$. Therefore $t = |H| = 1$ implying $G = K \cong P \cong C_p(3)$ and this is one of the possibilities in our main theorem.

If $P \cong B_{p'}(3)$, then again $p' = p$ and similarly $G = K \cong P \cong B_p(3)$, which is another possibility in our main theorem. We will prove no other possibility can occur by considering further simple groups P from Tables 1, 2 and 3.

Case 10. $P \cong A_{p'-1}(q),\ (p', q) \neq (3, 2), (3, 4);\ A_{p'}(q),\ q - 1 \mid p' + 1;\ {}^2A_{p'-1}(q)$ or ${}^2A_{p'}(q),\ q - 1 \mid p' + 1$.

In these case because of the similarity in argument we produce the proof of impossibility of $P \cong A_{p'}(q),\ q - 1 \mid p' + 1$. The odd order component of $A_{p'}(q)$

is $\frac{q^{p'}-1}{q-1}$ and hence $\frac{q^{p'}-1}{q-1} = \frac{3^p-1}{2}$. From this equation we obtain

$$(*) \qquad q^{p'-1} + q^{p'-2} + \cdots + q = 3(3^{p-1} - 1)/2.$$

If $q = 3$, then from $(*)$ we will obtain $p' = p$. Then the order of $P \cong A_p(3)$ has 3-part equal to $3^{\binom{p+1}{2}}$ which does not divide 3^{p^2}. Therefore $q \neq 3$ and from $(*)$ we deduce that $3 \nmid q$. Hence $q \equiv \pm 1 \pmod 3$. If $q \equiv 1 \pmod 3$, then from $(*)$ we obtain $p' \equiv 1 \pmod 3$. But from $q - 1 \mid p' + 1$ we got $p' \equiv -1 \pmod 3$, a contradiction. Therefore we may assume $q \equiv -1 \pmod 3$. Now $(*)$ may be rewritten as $q^{p'-1} + q^{p'-2} + \cdots + q = q^{p'-1} - 1 + q^{p'-2} + 1 \cdots + q^2 - 1 + q + 1 = 3(3^{p-1} - 1)/2$, from which we deduce that $q + 1$ is not a multiple of 9. Now $q \equiv -1 \pmod 3$ implies $q \equiv 2$ or $5 \pmod 9$. From now on the argument is the same as what was used in the proof of Case 7. We omit the calculations and end the impossibility of this case.

As it is seen all of the simple groups in Table 1 have been considered. The Gruenberg-Kegel graph of these groups has two components. Next we will consider simple groups P with $s(P) \geq 3$ which are tabulated in Tables 2 and 3. Some of these groups have already been considered. Therefore we deal with those which have not been considered.

Case 11. $P \cong A_1(q)$, $3 < q \equiv \epsilon \pmod 4$, $\epsilon = \pm 1$.

In this case the odd order component is either q or $\frac{q+\epsilon}{2}$. First we deal with $q = \frac{3^p-1}{2}$. Clearly $q \equiv 1 \pmod 4$, hence $\epsilon = 1$. If we set $\left|\frac{G}{K}\right| = t$, then $t\,|H|\,|P| = |G|$, where $t \mid |\mathrm{Out}(P)|$, and by [7] we have $|\mathrm{Out}(P)| = (2, q-1)f$, $q = r^f$, r a prime number. From $q = \frac{3^p-1}{2}$ it is easy to deduce that $q \equiv 4 \pmod 9$, hence $3 \nmid f$. Therefore t is prime to 3. Now from $t\,|H|\,|P| = |G|$ we obtain $t\,|H| \times \frac{1}{2}q(q^2-1) = |G|$ and simplifying both sides yields:

$$(*) \qquad 3t\,|H| = 8 \times 3^{p^2}(3^{p-1}+1)\prod_{i=1}^{p-2}(3^{2i}-1)$$

since $(t, 3) = 1$ from $(*)$ it follows that a Sylow 3-subgroup S of H has order 3^{p^2-1}. We have $S \trianglelefteq G$, because H is a nilpotent group, hence $\frac{3^p-1}{2} \mid |S| - 1 = 3^{p^2-1} - 1$ which is impossible. Therefore the odd order component q of P cannot be equal to $\frac{3^p-1}{2}$.

Secondly we assume $\frac{q+\epsilon}{2} = \frac{3^p-1}{2}$ which implies $q = 3^p - 1 - \epsilon$. Therefore $q = 3^p - 2$ or 3^p according to $\epsilon = +1$ or $\epsilon = +1$, respectively. With the same notation used above, from $t\,|H|\,|P| = |G|$ we obtain the following:

$$(**) \qquad 3t\,|H|\,(3^p-2) = 3^{p^2}(3^p+1)(3^{p-1}+1)\prod_{i=1}^{p-2}(3^{2i}-1), \text{ if } \epsilon = 1$$

$$(***) \qquad 3t\,|H| = 3^{p^2}\prod_{i=1}^{p-1}(3^{2i}-1), \text{ if } \epsilon = -1.$$

If $p = 3$, then from $(**)$ and $(***)$ evidently the order of a Sylow 7 and 5-subgroup S of H has order 7 and 5, respectively. Now $m_2 = 25$ and using Lemma 3 a contradiction is obtained.

Therefore we assume $p > 3$. In the case of $(***)$ since $q = 3^p$, we obtain $t \mid 2p$, thus $3^{2(p-1)} - 1 \mid |H|$. Now if r is a Zsigmondy prime for $3^{2p-1} - 1$, then $r \mid 3^{p-1} + 1$, and a Sylow r-subgroup S of H has the property $|S| \leq 3^{p-1} + 1$. Lemma 3 implies $m_2 = \frac{3^p-1}{2} \mid |S| - 1$, hence $|S| \geq \frac{3^p+1}{2}$, contradicting $|S| \leq 3^{p-1} + 1$. Therefore equation $(***)$ is ruled out. For equation $(**)$ the same reasoning as used in the proof of Case 7 works. We will not give the details because of its similarity to Case 7.

Case 12. $P \cong A_1(q)$, $q > 2$ is even.

The odd order components are $q - 1$ and $q + 1$. If $q + 1 = \frac{3^p-1}{2}$, then $q = \frac{3(3^p-1)}{2}$ is impossible because q is even. If $q - 1 = \frac{3^p-1}{2}$, then $q = \frac{3^p+1}{2}$ and by [8] there is no solution to the equation.

Case 13. $P \cong {}^2D_{p'}(3)$, $p' = 2^m + 1 \geq 5$; $P \cong {}^2D_{p'+1}(2)$, $p' = 2^n - 1$, $n \geq 2$.

The odd order components of ${}^2D_{p'}(3)$ are $\frac{3^{p'-1}+1}{2}$ and $\frac{3^{p'}+1}{4}$. If we equate each of the above numbers with $\frac{3^p-1}{2}$, an easy contradiction is obtained.

The odd order components of ${}^2D_{p'+1}(2)$ are $2^{p'} + 1$ and $2^{p'+1} + 1$. If $m_2 = 2^{p'} + 1$ or $2^{p'+1} + 1$, then $2^{p'+1} = 3^p - 3$ or $2^{p'+2} = 3^p - 3$, respectively, and obviously both equations are impossible.

Case 14. $P \cong G_2(q)$, $3 \mid q$; ${}^2G_2(q)$, $q = 3^{2m+1} > 3$.

The proof of impossibility of both cases is the same. Therefore we assume $P \cong {}^2G_2(q)$. The odd order components of P are $q - \sqrt{3q} + 1$ and $q + \sqrt{3q} + 1$. If $q \pm \sqrt{3q} + 1 = m_2$, then $3^{2m+1} \pm 3^{m+1} = \frac{3(3^{p-1}-1)}{2}$, which is obviously impossible.

Case 15. $P \cong F_4(q)$, q even.

The odd order components of $F_4(q)$ are $q^4 + 1$ and $q^4 - q^2 + 1$. If $q^4 + 1 = \frac{3^p-1}{2}$, then $q^4 = \frac{3(3^{p-1}-1)}{2}$, which is impossible because q is a power of 2.

If $q^4 - q^2 + 1 = \frac{3^p-1}{2}$, then $q^2(q^2 - 1) = \frac{3(3^{p-1}-1)}{2}$. Now the same argument as used in Cases 4 or 5 works, so we omit the details.

Case 16. $P \cong {}^2F_4(q)$, $q = 2^{2m+1} > 2$.

By Table 2 the odd order components of ${}^2F_4(q)$ are

$$q^2 - \sqrt{2q^3} + q - \sqrt{2q} + 1 \quad \text{and} \quad q^2 + \sqrt{2q^3} + q + \sqrt{2q} + 1.$$

Because of similarity we deal with one of these numbers. Therefore we assume $q^2 - \sqrt{2q^3} + q - \sqrt{2q} + 1 = \frac{3^p-1}{2}$ and $q = 2^{2m+1}$ to obtain

$$(*) \qquad 2^{m+2}(2^m - 1)(2^{2m+1} + 1) = 3(3^{p-1} - 1).$$

Since $2^{2m+1} + 1$ is a multiple of 3, from $(*)$ it follows that $m = 2k + 1$ must be an odd number. But $k \equiv 0,\ 1,\ 2 \pmod 3$ and considering the remainder of both sides of $(*)$ modulo 9 we obtain $k \equiv 1 \pmod 3$. If we set $k = 1 + 3l$, then $m = 6l + 3$.

If we set $\left|\frac{G}{K}\right| = t$, then $t \mid |\mathrm{Out}(P)|$, and $t \mid |H|\,|P| = |G|$, from which it follows:

$$(**) \qquad \begin{aligned} &t\,|H|\,q^{12}(q^4-1)(q^3+1)2^{m+1}(2^m+1)(2^{2m+1}+1) \\ &= 3^{p^2}(3^p+1)\prod_{i=1}^{p-1}(3^{2i}-1). \end{aligned}$$

Since $q = 2^{2m+1} = 2^{12l+7}$ and by [7] t is a divisor of $12l+7$, we deduce $3 \nmid t$. Now we have $(q^4-1)_3 = 3$, $(q^3+1)_3 = 3^2$, $(2^{2m+1}+1)_3 = 3$, and by setting $(2^m+1)_3 = 3^s$ we obtain $3^{p^2-(s+4)}$ for the order of a Sylow 3-subgroup S of H. By Lemma 3 we have $\frac{3^p-1}{2} \mid 3^{p^2-(s+4)} - 1$, from which it follows that $s+4$ should be a multiple of p. We set $s+4 = pt$ for some $t \in \mathbb{N}$.

If $t = 1$, then $p = s+4$, and since from $(*)$ we have $3^p > 81(2^m+1)$, hence $3^{p-4} = 3^s > 2^m+1$ contradicts $(2^m+1)_3 = 3^s$. Therefore we assume $t > 1$. Now we set $2^m+1 = 3^s u = 3^{pt-4}u$, from which it follows that $2^m - 1 > 3^p - 3$ which is against $(*)$. This final contradiction rules out the possibility of Case 16.

In the final step we must consider simple groups in Table 3 which are not considered so far. These are ${}^2B_2(q)$, $q = 2^{2m+1} > 2$ and $E_8(q)$, $q \equiv 2, 3 \pmod 5$ with 4 prime graph components and $E_8(q)$, $q \equiv 0, 1, 4 \pmod 5$ with 5 prime graph components. In all of the above cases the odd order component is a number of the form $qf(q)+1$, where f is a function of q. If $qf(q)+1 = \frac{3^p-1}{2}$, then $qf(q) = \frac{3(3^{p-1}-1)}{2}$, hence the same consideration as used so far works to obtain a contradiction. Because of similarity we don't present the detail. Finally since we have considered all the simple groups listed in Tables 1, 2 and 3, the main theorem is proved now. □

Acknowledgements. This research was done at the mathematics department of the university of North Carolina at Charlotte, USA, while the author had a visiting position in 2006-2007. The author deeply appreciate the kindness and hospitality of the chair and the cochair of the UNCC mathematics department. The support of the university of Tehran is also acknowledged.

References

[1] S. H. Alavi and A. Daneshkhah, *A new characterization of alternating and symmetric groups*, J. Appl. Math. Comput. **17** (2005), no. 1-2, 245–258.

[2] G. Y. Chen, *A new characterization of sporadic simple groups*, Algebra Colloq. **3** (1996), no. 1, 49–58.

[3] ———, *A new characterization of* $\mathrm{PSL}_2(q)$, Southeast Asian Bull. Math. **22** (1998), no. 3, 257–263.

[4] ———, *On Thompson's conjecture*, J. Algebra **185** (1996), no. 1, 184–193.

[5] ———, *On Frobenius and 2-Frobenius groups*, J. Southwest China Normal Univ. **20** (1995), no. 5, 485–487.

[6] G. Y. Chen and H. Shi, ${}^2D_n(3)(9 \le n = 2^m+1$ *not a prime) can be characterized by its order components*, J. Appl. Math. Comput. **19** (2005), no. 1-2, 353–362.

[7] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[8] P. Crescenzo, *A Diophantine equation which arises in the theory of finite groups*, Advances in Math. **17** (1975), no. 1, 25–29.
[9] A. Iranmanesh, S. H. Alavi, and B. Khosravi, *A characterization of* $\mathrm{PSL}(3,q)$ *where* q *is an odd prime power*, J. Pure Appl. Algebra **170** (2002), no. 2-3, 243–254.
[10] ———, *A characterization of* $\mathrm{PSL}(3,q)$ *for* $q = 2^m$, Acta Math. Sin. (Engl. Ser.) **18** (2002), no. 3, 463–472.
[11] A. Iranmanesh, B. Khosravi, and S. H. Alavi, *A characterization of* $\mathrm{PSU}_3(q)$ *for* $q > 5$, Southeast Asian Bull. Math. **26** (2002), no. 1, 33–44.
[12] A. Khosravi and B. Khosravi, *A new characterization of* $\mathrm{PSL}(p,q)$, Comm. Algebra **32** (2004), no. 6, 2325–2339.
[13] ———, *A new characterization of some alternating and symmetric groups*, Int. J. Math. Math. Sci. (2003), no. 45, 2863–2872.
[14] ———, *r-recognizability of $B_n(q)$ and $C_n(q)$ where $n = 2^m \geq 4$*, J. Pure Appl. Algebra **199** (2005), no. 1-3, 149–165.
[15] Behrooz Khosravi, Behnam Khosravi, and Bahman Khosravi, *The number of isomorphism classes of finite groups with the set of order components of* $C_4(q)$, Appl. Algebra Engrg. Comm. Comput. **15** (2005), no. 5, 349–359.
[16] A. S. Kondratev, *On prime graph components of finite simple groups*, Mat. Sb. 180 (1989), no. 6, 787–797.
[17] H. Shi and G. Y. Chen, ${}^2D_{p+1}(2)(5 \leq p \neq 2^m - 1)$ *can be characterized by its order components*, Kumamoto J. Math. **18** (2005), 1–8.
[18] J. S. Williams, *Prime graph components of finite groups*, J. Algebra **69** (1981), no. 2, 487–513.
[19] K. Zsigmondy, *Zur Theorie der Potenzreste*, Monatsh. Math. Phys. **3** (1892), no. 1, 265–284.

Department of Mathematics, Statistics and Computer Science
Faculty of Science, University of Tehran
Tehran, Iran
E-mail address: darafsheh@ut.ac.ir