

FOCK SPACE REPRESENTATIONS OF QUANTUM AFFINE ALGEBRAS AND GENERALIZED LASCoux-LECLERC-THIBON ALGORITHM

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ABSTRACT. We construct the Fock space representations of classical quantum affine algebras using combinatorics of Young walls. We also show that the crystal graphs of the Fock space representations can be realized as the abstract crystal consisting of proper Young walls. Finally, we give a generalized version of Lascoux-Leclerc-Thibon algorithm for computing the global bases of the basic representations of classical quantum affine algebras.

Introduction

The *crystal basis theory* developed by Kashiwara ([9, 10]) provides us with a very powerful combinatorial method of studying the structure of integrable modules over quantum groups. Let M be an integrable module over a quantum group $U_q(\mathfrak{g})$ in the category \mathcal{O}_{int} and let \mathbb{A}_0 denote the subring of $\mathbb{Q}(q)$ consisting of regular functions at $q = 0$. A *crystal basis* of M is a pair (L, B) , where L is an \mathbb{A}_0 -lattice of M and B is a \mathbb{Q} -basis of L/qL satisfying certain properties involving *Kashiwara operators*. Thus a crystal basis can be understood as a basis of M at $q = 0$ and the set B is given a structure of colored oriented graph, called the *crystal graph*, that reflects the combinatorial structure of M .

It is known that every $U_q(\mathfrak{g})$ -module M in the category \mathcal{O}_{int} is a direct sum of irreducible highest weight modules with dominant integral highest weights and that the crystal bases are preserved under this decomposition (see, for example, [4, 10]). Hence it is a very natural problem to find a concrete realization of the crystal graph $B(\lambda)$ of an irreducible highest weight $U_q(\mathfrak{g})$ -module $V(\lambda)$ with dominant integral highest weight λ .

Moreover, one can *globalize* the main idea of crystal basis theory. Let $V(\lambda)$ be an irreducible highest weight $U_q(\mathfrak{g})$ -module with dominant integral highest

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weight λ , and let $(L(\lambda), B(\lambda))$ be the crystal basis of $V(\lambda)$. Consider the \mathbb{Q} -algebra automorphism of $U_q(\mathfrak{g})$ defined by

$$\bar{q} = q^{-1}, \quad \bar{e}_i = e_i, \quad \bar{f}_i = f_i, \quad \bar{q}^h = q^{-h} \quad \text{for } i \in I, h \in P^\vee.$$

Then we get a \mathbb{Q} -linear automorphism of $V(\lambda)$ given by

$$Pv_\lambda \longmapsto \bar{P}v_\lambda \quad \text{for } P \in U_q(\mathfrak{g}),$$

where v_λ denotes the highest weight vector of $V(\lambda)$. In [10] and [16], Kashiwara and Lusztig independently showed that there exists a unique *global basis* (or *canonical basis*) $G(\lambda) = \{G(b) \mid b \in B(\lambda)\}$ of $V(\lambda)$ satisfying the properties:

$$\overline{G(b)} = G(b), \quad G(b) \equiv b \pmod{qL(\lambda)} \quad \text{for all } b \in B(\lambda).$$

Therefore one naturally gets interested in the following problem: Given a realization of the crystal graph $B(\lambda)$ of $V(\lambda)$, can we find an effective algorithm for constructing the global basis $G(\lambda)$?

In [7], Kang introduced the notion of *Young walls* as a new combinatorial scheme for realizing the crystal bases for quantum affine algebras. The Young walls consist of colored blocks with various shapes that are built on a given *ground-state wall*, and they can be viewed as generalizations of colored Young diagrams. More precisely, let $U_q(\mathfrak{g})$ be a classical quantum affine algebra of type $A_n^{(1)}$ ($n \geq 1$), $A_{2n-1}^{(2)}$ ($n \geq 3$), $D_n^{(1)}$ ($n \geq 4$), $A_{2n}^{(2)}$ ($n \geq 1$), $D_{n+1}^{(2)}$ ($n \geq 2$) and $B_n^{(1)}$ ($n \geq 3$), and let Λ be a dominant integral weight of level 1. Then the description of ground-state wall Y_Λ , the rules and patterns for building Young walls, and the action of Kashiwara operators are given explicitly in terms of combinatorics of Young walls. In this way, the set $\mathcal{Z}(\Lambda)$ of *proper Young walls* is given a structure of *abstract crystal* for the quantum affine algebra $U_q(\mathfrak{g})$, and the crystal $B(\Lambda)$ of the basic representation $V(\Lambda)$ is realized as the abstract crystal $\mathcal{Y}(\Lambda)$ consisting of *reduced proper Young walls* (see [7] for more details).

The goal of this paper is to find an effective algorithm for computing the global basis element $G(Y)$ in $G(\Lambda)$ for each reduced proper Young wall Y in $\mathcal{Y}(\Lambda)$. For this purpose, we first construct the *Fock space representations* of quantum affine algebras in a purely combinatorial way. We take the Fock space $\mathcal{F}(\Lambda)$ to be the $\mathbb{Q}(q)$ -vector space with a basis consisting of proper Young walls, and define the $U_q(\mathfrak{g})$ -action on $\mathcal{F}(\Lambda)$ in terms of combinatorics of Young walls. Then the Fock space $\mathcal{F}(\Lambda)$ becomes an integrable $U_q(\mathfrak{g})$ -module in the category \mathcal{O}_{int} (see Section 5).

We then show that the crystal of $\mathcal{F}(\Lambda)$ is isomorphic to the crystal $\mathcal{Z}(\Lambda)$ consisting of proper Young walls (Theorem 6.1). As a corollary, we get an explicit decomposition of the Fock space $\mathcal{F}(\Lambda)$ into a direct sum of irreducible highest weight $U_q(\mathfrak{g})$ -modules by locating the maximal vectors in the crystal $\mathcal{Z}(\Lambda)$ (Corollary 6.2).

In [13], Kashiwara, Miwa, Petersen and Yung gave a more abstract construction of the Fock space representations of quantum affine algebras. For a dominant integral weight Λ of level $l \geq 1$, the Fock space $\mathcal{F}(\Lambda)$ was realized

as the inductive limit of q -deformed wedge spaces arising from a level l perfect representation. We expect one can construct the higher level Fock space representations of quantum affine algebras using combinatorics of Young walls.

Finally, we give a generalized version of Lascoux-Leclerc-Thibon algorithm ([14, 15]) for constructing the global bases of basic representations of classical quantum affine algebras of type $A_n^{(1)}$ ($n \geq 1$), $A_{2n-1}^{(2)}$ ($n \geq 3$), $D_n^{(1)}$ ($n \geq 4$), $A_{2n}^{(2)}$ ($n \geq 1$), $D_{n+1}^{(2)}$ ($n \geq 2$) and $B_n^{(1)}$ ($n \geq 3$). More precisely, for each reduced proper Young wall Y in $\mathcal{Y}(\Lambda)$, we obtain an effective algorithm for computing the global basis element $G(Y)$ in $G(\Lambda)$ that can be expressed as a $\mathbb{Z}[q]$ -linear combination of proper Young walls (Theorem 7.13):

$$G(Y) = \sum_{Z \in \mathcal{Z}(\Lambda)} G_{Y,Z}(q)Z \quad \text{for some } G_{Y,Z}(q) \in \mathbb{Z}[q].$$

By construction, the matrix coefficients $G_{Y,Z}(q)$ satisfy certain unitriangular conditions. We expect that there exist some interesting algebraic structures such that the irreducible modules at some specializations are parametrized by reduced proper Young walls and that the decomposition matrices are determined by the polynomials $G_{Y,Z}(q)$ giving the global basis elements (cf. [1, 2, 3]).

1. Quantum groups

Let I be a finite index set. A square matrix $A = (a_{ij})_{i,j \in I}$ is called a *generalized Cartan matrix* if it satisfies: (i) $a_{ii} = 2$ for all $i \in I$, (ii) $a_{ij} \in \mathbb{Z}_{\leq 0}$ for all $i, j \in I$, (iii) $a_{ij} = 0$ implies $a_{ji} = 0$. In this paper, we assume that A is *symmetrizable*; i.e., there is a diagonal matrix $D = \text{diag}(s_i \in \mathbb{Z}_{>0} \mid i \in I)$ with positive integral entries such that DA is symmetric.

Consider the free abelian group

$$(1.1) \quad P^\vee = \left(\bigoplus_{i \in I} \mathbb{Z}h_i \right) \oplus \left(\bigoplus_{j=1}^{\text{corank } A} \mathbb{Z}d_j \right),$$

and let $\mathfrak{h} = \mathbb{Q} \otimes_{\mathbb{Z}} P^\vee$. The free abelian group P^\vee is called the *dual weight lattice* and the \mathbb{Q} -vector space \mathfrak{h} is called the *Cartan subalgebra*.

The *weight lattice* and the set of *simple coroots* are defined to be

$$(1.2) \quad P = \{ \lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbb{Z} \}, \quad \Pi^\vee = \{ h_i \mid i \in I \}.$$

We denote by $\Pi = \{ \alpha_i \mid i \in I \}$ the set of *simple roots*, which is a linearly independent subset of \mathfrak{h}^* satisfying

$$(1.3) \quad \alpha_i(h_j) = a_{ji} \quad \text{for all } i, j \in I.$$

Definition 1.1. The quintuple $(A, P^\vee, P, \Pi^\vee, \Pi)$ defined above is called a *Cartan datum* associated with A .

We denote by $P^+ = \{ \lambda \in P \mid \lambda(h_i) \geq 0 \text{ for all } i \in I \}$ the set of *dominant integral weights*. The free abelian group $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is called the *root lattice*.

We set $Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$ and $Q_- = -Q_+$. There is a partial ordering on \mathfrak{h}^* defined by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q_+$. Since the generalized Cartan matrix A is symmetrizable, there is a nondegenerate symmetric bilinear form (\mid) on \mathfrak{h}^* satisfying

$$s_i = \frac{(\alpha_i|\alpha_i)}{2} \quad \text{and} \quad \frac{2(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)} = a_{ij} \quad \text{for all } i, j \in I.$$

For an indeterminate q , set $q_i = q^{s_i}$ and define

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_i! = \prod_{k=1}^n [k]_i, \quad \begin{bmatrix} m \\ n \end{bmatrix}_i = \frac{[m]_i!}{[m-n]_i! [n]_i!}.$$

Definition 1.2. The *quantum group* $U_q(\mathfrak{g})$ associated with a Cartan datum $(A, P^\vee, P, \Pi^\vee, \Pi)$ is the associative algebra over $\mathbb{Q}(q)$ with 1 generated by the symbols e_i, f_i ($i \in I$) and q^h ($h \in P^\vee$) subject to the following defining relations:

$$(1.4) \quad \begin{aligned} q^0 &= 1, \quad q^h q^{h'} = q^{h+h'} \quad (h, h' \in P^\vee), \\ q^h e_i q^{-h} &= q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \quad (h \in P^\vee, i \in I), \\ e_i f_j - f_j e_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad \text{where } K_i = q^{s_i h_i}, \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i e_i^{1-a_{ij}-k} e_j e_i^k &= 0 \quad (i \neq j), \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i f_i^{1-a_{ij}-k} f_j f_i^k &= 0 \quad (i \neq j). \end{aligned}$$

The quantum group $U_q(\mathfrak{g})$ has a *Hopf algebra* structure with the comultiplication Δ , counit ε , and antipode S defined by

$$(1.5) \quad \begin{aligned} \Delta(q^h) &= q^h \otimes q^h, \\ \Delta(e_i) &= e_i \otimes K_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i, \\ \varepsilon(q^h) &= 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0, \\ S(q^h) &= q^{-h}, \quad S(e_i) = -e_i K_i, \quad S(f_i) = -K_i^{-1} f_i \end{aligned}$$

for $h \in P^\vee$ and $i \in I$.

Let U^+ (resp. U^-) be the subalgebra of $U_q(\mathfrak{g})$ generated by the elements e_i (resp. f_i) for $i \in I$, and let U^0 be the subalgebra of $U_q(\mathfrak{g})$ generated by q^h ($h \in P^\vee$). Then we have the *triangular decomposition*:

$$U_q(\mathfrak{g}) \cong U^- \otimes U^0 \otimes U^+.$$

A $U_q(\mathfrak{g})$ -module V is called a *weight module* if it admits a *weight space decomposition* $V = \bigoplus_{\mu \in P} V_\mu$, where $V_\mu = \{v \in V \mid q^h v = q^{\mu(h)} v \text{ for all } h \in P^\vee\}$.

$P^\vee\}$. If $\dim_{\mathbb{Q}(q)} V_\mu < \infty$ for all $\mu \in P$, we define the *character* of V by

$$\text{ch} V = \sum_{\mu \in P} (\dim_{\mathbb{Q}(q)} V_\mu) e^\mu,$$

where e^μ are basis elements of the group algebra $\mathbb{Q}(q)[P]$ with the multiplication given by $e^\mu e^\nu = e^{\mu+\nu}$ for all $\mu, \nu \in P$.

A weight module V over $U_q(\mathfrak{g})$ is called a *highest weight module with highest weight* λ ($\lambda \in P$) if there exists a non-zero vector $v_\lambda \in V$ (called the *highest weight vector*) such that (i) $e_i v_\lambda = 0$ for all $i \in I$, (ii) $q^h v_\lambda = q^{\lambda(h)} v_\lambda$ for all $h \in P^\vee$, (iii) $V = U_q(\mathfrak{g}) v_\lambda$.

For example, let $J(\lambda)$ denote the left ideal of $U_q(\mathfrak{g})$ generated by $e_i, q^h - q^{\lambda(h)} 1$ for $i \in I, h \in P^\vee$, and set $M(\lambda) = U_q(\mathfrak{g})/J(\lambda)$. Then, via left multiplication, $M(\lambda)$ becomes a highest weight $U_q(\mathfrak{g})$ -module with highest weight λ , called the *Verma module*, and it satisfies the following properties:

Proposition 1.3 (cf. [4]). (a) $M(\lambda)$ is a free U^- -module of rank 1.

(b) Every highest weight $U_q(\mathfrak{g})$ -module with highest weight λ is a homomorphic image of $M(\lambda)$.

(c) $M(\lambda)$ contains a unique maximal submodule $R(\lambda)$.

The unique irreducible quotient $V(\lambda) = M(\lambda)/R(\lambda)$ is called the *irreducible highest weight* $U_q(\mathfrak{g})$ -module with highest weight λ .

Definition 1.4. The category \mathcal{O}_{int} consists of $U_q(\mathfrak{g})$ -modules M satisfying the following properties:

(i) M is a weight module,

(ii) there exist finitely many $\lambda_1, \dots, \lambda_s \in P$ such that

$$\text{wt}(M) := \{\mu \in P \mid M_\mu \neq 0\} \subset \bigcup_{j=1}^s (\lambda_j - Q_+),$$

(iii) e_i and f_i ($i \in I$) are locally nilpotent on M .

The basic properties of the category \mathcal{O}_{int} are given in the following proposition.

Proposition 1.5 (cf. [4]). (a) For each $i \in I$, let $U_{(i)}$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i, f_i, K_i^{\pm 1}$, which is isomorphic to the quantum group $U_q(\mathfrak{sl}_2)$. Then every $U_q(\mathfrak{g})$ -module M in the category \mathcal{O}_{int} is a direct sum of finite dimensional irreducible $U_{(i)}$ -submodules.

(b) The category \mathcal{O}_{int} is semisimple. Moreover, every irreducible object in the category \mathcal{O}_{int} has the form $V(\lambda)$ with $\lambda \in P^+$.

2. Crystal bases

In this section, we briefly review the *crystal basis theory* for quantum groups developed by Kashiwara ([9, 10]). We will also use the following notation for

divided powers:

$$e_i^{(n)} = e_i^n / [n]_i!, \quad f_i^{(n)} = f_i^n / [n]_i!.$$

Fix an index $i \in I$ and let $M = \bigoplus_{\lambda \in P} M_\lambda$ be a $U_q(\mathfrak{g})$ -module in the category \mathcal{O}_{int} . By the representation theory of $U_q(\mathfrak{sl}_2)$, every element $v \in M_\lambda$ can be written uniquely as

$$v = \sum_{k \geq 0} f_i^{(k)} v_k,$$

where $k \geq -\lambda(h_i)$ and $v_k \in \ker e_i \cap M_{\lambda + k\alpha_i}$. We define the endomorphisms \tilde{e}_i and \tilde{f}_i on M , called the *Kashiwara operators*, by

$$(2.1) \quad \tilde{e}_i v = \sum_{k \geq 1} f_i^{(k-1)} v_k, \quad \tilde{f}_i v = \sum_{k \geq 0} f_i^{(k+1)} v_k.$$

Let $\mathbb{A}_0 = \{f/g \in \mathbb{Q}(q) \mid f, g \in \mathbb{Q}[q], g(0) \neq 0\}$ be the subring of $\mathbb{Q}(q)$ consisting of the rational functions in q that are regular at $q = 0$.

Definition 2.1. A *crystal basis* of M is a pair (L, B) , where

- (i) L is a free \mathbb{A}_0 -submodule M such that $M \cong \mathbb{Q}(q) \otimes_{\mathbb{A}_0} L$,
- (ii) B is a basis of the \mathbb{Q} -vector space L/qL ,
- (iii) $L = \bigoplus_{\lambda \in P} L_\lambda$, where $L_\lambda = L \cap M_\lambda$,
- (iv) $B = \bigsqcup_{\lambda \in P} B_\lambda$, where $B_\lambda = B \cap (L_\lambda/qL_\lambda)$,
- (v) $\tilde{e}_i L \subset L$, $\tilde{f}_i L \subset L$ for all $i \in I$,
- (vi) $\tilde{e}_i B \subset B \cup \{0\}$, $\tilde{f}_i B \subset B \cup \{0\}$ for all $i \in I$,
- (vii) for $b, b' \in B$, $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$.

The set B is given a colored oriented graph structure with the arrows defined by

$$b \xrightarrow{i} b' \quad \text{if and only if} \quad \tilde{f}_i b = b'.$$

The graph associated with B is called the *crystal graph* of M and it reflects the combinatorial structure of M . For instance, we have

$$\dim_{\mathbb{Q}(q)} M_\lambda = \#B_\lambda \quad \text{for all } \lambda \in P.$$

Let (L, B) be a crystal basis of a $U_q(\mathfrak{g})$ -module M in the category \mathcal{O}_{int} . For each $b \in B$ and $i \in I$, we define

$$\varepsilon_i(b) = \max\{k \geq 0 \mid \tilde{e}_i^k b \in B\}, \quad \varphi_i(b) = \max\{k \geq 0 \mid \tilde{f}_i^k b \in B\}.$$

Then the set B satisfies the following properties.

Proposition 2.2 ([10, 11, 12]). (a) For all $i \in I$ and $b \in B$, we have

$$\begin{aligned} \varphi_i(b) &= \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle, \\ \text{wt}(\tilde{e}_i b) &= \text{wt}(b) + \alpha_i, \\ \text{wt}(\tilde{f}_i b) &= \text{wt}(b) - \alpha_i. \end{aligned}$$

(b) If $\tilde{e}_i b \in B$, then

$$\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1, \quad \varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1.$$

(c) If $\tilde{f}_i b \in B$, then

$$\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \quad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1.$$

Moreover, the crystal bases have extremely simple behavior with respect to taking the tensor product.

Proposition 2.3 ([9, 10]). *Let M_j ($j = 1, 2$) be a $U_q(\mathfrak{g})$ -module in the category \mathcal{O}_{int} and let (L_j, B_j) be its crystal basis. Set*

$$L = L_1 \otimes_{A_0} L_2, \quad B = B_1 \times B_2.$$

Then (L, B) is a crystal basis of $M_1 \otimes_{\mathbb{Q}(q)} M_2$ with the Kashiwara operators on B given by

$$\begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \end{aligned}$$

The set $B_1 \times B_2$ will be denoted by $B_1 \otimes B_2$. The *tensor product rule* in Proposition 2.3 gives a very convenient combinatorial description of the action of Kashiwara operators on the multi-fold tensor product of crystal basis. Let M_j be a $U_q(\mathfrak{g})$ -module in the category \mathcal{O}_{int} with a crystal basis (L_j, B_j) ($j = 1, \dots, N$). Fix an index $i \in I$ and consider a vector $b = b_1 \otimes \cdots \otimes b_N \in B_1 \otimes \cdots \otimes B_N$. To each $b_j \in B_j$ ($j = 1, \dots, N$), we assign a sequence of $-$'s and $+$'s with as many $-$'s as $\varepsilon_i(b_j)$ followed by as many $+$'s as $\varphi_i(b_j)$:

$$\begin{aligned} b &= b_1 \otimes b_2 \otimes \cdots \otimes b_N \\ &\mapsto (\underbrace{-, \dots, -}_{\varepsilon_i(b_1)}, \underbrace{+, \dots, +}_{\varphi_i(b_1)}, \dots, \underbrace{-, \dots, -}_{\varepsilon_i(b_N)}, \underbrace{+, \dots, +}_{\varphi_i(b_N)}). \end{aligned}$$

In this sequence, we cancel out all the $(+, -)$ -pairs to obtain a sequence of $-$'s followed by $+$'s:

$$(2.2) \quad i\text{-sgn}(b) = (-, -, \dots, -, +, +, \dots, +).$$

The sequence $i\text{-sgn}(b)$ is called the *i-signature* of b .

Now the tensor product rule tells that \tilde{e}_i acts on b_j corresponding to the rightmost $-$ in the i -signature of b and \tilde{f}_i acts on b_k corresponding to the leftmost $+$ in the i -signature of b :

$$(2.3) \quad \begin{aligned} \tilde{e}_i b &= b_1 \otimes \cdots \otimes \tilde{e}_i b_j \otimes \cdots \otimes b_N, \\ \tilde{f}_i b &= b_1 \otimes \cdots \otimes \tilde{f}_i b_k \otimes \cdots \otimes b_N. \end{aligned}$$

We define $\tilde{e}_i b = 0$ (resp. $\tilde{f}_i b = 0$) if there is no $-$ (resp. $+$) in the i -signature of b .

By extracting the properties of crystal graphs, we define the notion of abstract *crystals* as follows ([11, 12]).

Definition 2.4. Let $(A, P^\vee, P, \Pi^\vee, \Pi)$ be a Cartan datum and let $U_q(\mathfrak{g})$ be the corresponding quantum group.

A *crystal associated with* $(A, P^\vee, P, \Pi^\vee, \Pi)$ (or a $U_q(\mathfrak{g})$ -crystal) is a set B together with the maps $\text{wt} : B \rightarrow P$, $\varepsilon_i : B \rightarrow \mathbb{Z} \cup \{-\infty\}$, $\varphi_i : B \rightarrow \mathbb{Z} \cup \{-\infty\}$, $\bar{e}_i : B \rightarrow B \cup \{0\}$, and $\tilde{f}_i : B \rightarrow B \cup \{0\}$ satisfying the following conditions:

(i) for all $i \in I$, $b \in B$, we have

$$\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle,$$

$$\text{wt}(\bar{e}_i b) = \text{wt}(b) + \alpha_i,$$

$$\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i,$$

(ii) if $\bar{e}_i b \in B$, then

$$\varepsilon_i(\bar{e}_i b) = \varepsilon_i(b) - 1, \quad \varphi_i(\bar{e}_i b) = \varphi_i(b) + 1,$$

(iii) if $\tilde{f}_i b \in B$, then

$$\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1, \quad \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1,$$

(iv) $\tilde{f}_i b = b'$ if and only if $b = \bar{e}_i b'$ for all $i \in I$, $b, b' \in B$,

(v) if $\varepsilon_i(b) = -\infty$, then $\bar{e}_i b = \tilde{f}_i b = 0$.

For instance, if (L, B) is a crystal basis of a $U_q(\mathfrak{g})$ -module in the category \mathcal{O}_{int} , then B is a $U_q(\mathfrak{g})$ -crystal.

Definition 2.5. Let B_1 and B_2 be crystals. A *morphism of crystals* (or a *crystal morphism*) $\psi : B_1 \rightarrow B_2$ is a map $\psi : B_1 \cup \{0\} \rightarrow B_2 \cup \{0\}$ satisfying the conditions:

(i) $\psi(0) = 0$,

(ii) if $b \in B_1$ and $\psi(b) \in B_2$, then

$$\text{wt}(\psi(b)) = \text{wt}(b), \quad \varepsilon_i(\psi(b)) = \varepsilon_i(b), \quad \varphi_i(\psi(b)) = \varphi_i(b),$$

(iii) if $b, b' \in B_1$, $\psi(b), \psi(b') \in B_2$ and $\tilde{f}_i b = b'$, then $\tilde{f}_i \psi(b) = \psi(b')$.

Definition 2.6. The *tensor product* $B_1 \otimes B_2$ of the crystals B_1 and B_2 is defined to be the set $B_1 \times B_2$ whose crystal structure is given by

$$\begin{aligned} \text{wt}(b_1 \otimes b_2) &= \text{wt}(b_1) + \text{wt}(b_2), \\ \varepsilon_i(b_1 \otimes b_2) &= \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, \text{wt}(b_1) \rangle), \\ \varphi_i(b_1 \otimes b_2) &= \max(\varphi_i(b_2), \varepsilon_i(b_1) + \langle h_i, \text{wt}(b_2) \rangle), \\ (2.4) \quad \bar{e}_i(b_1 \otimes b_2) &= \begin{cases} \bar{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \bar{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \end{aligned}$$

Here, we denote $b_1 \otimes b_2 = (b_1, b_2)$ and use the convention that $b_1 \otimes 0 = 0 \otimes b_2 = 0$. With the above definitions, we can check that the category of crystals becomes a *tensor category*.

The existence and uniqueness theorem for crystal bases is given as follows.

Theorem 2.7 ([10]). *Let $V(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$ -module with highest weight $\lambda \in P^+$ and highest weight vector v_λ . Let $L(\lambda)$ be the free \mathbb{A}_0 -submodule of $V(\lambda)$ spanned by the vectors of the form $\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_\lambda$ ($i_k \in I, r \in \mathbb{Z}_{\geq 0}$) and set*

$$B(\lambda) = \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_\lambda + qL(\lambda) \in L(\lambda)/qL(\lambda)\} \setminus \{0\}.$$

Then $(L(\lambda), B(\lambda))$ is a crystal basis of $V(\lambda)$ and every crystal basis of $V(\lambda)$ is isomorphic to $(L(\lambda), B(\lambda))$.

One can globalize the main idea of crystal basis theory. Consider the \mathbb{Q} -algebra automorphism of $U_q(\mathfrak{g})$ defined by

$$(2.5) \quad \bar{q} = q^{-1}, \quad \bar{e}_i = e_i, \quad \bar{f}_i = f_i, \quad \bar{q}^h = q^{-h} \quad \text{for } i \in I, h \in P^\vee.$$

This induces a \mathbb{Q} -linear automorphism of $V(\lambda)$ given by

$$(2.6) \quad Pv_\lambda \longmapsto \bar{P}v_\lambda \quad \text{for } P \in U_q(\mathfrak{g}),$$

where v_λ denotes the highest weight vector of $V(\lambda)$. Let $\mathbb{A} = \mathbb{Q}[q, q^{-1}]$ and define $V(\lambda)_\mathbb{A} = U_\mathbb{A}^- v_\lambda$, where $U_\mathbb{A}^-$ is the \mathbb{A} -subalgebra of $U_q(\mathfrak{g})$ generated by $f_i^{(n)}$ ($i \in I, n \in \mathbb{Z}_{\geq 0}$). Then we have

Theorem 2.8 ([10, 16]). *There exists a unique \mathbb{A} -basis $G(\lambda) = \{G(b) \in V(\lambda)_\mathbb{A} \cap L(\lambda) \mid b \in B(\lambda)\}$ of $V(\lambda)_\mathbb{A}$ such that*

$$\overline{G(b)} = G(b), \quad G(b) \equiv b \pmod{qL(\lambda)} \quad \text{for all } b \in B(\lambda).$$

The basis $G(\lambda)$ of $V(\lambda)$ given in Theorem 2.8 is called the *global basis* or the *canonical basis* of $V(\lambda)$ associated with the crystal basis $(L(\lambda), B(\lambda))$.

3. Quantum affine algebras

Let $I = \{0, 1, \dots, n\}$ be an index set and let $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix of *affine type*. We denote by

$$P^\vee = \mathbb{Z}h_0 \oplus \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n \oplus \mathbb{Z}d$$

the *dual weight lattice* and $\Pi^\vee = \{h_i \mid i \in I\}$ the *simple coroots*. The *simple roots* α_i and the *fundamental weights* Λ_i are given by

$$\begin{aligned} \alpha_i(h_j) &= a_{ji}, & \alpha_i(d) &= \delta_{0,i}, \\ \Lambda_i(h_j) &= \delta_{ij}, & \Lambda_i(d) &= 0 \quad (i, j \in I). \end{aligned}$$

We define the *affine weight lattice* to be

$$P = \{\lambda \in \mathfrak{h}^* \mid \lambda(P^\vee) \subset \mathbb{Z}\}.$$

The quintuple $(A, \Pi, \Pi^\vee, P, P^\vee)$ is called an *affine Cartan datum*. To each affine Cartan datum, we can associate an infinite dimensional Lie algebra \mathfrak{g} called the *affine Kac-Moody algebra* ([6]). The center of the affine Kac-Moody algebra \mathfrak{g} is 1-dimensional and is generated by the *canonical central element*

$$c = c_0 h_0 + c_1 h_1 + \cdots + c_n h_n.$$

Moreover, the imaginary roots of \mathfrak{g} are nonzero integral multiples of the *null root*

$$\delta = d_0 \alpha_0 + d_1 \alpha_1 + \cdots + d_n \alpha_n.$$

Here, c_i and d_i ($i \in I$) are the non-negative integers given in [6].

Using the fundamental weights and the null root, the affine weight lattice can be written as

$$P = \mathbb{Z}\Lambda_0 \oplus \mathbb{Z}\Lambda_1 \oplus \cdots \oplus \mathbb{Z}\Lambda_n \oplus \mathbb{Z}\delta.$$

Set

$$P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I\}.$$

The elements of P (resp. P^+) are called the *affine weights* (resp. *affine dominant integral weights*). The *level* of an affine dominant integral weight $\lambda \in P^+$ is defined to be the nonnegative integer $\lambda(c)$.

Definition 3.1. The *quantum affine algebra* $U_q(\mathfrak{g})$ is the quantum group associated with the affine Cartan datum $(A, \Pi, \Pi^\vee, P, P^\vee)$.

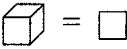







The subalgebra of $U_q(\mathfrak{g})$ generated by $e_i, f_i, K_i^{\pm 1}$ ($i \in I$) is denoted by $U'_q(\mathfrak{g})$, and is also called the *quantum affine algebra*.

In this paper, we will focus on the quantum affine algebras of type $A_n^{(1)}$ ($n \geq 1$), $A_{2n-1}^{(2)}$ ($n \geq 3$), $D_n^{(1)}$ ($n \geq 4$), $A_{2n}^{(2)}$ ($n \geq 1$), $D_{n+1}^{(2)}$ ($n \geq 2$) and $B_n^{(1)}$ ($n \geq 3$).

4. Combinatorics of Young walls

In [7], Kang introduced a new family of combinatorial objects called the *Young walls* which can be viewed as generalizations of colored Young diagrams, and gave a realization of crystals $B(\Lambda)$ for the basic representations of quantum affine algebras in terms of *reduced proper Young walls*. In this section, we briefly explain the combinatorics of Young walls.

The Young walls are built of colored blocks of three different shapes. They are called the blocks of *type I*, *type II*, and *type III*, respectively. For each type of quantum affine algebras, we use different sets of colored blocks.

Type	Shape	Width	Thickness	Height	Volume
I	 = 	1	1	1	1
II	 = 	1	1	$\frac{1}{2}$	$\frac{1}{2}$
III	 =  ,  = 	1	$\frac{1}{2}$	1	$\frac{1}{2}$

For each dominant integral weight Λ of level 1, we fix a frame Y_Λ called the *ground-state wall of weight Λ* , and on this frame, we build a wall of thickness less than or equal to one unit which extends infinitely to the left. The rules for building the walls are as follows:

- (1) The colored blocks should be stacked in columns. No block can be placed on top of a column of half-unit thickness.
- (2) Except for the right-most column, there should be no free space to the right of any block.
- (3) The colored blocks should be stacked in a specified pattern which is determined by the type of the quantum affine algebra $U_q(\mathfrak{g})$ and the level 1 dominant integral weight Λ .

The coloring of blocks, the description of ground-state walls and the patterns for building the walls are given in [7] (see also Appendix).

Example 4.1. If $\mathfrak{g} = B_3^{(1)}$ and $\Lambda = \Lambda_0$, we use the colored blocks

$$\begin{array}{|c|} \hline 0 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 1 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 2 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 3 \\ \hline \end{array}.$$

The walls are built on the ground-state wall

$$Y_{\Lambda_0} = \dots \begin{array}{|c|c|c|c|} \hline 0 & 1 & 0 & 1 \\ \hline \end{array} = \dots \begin{array}{|c|c|c|c|} \hline \diagdown & \diagdown & \diagdown & \diagdown \\ \hline \end{array}$$

following the pattern given below.

	$\begin{array}{ c } \hline 1 \\ \hline \end{array}$	$\begin{array}{ c } \hline 0 \\ \hline \end{array}$	$\begin{array}{ c } \hline 1 \\ \hline \end{array}$	$\begin{array}{ c } \hline 0 \\ \hline \end{array}$
	$\begin{array}{ c } \hline 0 \\ \hline \end{array}$	$\begin{array}{ c } \hline 1 \\ \hline \end{array}$	$\begin{array}{ c } \hline 0 \\ \hline \end{array}$	$\begin{array}{ c } \hline 1 \\ \hline \end{array}$
	2	2	2	2
	3	3	3	3
	3	3	3	3
	2	2	2	2
	$\begin{array}{ c } \hline 1 \\ \hline \end{array}$	$\begin{array}{ c } \hline 0 \\ \hline \end{array}$	$\begin{array}{ c } \hline 1 \\ \hline \end{array}$	$\begin{array}{ c } \hline 0 \\ \hline \end{array}$
	$\begin{array}{ c } \hline 0 \\ \hline \end{array}$	$\begin{array}{ c } \hline 1 \\ \hline \end{array}$	$\begin{array}{ c } \hline 0 \\ \hline \end{array}$	$\begin{array}{ c } \hline 1 \\ \hline \end{array}$

A wall Y built on the ground-state wall Y_Λ following the rules given above is called a *Young wall on Y_Λ* , if the heights of its columns are weakly decreasing as we proceed from right to left. We often write $Y = (y_k)_{k=0}^\infty = (\dots, y_2, y_1, y_0)$ as an infinite sequence of its columns.

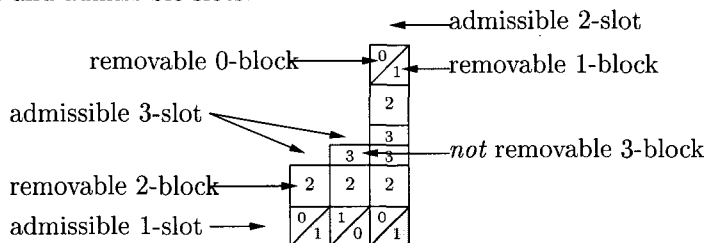
Definition 4.2. (1) A column of a Young wall is called a *full column* if its height is a multiple of the unit length and its top is of unit thickness.
 (2) For quantum affine algebras of type $A_{2n-1}^{(2)}$ ($n \geq 3$), $D_n^{(1)}$ ($n \geq 4$), $A_{2n}^{(2)}$ ($n \geq 1$), $D_{n+1}^{(2)}$ ($n \geq 2$) and $B_n^{(1)}$ ($n \geq 3$), a Young wall is said to be *proper* if none of the full columns have the same heights.
 (3) For quantum affine algebras of type $A_n^{(1)}$ ($n \geq 1$), every Young wall is defined to be *proper*.

Let $\mathcal{Z}(\Lambda)$ denote the set of all proper Young walls on Y_Λ . Then $\mathcal{Z}(\Lambda)$ can be given a $U_q(\mathfrak{g})$ -crystal structure as follows.

Definition 4.3. Let $Y = (y_k)_{k=0}^\infty$ be a proper Young wall on Y_Λ .

- (1) A block of color i (in short, an i -block) in Y is called a *removable i -block* if Y remains a proper Young wall after removing the block. A column in Y is said to be *i -removable* if the column has a removable i -block.
- (2) A place in Y is called an *admissible i -slot* if one may add an i -block to obtain another proper Young wall. A column in Y is said to be *i -admissible* if the column has an admissible i -slot.

Example 4.4. In the following figure, we consider a proper Young wall for $\mathfrak{g} = B_3^{(1)}$ built on the ground-state wall Y_{Λ_0} and indicate all the removable blocks and admissible slots.



Fix an index $i \in I$ and let $Y = (y_k)_{k=0}^\infty$ be a proper Young wall on Y_Λ . To each column y_k of Y , we assign its *i -signature* as follows:

- (1) we assign $--$ if the column y_k is twice i -removable (the i -block will be of type II);
- (2) we assign $-$ if the column is once i -removable, but not i -admissible (the i -block may be of type I, II, III);
- (3) we assign $-+$ if the column is once i -removable and once i -admissible (the i -block will be of type II);
- (4) we assign $+$ if the column is once i -admissible, but not i -removable (the i -block may be of type I, II, III);

- (5) we assign $++$ if the column is twice i -admissible (the i -block will be of type II).

Then we get a finite sequence of $+$'s and $-$'s for Y . From this sequence, we cancel out every $(+, -)$ -pair to obtain a finite sequence of $-$'s followed by $+$'s, reading from left to right. This sequence is called the i -signature of Y .

We now define the *abstract Kashiwara operators* \tilde{E}_i, \tilde{F}_i ($i \in I$) on $\mathcal{Z}(\Lambda)$ as follows.

- (1) We define $\tilde{E}_i Y$ to be the proper Young wall obtained from Y by removing the i -block corresponding to the rightmost $-$ in the i -signature of Y . We define $\tilde{E}_i Y = 0$ if there exists no $-$ in the i -signature of Y .
- (2) We define $\tilde{F}_i Y$ to be the proper Young wall obtained from Y by adding an i -block to the column corresponding to the leftmost $+$ in the i -signature of Y . We define $\tilde{F}_i Y = 0$ if there exists no $+$ in the i -signature of Y .

Example 4.5. (a) Let $\mathfrak{g} = B_3^{(1)}$ and let

$$Y = \begin{array}{cccccccc} & & & & & & & 0 & 1 \\ & & & & & & 2 & 2 & 2 \\ & & & & & 3 & 3 & 3 \\ & & & & 3 & 3 & 3 \\ & & 2 & 2 & 2 & 2 & 2 \\ & 1 & 0 & 1 & 1 & 0 & 1 \\ & 2 & 2 & 2 & 2 & 2 \\ & & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}$$

. . . . + . . . - ++

If $i = 3$, we first get the sequence $(\dots, +, \cdot, \cdot, -, ++, \cdot, \cdot, \cdot)$. After cancelling out $(+, -)$ -pairs, we obtain the 3-signature $(+, +)$ of Y .

Therefore, we have $\tilde{E}_3 Y = 0$ and

$$\tilde{F}_3 Y = \begin{array}{cccccccc} & & & & & & & 0 & 1 \\ & & & & & & 2 & 2 & 2 \\ & & & & & 3 & 3 & 3 \\ & & & & 3 & 3 & 3 \\ & & 2 & 2 & 2 & 2 & 2 \\ & 1 & 0 & 1 & 1 & 0 & 1 \\ & 2 & 2 & 2 & 2 & 2 \\ & & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}$$

- (b) Let $\mathfrak{g} = A_4^{(2)}$ and let

$$Y = \begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & 1 \\ & & & & & & 2 \\ & & & & & & 1 \\ & & & & & & 0 \\ & & & & 0 & 0 & 0 \\ & & & 1 & 1 & 1 & 1 \\ & & 2 & 2 & 2 & 2 \\ & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

. . . - - . + -+ .

If $i = 0$, we first get the sequence $(\dots, -, -, \cdot, +, -+)$. After cancelling out $(+, -)$ -pairs, we obtain the 0-signature $(-, -, +)$ of Y .

Therefore, we have

$$\tilde{E}_0 Y = \begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & 1 \\ & & & & & & 2 \\ & & & & & & 1 \\ & & & & & & 0 \\ & & & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}, \quad \tilde{F}_0 Y = \begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & 0 \\ & & & & & & 1 \\ & & & & & & 2 \\ & & & & & & 1 \\ & & & & & & 0 \\ & & & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}.$$

Next, we define the maps

$$\text{wt} : \mathcal{Z}(\Lambda) \longrightarrow P, \quad \varepsilon_i : \mathcal{Z}(\Lambda) \longrightarrow \mathbb{Z}, \quad \varphi_i : \mathcal{Z}(\Lambda) \longrightarrow \mathbb{Z}$$

by

$$(4.1) \quad \begin{aligned} \text{wt}(Y) &= \Lambda - \sum_{i \in I} k_i \alpha_i, \\ \varepsilon_i(Y) &= \text{the number of } -\text{'s in the } i\text{-signature of } Y, \\ \varphi_i(Y) &= \text{the number of } +\text{'s in the } i\text{-signature of } Y, \end{aligned}$$

where k_i is the number of i -blocks in Y that have been added to Y_Λ .

Then we have :

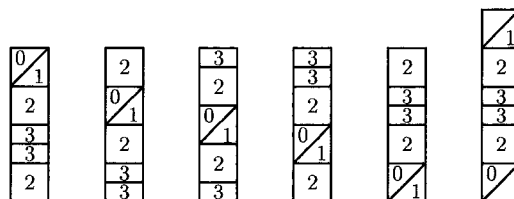
Theorem 4.6 ([7]). *The set $\mathcal{Z}(\Lambda)$ together with the maps $\text{wt} : \mathcal{Z}(\Lambda) \longrightarrow P$, $\tilde{E}_i, \tilde{F}_i : \mathcal{Z}(\Lambda) \longrightarrow \mathcal{Z}(\Lambda) \cup \{0\}$, and $\varepsilon_i, \varphi_i : \mathcal{Z}(\Lambda) \longrightarrow \mathbb{Z}$ ($i \in I$) becomes a $U_q(\mathfrak{g})$ -crystal.*

Let $\delta = d_0 \alpha_0 + \dots + d_n \alpha_n$ be the null root of $U_q(\mathfrak{g})$, and set

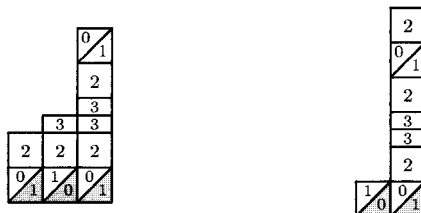
$$(4.2) \quad a_i = \begin{cases} d_i & \text{if } \mathfrak{g} \neq D_{n+1}^{(2)}, \\ 2d_i & \text{if } \mathfrak{g} = D_{n+1}^{(2)}. \end{cases}$$

- Definition 4.7.** (1) The part of a column in a proper Young wall is called a δ -column if it contains a_0 -many 0-blocks, a_1 -many 1-blocks, \dots , a_n -many n -blocks in some cyclic order.
- (2) A δ -column in a proper Young wall is called *removable* if it can be removed from the top to yield another proper Young wall.
- (3) A proper Young wall is said to be *reduced* if none of its columns contain a removable δ -column.

Example 4.8. (a) The following are δ -columns for $\mathfrak{g} = B_3^{(1)}$.



(b) Consider the following proper Young walls for $\mathfrak{g} = B_3^{(1)}$. The first one is reduced, but the second one is not. Note that the second Young wall contains a removable δ -column.



Let Δ be the volume of the δ -column. We list the value of Δ for each quantum affine algebra $U_q(\mathfrak{g})$ in the following table.

$U_q(\mathfrak{g})$	Δ
$A_n^{(1)}$	n
$A_{2n-1}^{(2)}$	$2n - 2$
$D_n^{(1)}$	$2n - 4$
$A_{2n}^{(2)}$	$2n$
$D_{n+1}^{(2)}$	$2n$
$B_n^{(1)}$	$2n - 2$

Note that Δ is not necessarily equal to the Coxeter number or to the dual Coxeter number for \mathfrak{g} (cf. [6]).

Let $\mathcal{Y}(\Lambda) \subset \mathcal{Z}(\Lambda)$ be the set of all reduced proper Young walls on Y_Λ . Then we have:

Theorem 4.9 ([7]). *For all $i \in I$ and $Y \in \mathcal{Y}(\Lambda)$, we have*

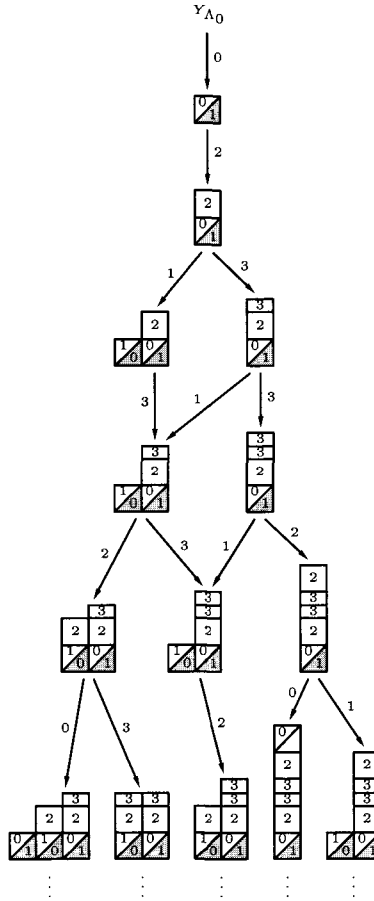
$$\tilde{E}_i Y \in \mathcal{Y}(\Lambda) \cup \{0\}, \quad \tilde{F}_i Y \in \mathcal{Y}(\Lambda) \cup \{0\}.$$

Hence $\mathcal{Y}(\Lambda)$ is a $U_q(\mathfrak{g})$ -crystal. Moreover, there exists an isomorphism of $U_q(\mathfrak{g})$ -crystals

$$\mathcal{Y}(\Lambda) \xrightarrow{\sim} B(\Lambda) \quad \text{given by } Y_\Lambda \mapsto u_\Lambda,$$

where $B(\Lambda)$ is the crystal of the basic representation $V(\Lambda)$ of $U_q(\mathfrak{g})$ and u_Λ is the highest weight vector in $B(\Lambda)$.

Example 4.10. The crystal $\mathcal{Y}(\Lambda_0)$ for $U_q(B_3^{(1)})$ is given as follows



Remark 4.11. When ζ is a primitive n -th root of unity, the finite dimensional irreducible representations of the (finite) Hecke algebra $\mathcal{H}_N(\zeta)$ can be parametrized by n -reduced colored Young diagrams. Observe that they are the same as the reduced proper Young walls of type $A_{n-1}^{(1)}$. We expect that for each type of classical quantum affine algebras and level 1 dominant integral weights, there exist some interesting algebraic structures whose irreducible representations (at some specialization) are parametrized by reduced proper Young walls. In [2], Brundan and Kleshchev verified this idea by showing that the

irreducible representations of the Hecke-Clifford superalgebra $\mathcal{H}_N(\zeta)$ with ζ a primitive $(2n + 1)$ -th root of unity are parametrized by the set of reduced proper Young walls of type $A_{2n}^{(2)}$ with N blocks.

5. Fock space representation

Let $\mathcal{F}(\Lambda) = \bigoplus_{Y \in \mathcal{Z}(\Lambda)} \mathbb{Q}(q)Y$ be the $\mathbb{Q}(q)$ -vector space with a basis $\mathcal{Z}(\Lambda)$ consisting of proper Young walls. The goal of this section is to define a $U_q(\mathfrak{g})$ -module structure on $\mathcal{F}(\Lambda)$, the *Fock space representation of $U_q(\mathfrak{g})$* .

For this purpose, we introduce some terminology. Let $Y = (y_k)_{k=0}^\infty$ be a proper Young wall on Y_Λ , and let $|y_k|$ denote the number of blocks in y_k added to Y_Λ . We define the *associated partition of Y* to be $|Y| = (\dots, |y_k|, \dots, |y_1|, |y_0|)$. For proper Young walls $Y = (y_k)_{k=0}^\infty$ and $Z = (z_k)_{k=0}^\infty$ in $\mathcal{Z}(\Lambda)$, we define $|Y| \supseteq |Z|$ if and only if $\sum_{k=l}^\infty |y_k| \geq \sum_{k=l}^\infty |z_k|$ for all $l \geq 0$. For example, if

$$Y = \begin{array}{c} \begin{array}{|c|} \hline 0 \\ \hline \end{array} \\ \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} \\ \begin{array}{|c|c|c|} \hline 3 & 3 & 3 \\ \hline \end{array} \\ \begin{array}{|c|c|c|} \hline 2 & 2 & 2 \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|} \hline 0 & 1 & 0 & 0 \\ \hline \end{array} \end{array} \quad \text{and} \quad Z = \begin{array}{c} \begin{array}{|c|} \hline 3 \\ \hline \end{array} \\ \begin{array}{|c|} \hline 2 \\ \hline \end{array} \\ \begin{array}{|c|} \hline 0 \\ \hline \end{array} \\ \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} \\ \begin{array}{|c|c|} \hline 3 & 3 \\ \hline \end{array} \\ \begin{array}{|c|c|} \hline 2 & 2 \\ \hline \end{array} \\ \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline \end{array} \end{array},$$

then we have $|Y| = (3, 5, 7) \supseteq |Z| = (1, 5, 9)$.

Note that it is not a partial ordering on $\mathcal{Z}(\Lambda)$ since there exist $Y \neq Z$ in $\mathcal{Z}(\Lambda)$ such that $|Y| = |Z|$. However, it induces a partial ordering on the set of associated partitions. (The readers may want to compare it with the usual dominance ordering. See, for example, [17])

Let $S = \{k \mid s \leq k < t\}$ for some $0 \leq s < t \leq \infty$. Then S is a finite or an infinite interval in $\mathbb{Z}_{\geq 0}$. We define the (S) -part of Y to be $Y_S = (y_k)_{k \in S}$. For example, if $S = \{k \in \mathbb{Z}_{\geq 0} \mid k \geq s\}$, the $Y_S = (y_k)_{k=s}^\infty$ is itself a proper Young wall in $\mathcal{Z}(\Lambda')$ for some level 1 dominant integral weight Λ' . On the other hand, if $S = \{k \mid s \leq k < t < \infty\}$ is a finite interval, Y_S is no more a proper Young wall, but a finite collection of columns in Y . By restricting our attentions to the columns in Y_S , we may define the notions of admissible i -slots, removable i -blocks, the i -signature of Y_S , $\varepsilon_i(Y_S)$, and $\varphi_i(Y_S)$ (however, wt can be defined for proper Young walls only).

Example 5.1. When $\mathfrak{g} = A_4^{(2)}$ and $\Lambda = \Lambda_0$, consider

$$Y = (y_k)_{k=0}^\infty = \begin{array}{c} \begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|c|c|} \hline 2 & 2 & 2 & 2 & 2 & 2 \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \\ \begin{array}{|c|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \end{array}.$$

Note that the 0-signature of Y is $+$ and hence $\varepsilon_0(Y) = 0$, $\varphi_0(Y) = 1$. If $S = \{0, 1, 2, 3, 4\}$, then we have

$$Y_S = \begin{array}{|c|c|c|c|c|} \hline & & & & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline \end{array},$$

$\varepsilon_0(Y_S) = 2$ and $\varphi_0(Y_S) = 0$ since the 0-signature of $Y_S = --$.

We will now define the action of $U_q(\mathfrak{g})$ on $\mathcal{Z}(\Lambda)$. Since the action of q^h ($h \in P^\vee$) is easily defined by

$$(5.1) \quad q^h Y = q^{\text{wt}(Y)(h)} Y \quad \text{for } Y \in \mathcal{Z}(\Lambda),$$

we will focus on the actions of e_i and f_i ($i \in I$).

Case 1. Suppose that the i -blocks are of type I.

Let b be a removable i -block in y_k of Y . We define $Y_R(b) = (y_{k-1}, \dots, y_1, y_0)$ to be the part of Y consisting of the columns lying at the right of b , and set

$$(5.2) \quad R_i(b; Y) = \varphi_i(Y_R(b)) - \varepsilon_i(Y_R(b)).$$

If $k = 0$, we understand $Y_R(b) = \emptyset$ and $R_i(b; Y) = 0$. We denote by $Y \nearrow b$ the Young wall obtained by removing b from Y . Then we define

$$(5.3) \quad e_i Y = \sum_b q_i^{-R_i(b; Y)} (Y \nearrow b),$$

where b runs over all removable i -blocks in Y .

On the other hand, if b is an admissible i -slot in y_k of Y , then we define $Y_L(b) = (\dots, y_{k+2}, y_{k+1})$ to be the Young wall consisting of the columns in Y lying at the left of b , and set

$$(5.4) \quad L_i(b; Y) = \varphi_i(Y_L(b)) - \varepsilon_i(Y_L(b)).$$

Here, the wall $Y_L(b)$ is a proper Young wall on $Y_{\Lambda'}$ for some level 1 dominant integral weight Λ' . We denote by $Y \swarrow b$ the Young wall obtained by adding an i -block at b . Then we define

$$(5.5) \quad f_i Y = \sum_b q_i^{L_i(b; Y)} (Y \swarrow b),$$

where b runs over all admissible i -slots in Y .

Example 5.2. (a) If $\mathfrak{g} = A_2^{(1)}$, $\Lambda = \Lambda_0$ and $i = 0$, then $q_0 = q$ and we have

$$e_0 = \begin{array}{|c|c|c|c|} \hline & & 2 & \\ \hline & 0 & 1 & \\ \hline & 2 & 0 & \\ \hline 0 & 1 & 2 & \\ \hline 2 & 0 & 1 & \\ \hline 0 & 1 & 2 & 0 \\ \hline \end{array} = q \begin{array}{|c|c|c|c|} \hline & & 2 & \\ \hline & 0 & 1 & \\ \hline & 2 & 0 & \\ \hline 0 & 1 & 2 & \\ \hline 2 & 0 & 1 & \\ \hline 1 & 2 & 0 & \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & & 2 & \\ \hline & 0 & 1 & \\ \hline & 2 & 0 & \\ \hline & 1 & 2 & \\ \hline 2 & 0 & 1 & \\ \hline 0 & 1 & 2 & 0 \\ \hline \end{array} + q^{-1} \begin{array}{|c|c|c|c|} \hline & & 2 & \\ \hline & & 1 & \\ \hline & 2 & 0 & \\ \hline & 0 & 1 & 2 \\ \hline & 2 & 0 & 1 \\ \hline 0 & 1 & 2 & 0 \\ \hline \end{array},$$

$$f_0 = \begin{array}{|c|c|c|c|} \hline & & 2 & \\ \hline & & 1 & \\ \hline & 2 & 0 & \\ \hline & 1 & 2 & \\ \hline 2 & 0 & 1 & \\ \hline 0 & 1 & 2 & 0 \\ \hline \end{array} = q^{-1} \begin{array}{|c|c|c|c|} \hline & & 2 & \\ \hline & & 1 & \\ \hline & 2 & 0 & \\ \hline & 0 & 1 & 2 \\ \hline & 2 & 0 & 1 \\ \hline 0 & 1 & 2 & 0 \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & & 2 & \\ \hline & 0 & 1 & \\ \hline & 2 & 0 & \\ \hline & 1 & 2 & \\ \hline 2 & 0 & 1 & \\ \hline 0 & 1 & 2 & 0 \\ \hline \end{array} + q \begin{array}{|c|c|c|c|} \hline & & 0 & \\ \hline & & 2 & \\ \hline & & 1 & \\ \hline & 2 & 0 & \\ \hline & 1 & 2 & \\ \hline 0 & 1 & 2 & 0 \\ \hline \end{array}.$$

(b) If $\mathfrak{g} = A_5^{(2)}$, $\Lambda = \Lambda_0$ and $i = 2$, then $q_2 = q$ and we have

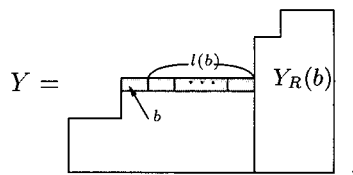
$$e_2 = \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline & 2 & 2 & \\ \hline & 1 & 0 & 1 \\ \hline 2 & 2 & 2 & \\ \hline 3 & 3 & 3 & \\ \hline 2 & 2 & 2 & 2 \\ \hline 1 & 0 & 1 & 0 & 1 \\ \hline \end{array} = q \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline & 2 & 2 & \\ \hline & 1 & 0 & 1 \\ \hline 2 & 2 & 2 & \\ \hline 3 & 3 & 3 & \\ \hline 2 & 2 & 2 & \\ \hline 1 & 0 & 1 & 0 & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline & 2 & 2 & \\ \hline & 1 & 0 & 1 \\ \hline 2 & 2 & 2 & \\ \hline 3 & 3 & 3 & \\ \hline 2 & 2 & 2 & 2 \\ \hline 1 & 0 & 1 & 0 & 1 \\ \hline \end{array} + q^{-1} \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline & & 2 & \\ \hline & 1 & 0 & 1 \\ \hline 2 & 2 & 2 & \\ \hline 3 & 3 & 3 & \\ \hline 2 & 2 & 2 & 2 \\ \hline 1 & 0 & 1 & 0 & 1 \\ \hline \end{array},$$

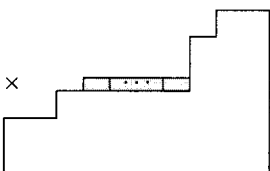
$$f_2 = \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline & & 2 & \\ \hline & 1 & 0 & 1 \\ \hline 2 & 2 & 2 & \\ \hline 3 & 3 & 3 & \\ \hline 2 & 2 & 2 & 2 \\ \hline 1 & 0 & 1 & 0 & 1 \\ \hline \end{array} = q^{-1} \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline & & 2 & \\ \hline & 1 & 0 & 1 \\ \hline 2 & 2 & 2 & \\ \hline 3 & 3 & 3 & \\ \hline 2 & 2 & 2 & 2 \\ \hline 1 & 0 & 1 & 0 & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline & & 3 & \\ \hline & 2 & 2 & \\ \hline & 1 & 0 & 1 \\ \hline 2 & 2 & 2 & \\ \hline 3 & 3 & 3 & \\ \hline 2 & 2 & 2 & 2 \\ \hline 1 & 0 & 1 & 0 & 1 \\ \hline \end{array} + q \begin{array}{|c|c|c|c|} \hline & & 2 & \\ \hline & & 3 & \\ \hline & & 2 & \\ \hline & 1 & 0 & 1 \\ \hline 2 & 2 & 2 & \\ \hline 3 & 3 & 3 & \\ \hline 2 & 2 & 2 & 2 \\ \hline 1 & 0 & 1 & 0 & 1 \\ \hline \end{array}.$$

Case 2. Suppose that the i -blocks are of type II.

In this case, we have $q_i = q$. Let b be a removable i -block in y_k of Y . If the i -signature of y_k is $--$, or if the i -signature of y_k is $-$ and there is another i -block beneath b , we define $Y \nearrow b$ to be the Young wall obtained by removing the block b from Y . If the i -signature of y_k is $-+$, or if the i -signature of y_k is $-$ and there is no i -block beneath b , then we define $Y \nearrow b = q^{-1}(1 - (-q^2)^{l(b)+1})Z$,

where Z is the Young wall obtained by removing the block b from Y and $l(b)$ is the number of y_l 's with $l < k$ such that $|y_l| = |y_k|$. That is, if



then $Y \nearrow b = \frac{(1 - (-q^2)^{l(b)+1})}{q} \times$ 

In either case, we define $Y_R(b) = (y_l, \dots, y_0)$, where l is the integer such that $|y_k| = |y_{k-1}| = \dots = |y_{l+1}| < |y_l|$, and set

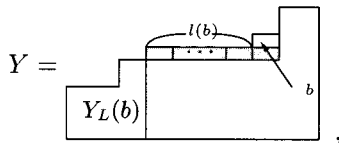
$$(5.6) \quad R_i(b; Y) = \varphi_i(Y_R(b)) - \varepsilon_i(Y_R(b)).$$

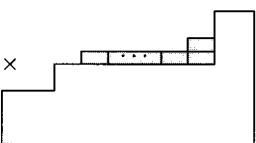
If $k = 0$, we understand $Y_R(b) = \emptyset$ and $R_i(b; Y) = 0$. Then we define

$$(5.7) \quad e_i Y = \sum_b q_i^{-R_i(b; Y)} (Y \nearrow b),$$

where b runs over all removable i -blocks in Y .

On the other hand, suppose that b is an admissible i -slot in y_k of Y . If the i -signature of y_k is $++$, or if the i -signature of y_k is $+$ and there is no i -block beneath b , then we define $Y \swarrow b$ to be the Young wall obtained by adding an i -block at b . If the i -signature of y_k is $-+$, or if the i -signature of y_k is $+$ and there is another i -block beneath b , then we define $Y \swarrow b = q^{-1}(1 - (-q^2)^{l(b)+1})Z$, where Z is the Young wall obtained by adding an i -block at b and $l(b)$ is the number of y_l 's with $l > k$ such that $|y_l| = |y_k|$. That is, if



then $Y \swarrow b = \frac{(1 - (-q^2)^{l(b)+1})}{q} \times$ 

In either case, we define $Y_L(b) = (\dots, y_{l+2}, y_{l+1})$, where l is the integer such that $|y_{l+1}| < |y_l| = |y_{l-1}| = \dots = |y_k|$, and set

$$(5.8) \quad L_i(b; Y) = \varphi_i(Y_L(b)) - \varepsilon_i(Y_L(b)).$$

Then we define

$$(5.9) \quad f_i Y = \sum_b q_i^{L_i(b; Y)} (Y \swarrow b),$$

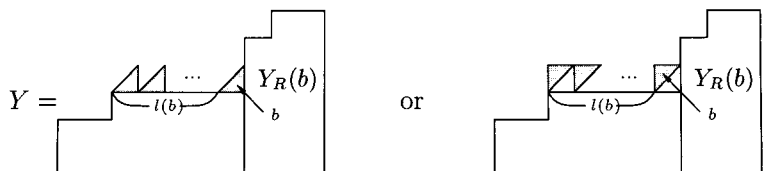
where b runs over all admissible i -slots in Y .

Example 5.3. If $\mathfrak{g} = A_4^{(2)}$, $\Lambda = \Lambda_0$ and $i = 0$, then we have

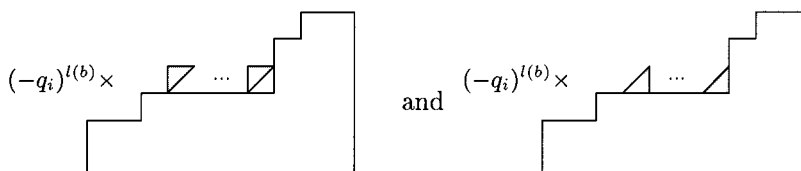
$$\begin{aligned}
 e_0 & \begin{array}{|c|c|c|c|c|c|} \hline & & & & & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} = q^2 \begin{array}{|c|c|c|c|c|c|} \hline & & & & & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \\
 & + q(1 + q^{10}) \begin{array}{|c|c|c|c|c|c|} \hline & & & & & 0 \\ \hline & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|c|} \hline & & & & & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}, \\
 f_0 & \begin{array}{|c|c|c|c|c|c|} \hline & & & & & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \\
 & = q^{-1} \begin{array}{|c|c|c|c|c|c|} \hline & & & & & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} + (1 + q^{10}) \begin{array}{|c|c|c|c|c|c|} \hline & & & & & 0 \\ \hline & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}.
 \end{aligned}$$

Case 3. Suppose that the i -blocks are of type III.

If b is a removable i -block in y_k of Y , then we define $Y \nearrow b$ to be the Young wall obtained by removing the block b from Y . We also consider the following i -block b in y_k of Y , which we call a *virtually removable i -block*.



In this case, we define $Y \nearrow b$ to be



respectively, where $l(b) \geq 1$ is given in the above figure. Note that, unlike Case 1 and Case 2, we need to *shift* the blocks from left to right and from back to front (resp. from front to back). In either case, we define $Y_R(b) = (y_{k-1}, \dots, y_0)$, and set

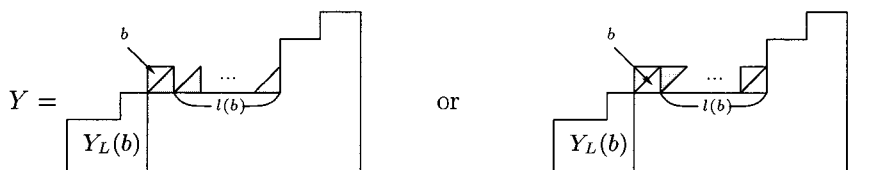
$$(5.10) \quad R_i(b; Y) = \varphi_i(Y_R(b)) - \varepsilon_i(Y_R(b)).$$

If $k = 0$, we understand $Y_R(b) = \emptyset$ and $R_i(b; Y) = 0$. Then we define

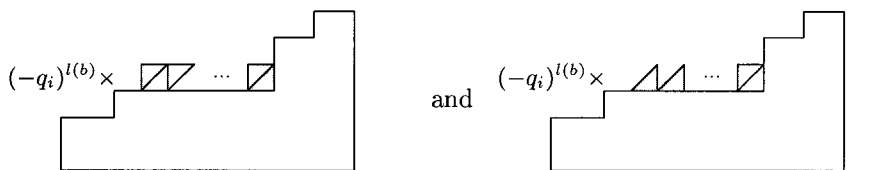
$$(5.11) \quad e_i Y = \sum_b q_i^{-R_i(b; Y)} (Y \nearrow b),$$

where b runs over all removable and virtually removable i -blocks in Y .

On the other hand, if b is an admissible i -slot in y_k of Y , then we define $Y \swarrow b$ to be the Young wall obtained by adding an i -block at b . We also consider the following i -slot b in y_k of Y , which we call a *virtually admissible i -slot*:



In this case, we define $Y \swarrow b$ to be



respectively, where $l(b) \geq 1$ is given in the above figure. Here again, one can observe that we need to *shift* the blocks from left to right and from back to front (resp. from front to back). In either case, we define $Y_L(b) = (\dots, y_{k+2}, y_{k+1})$ and set

$$(5.12) \quad L_i(b; Y) = \varphi_i(Y_L(b)) - \varepsilon_i(Y_L(b)).$$

Then we define

$$(5.13) \quad f_i Y = \sum_b q_i^{L_i(b; Y)} (Y \swarrow b),$$

where b runs over all admissible and virtually admissible i -slots in Y .

Example 5.4. If $\mathfrak{g} = B_3^{(1)}$, $\Lambda = \Lambda_0$ and $i = 0$, then $q_0 = q^2$ and we have

$$\begin{aligned}
 e_0 \begin{array}{|c|c|c|c|} \hline \diagup 0 \diagdown & \diagup 1 \diagdown & \diagup 0 \diagdown & \diagup 0 \diagdown \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline 3 & 3 & 3 & 3 \\ \hline 2 & 2 & 2 & 2 \\ \hline \diagup 1 \diagdown & \diagup 0 \diagdown & \diagup 1 \diagdown & \diagup 0 \diagdown \\ \hline \end{array} &= q^2 \begin{array}{|c|c|c|c|} \hline \diagup 1 \diagdown & \diagup 0 \diagdown & \diagup 0 \diagdown & \diagup 1 \diagdown \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline 3 & 3 & 3 & 3 \\ \hline 2 & 2 & 2 & 2 \\ \hline \diagup 1 \diagdown & \diagup 0 \diagdown & \diagup 1 \diagdown & \diagup 0 \diagdown \\ \hline \end{array} \\
 &+ q^6 \begin{array}{|c|c|c|c|} \hline \diagup 0 \diagdown & \diagup 1 \diagdown & \diagup 0 \diagdown & \diagup 1 \diagdown \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline 3 & 3 & 3 & 3 \\ \hline 2 & 2 & 2 & 2 \\ \hline \diagup 1 \diagdown & \diagup 0 \diagdown & \diagup 1 \diagdown & \diagup 0 \diagdown \\ \hline \end{array} + \begin{array}{|c|c|c|c|} \hline \diagup 0 \diagdown & \diagup 1 \diagdown & \diagup 0 \diagdown & \diagup 1 \diagdown \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline 3 & 3 & 3 & 3 \\ \hline 2 & 2 & 2 & 2 \\ \hline \diagup 1 \diagdown & \diagup 0 \diagdown & \diagup 1 \diagdown & \diagup 0 \diagdown \\ \hline \end{array}, \\
 \\
 f_0 \begin{array}{|c|c|c|c|} \hline \diagup 1 \diagdown & \diagup 0 \diagdown & \diagup 1 \diagdown & \diagup 1 \diagdown \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline 3 & 3 & 3 & 3 \\ \hline 2 & 2 & 2 & 2 \\ \hline \diagup 1 \diagdown & \diagup 0 \diagdown & \diagup 1 \diagdown & \diagup 0 \diagdown \\ \hline \end{array} &= \begin{array}{|c|c|c|c|} \hline \diagup 1 \diagdown & \diagup 0 \diagdown & \diagup 1 \diagdown & \diagup 1 \diagdown \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline 3 & 3 & 3 & 3 \\ \hline 2 & 2 & 2 & 2 \\ \hline \diagup 0 \diagdown & \diagup 1 \diagdown & \diagup 0 \diagdown & \diagup 1 \diagdown \\ \hline \end{array} + q^2 \begin{array}{|c|c|c|c|} \hline \diagup 0 \diagdown & \diagup 1 \diagdown & \diagup 0 \diagdown & \diagup 1 \diagdown \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline 3 & 3 & 3 & 3 \\ \hline 2 & 2 & 2 & 2 \\ \hline \diagup 1 \diagdown & \diagup 0 \diagdown & \diagup 1 \diagdown & \diagup 0 \diagdown \\ \hline \end{array} \\
 &+ q^8 \begin{array}{|c|c|c|c|} \hline \diagup 0 \diagdown & \diagup 1 \diagdown & \diagup 0 \diagdown & \diagup 1 \diagdown \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline 3 & 3 & 3 & 3 \\ \hline 2 & 2 & 2 & 2 \\ \hline \diagup 1 \diagdown & \diagup 0 \diagdown & \diagup 1 \diagdown & \diagup 0 \diagdown \\ \hline \end{array} + q^4 \begin{array}{|c|c|c|c|} \hline \diagup 1 \diagdown & \diagup 0 \diagdown & \diagup 0 \diagdown & \diagup 1 \diagdown \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline 3 & 3 & 3 & 3 \\ \hline 2 & 2 & 2 & 2 \\ \hline \diagup 1 \diagdown & \diagup 0 \diagdown & \diagup 1 \diagdown & \diagup 0 \diagdown \\ \hline \end{array}.
 \end{aligned}$$

With these actions, we have

Theorem 5.5. *The Fock space $\mathcal{F}(\Lambda)$ is a $U_q(\mathfrak{g})$ -module in the category \mathcal{O}_{int} .*

To prove this theorem, we need to verify that all the defining relations in (1.4) hold in $\mathcal{F}(\Lambda)$. First, it is straightforward to verify that the following

relations hold

$$\begin{aligned}
 (5.14) \quad & q^h q^{h'} Y = q^{h+h'} Y, \\
 & q^h e_i q^{-h} Y = q^{\alpha_i(h)} e_i Y, \\
 & q^h f_i q^{-h} Y = q^{-\alpha_i(h)} f_i Y
 \end{aligned}$$

for $Y \in \mathcal{Z}(\Lambda)$, $i \in I$ and $h, h' \in P^\vee$. Also, it is clear that e_i and f_i ($i \in I$) act locally nilpotently on $\mathcal{F}(\Lambda)$. Therefore, by Proposition B.1 in [13], we have only to show that

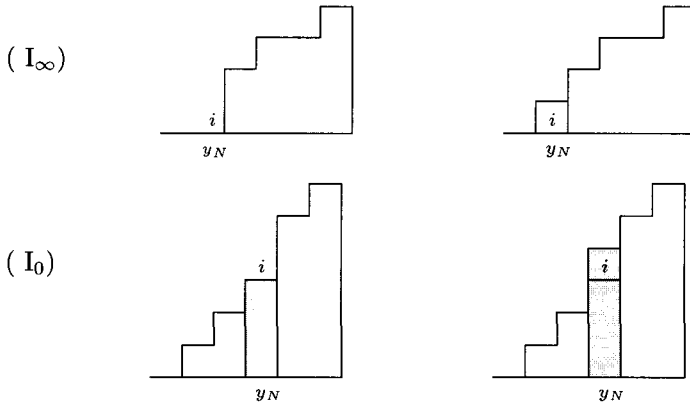
$$(5.15) \quad (e_i f_j - f_j e_i) Y = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} Y$$

for $Y \in \mathcal{Z}(\Lambda)$ and $i, j \in I$.

The rest of this subsection will be devoted to proving the relation (5.15). We first investigate the *local behavior* of $U_q(\mathfrak{g})$ -action on $\mathcal{F}(\Lambda)$.

Fix $i \in I$ and let Y be a proper Young wall. We will decompose Y into a sequence $Y = (Y_0, Y_1, \dots, Y_r)$ of parts, reading from left to right, which are called the *i*-component of Y .

Case 1. Suppose that *i*-blocks are of type I. Observe that the admissible *i*-slots and removable *i*-blocks in Y appear in one of the following situations:



In case I_∞ , let y_N be the column containing the admissible *i*-slot in the ground-state wall or the removable *i*-block that touches the ground-state wall as indicated in the above figure. We denote by Y_0 the part of Y consisting of y_N and the blocks in the ground-state wall lying in the left of y_N , and call it an *i*-component of type I_∞ .

In case I_0 , the column which is *i*-admissible or *i*-removable is called an *i*-component of type I_0 . If y_N is the left-most *i*-component of type I_0 and there is no *i*-component of type I_∞ , we denote by Y_0 the part of Y consisting of the blocks lying in the left of y_N . In this case, we call Y_0 a *trivial i*-component.

The parts of Y lying between two i -components of type I_∞ or I_0 will also be called the *trivial i -components*.

In this way, we obtain a unique decomposition $Y = (Y_0, Y_1, \dots, Y_r)$ of Y , where each of Y_k is an i -component of type I_∞ or I_0 , or a trivial i -component.

Example 5.6. Let $\mathfrak{g} = A_5^{(2)}$, $\Lambda = \Lambda_0$ and $i = 2$. If

$$Y = \begin{array}{cccc} & & & 3 \\ & & 2 & 2 \\ & 0 & 1 & 0 & 1 \\ 2 & 2 & 2 & 2 \\ & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 \\ 1 & 0 & 1 & 0 & 1 \end{array},$$

then we have $Y = (Y_0, Y_1, Y_2, Y_3)$, where

$$Y_0 = \dots \begin{array}{cccc} & & & 1 \\ & & 1 & 0 & 1 \\ & 0 & 1 & 0 & 1 \end{array} : \text{trivial } i\text{-component},$$

$$Y_1 = \begin{array}{c} 2 \\ 1 \\ 0 \end{array} : i\text{-component of type } I_0,$$

$$Y_2 = \begin{array}{cc} & 2 \\ 0 & 1 & 0 & 1 \\ 2 & 2 \\ 3 & 3 \\ 2 & 2 \\ 0 & 1 & 0 & 1 \end{array} : \text{trivial } i\text{-component and}$$

$$Y_3 = \begin{array}{c} 3 \\ 2 \\ 0 & 1 \\ 2 \\ 3 \\ 2 \\ 0 & 1 \end{array} : i\text{-component of type } I_0.$$

Let Y be a proper Young wall with the decomposition $Y = (Y_0, Y_1, \dots, Y_r)$ into i -components. To each Y_k , we associate a $U_{(i)}$ -module V_k as follows. Then we will view Y as $Y_0 \otimes Y_1 \otimes \dots \otimes Y_r$ inside $V_0 \otimes V_1 \otimes \dots \otimes V_r$.

If Y_k is a trivial i -component, then we associate the trivial representation $V_k = U = \mathbb{Q}(q)u$, and we identify Y_k with u . If Y_k is an i -component of

type I_∞ or I_0 , then we associate the 2-dimensional representation $V_k = V = \mathbb{Q}(q)v_0 \oplus \mathbb{Q}(q)v_1$, where the $U_{(i)}$ -action is given by

$$(5.16) \quad \begin{aligned} K_i v_0 &= q_i v_0, & K_i v_1 &= q_i^{-1} v_1, \\ e_i v_0 &= 0, & e_i v_1 &= v_0, \\ f_i v_0 &= v_1, & f_i v_1 &= 0. \end{aligned}$$

We identify the i -component Y_k with a basis element of V as follows :

$$(I_\infty) \quad v_0 \leftrightarrow \begin{array}{|c|} \hline \\ \hline \text{---} i \\ \hline \end{array} \quad v_1 \leftrightarrow \begin{array}{|c|} \hline \\ \hline i \\ \hline \end{array}$$

$$(I_0) \quad v_0 \leftrightarrow \begin{array}{|c|} \hline i \\ \hline \\ \hline \end{array} \quad v_1 \leftrightarrow \begin{array}{|c|} \hline i \\ \hline \\ \hline \end{array}.$$

Note that V is isomorphic to the 2-dimensional irreducible $U_{(i)}$ -module $V(1)$ with the crystal basis (L, B) , where

$$L = \mathbb{A}_0 v_0 \oplus \mathbb{A}_0 v_1, \quad B = \{\overline{v_0}, \overline{v_1}\},$$

and the crystal graph is given by

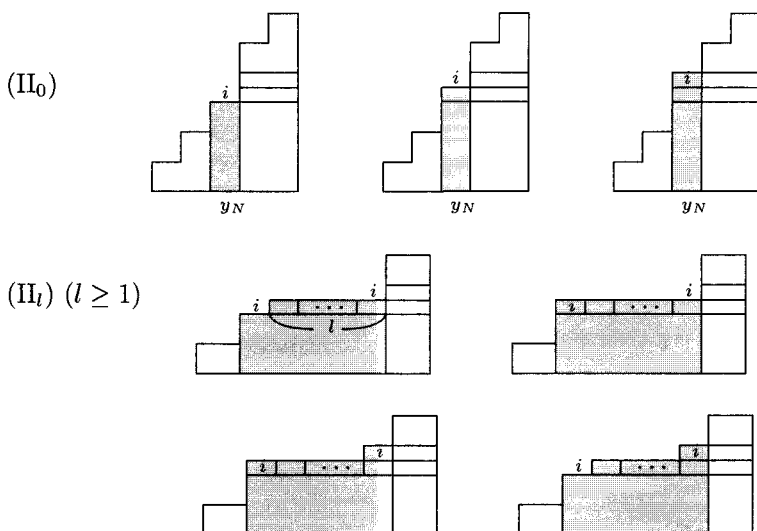
$$\overline{v_0} \xrightarrow{i} \overline{v_1}.$$

Example 5.7. In Example 5.6, we have

$$Y = \dots \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 0 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline 0 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 0 & 1 \\ \hline 2 & 2 \\ \hline 3 & 3 \\ \hline 2 & 2 \\ \hline 0 & 1 \\ \hline 1 & 0 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline 0 \\ \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 2 \\ \hline 0 \\ \hline 1 \\ \hline \end{array} \in U \otimes V \otimes U \otimes V.$$

Case 2. Suppose that the i -blocks are of type II. In this case, the admissible i -slots and the removable i -blocks in Y appear in one of the following situations:

$$(II_\infty) \quad \begin{array}{c} \begin{array}{|c|} \hline \\ \hline \end{array} \\ \begin{array}{|c|} \hline \\ \hline \end{array} \\ \begin{array}{|c|} \hline \\ \hline \end{array} \\ \begin{array}{|c|} \hline \\ \hline \end{array} \\ \begin{array}{|c|} \hline \\ \hline \end{array} \\ \begin{array}{|c|} \hline i \\ \hline \end{array} \\ \text{---} y_N \end{array} \quad \begin{array}{c} \begin{array}{|c|} \hline \\ \hline \end{array} \\ \begin{array}{|c|} \hline \\ \hline \end{array} \\ \begin{array}{|c|} \hline \\ \hline \end{array} \\ \begin{array}{|c|} \hline \\ \hline \end{array} \\ \begin{array}{|c|} \hline \\ \hline \end{array} \\ \begin{array}{|c|} \hline i \\ \hline \end{array} \\ \text{---} y_N \end{array}$$



In case Π_∞ , let y_N be the column containing the admissible i -slot in the ground-state wall or the removable i -block that touches the ground-state wall as indicated in the above figure. We denote by Y_0 the part of Y consisting of y_N and the blocks in the ground-state wall lying in the left of y_N , and call it an i -component of type Π_∞ .

In case Π_0 , the column which is i -admissible or i -removable is called an i -component of type Π_0 . In case Π_l , the whole shaded part containing an admissible i -slot or a removable i -block will be called an i -component of type Π_l .

Let Y_1 be the left-most i -component of type Π_l ($l \geq 0$). If there is no i -component of type Π_∞ , then we denote by Y_0 the part of Y consisting of the blocks lying in the left of Y_1 , and call it a *trivial i -component*. The parts of Y lying between two i -components of type Π_∞ or Π_l ($l \geq 0$) will also be called the *trivial i -components*.

In this way, we obtain a unique decomposition $Y = (Y_0, Y_1, \dots, Y_r)$ of Y , where each Y_k is an i -component of type Π_∞ , Π_l ($l \geq 0$), or a trivial i -component.

Example 5.8. Let $\mathfrak{g} = A_4^{(2)}$, $\Lambda = \Lambda_0$ and $i = 0$. If

$$Y = \begin{array}{cccccccc} & & & & & & & 0 \\ & & & & & & & 1 \\ & & & & & & 2 & 2 \\ & & & & 1 & 1 & 1 & \\ & & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array},$$

then we have $Y = (Y_0, Y_1, Y_2, Y_3)$, where

$$Y_0 = \dots \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & & & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} : i\text{-component of type } \Pi_\infty,$$

$$Y_1 = \begin{array}{|c|c|c|c|} \hline & & & 0 \\ & & 0 & 0 \\ & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \end{array} : i\text{-component of type } \Pi_3,$$

$$Y_2 = \begin{array}{|c|c|} \hline & 2 \\ \hline 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ \hline \end{array} : \text{trivial } i\text{-component, and}$$

$$Y_3 = \begin{array}{|c|} \hline 0 \\ 1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 1 \\ 2 \\ 1 \\ 0 \\ 0 \\ \hline \end{array} : i\text{-component of type } \Pi_0.$$

Let Y be a proper Young wall with the decomposition $Y = (Y_0, Y_1, \dots, Y_r)$ into i -components. To each Y_k , we associate a $U_{(i)}$ -module V_k as follows. Then we will view Y as $Y_0 \otimes Y_1 \otimes \dots \otimes Y_r$ inside $V_0 \otimes V_1 \otimes \dots \otimes V_r$.

If Y_k is a trivial i -component, then we associate the trivial representation $V_k = U = \mathbb{Q}(q)u$, and we identify Y_k with u .

If Y_k is an i -component of type Π_∞ , then we must have $k = 0$ and we associate the 2-dimensional representation $V_0 = V = \mathbb{Q}(q)v_0 \oplus \mathbb{Q}(q)v_1$, where $U_{(i)}$ -module action is given by (5.16). We identify Y_0 with the basis element of V as follows :

$$v_0 \leftrightarrow \dots \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & i \\ \hline i & i & i & i & i & i & i \\ \hline \end{array} \quad v_1 \leftrightarrow \dots \begin{array}{|c|c|c|c|c|c|c|} \hline & & & & & & i \\ \hline i & i & i & i & i & i & i \\ \hline \end{array} .$$

If Y_k is an i -component of type II_0 , then we associate the 3-dimensional representation $V_k = W_0 = \mathbb{Q}(q)w_0 \oplus \mathbb{Q}(q)w_1 \oplus \mathbb{Q}(q)w_2$, where the $U_{(i)}$ -module action is given by

$$(5.17) \quad \begin{aligned} K_i w_0 &= q_i^2 w_0, \quad K_i w_1 = w_1, \quad K_i w_2 = q_i^{-2} w_2, \\ e_i w_0 &= 0, \quad e_i w_1 = (q_i + q_i^{-1})w_0, \quad e_i w_2 = w_1, \\ f_i w_0 &= w_1, \quad f_i w_1 = (q_i + q_i^{-1})w_2, \quad f_i w_2 = 0. \end{aligned}$$

We identify the i -component Y_k with a basis element of W as follows :

$$w_0 \leftrightarrow \begin{array}{|c|} \hline \\ \hline \end{array} \quad w_1 \leftrightarrow \begin{array}{|c|} \hline i \\ \hline \end{array} \quad w_2 \leftrightarrow \begin{array}{|c|} \hline i \\ \hline i \\ \hline \end{array}.$$

Note that W is isomorphic to the 3-dimensional irreducible $U_{(i)}$ -module $V(2)$ with the crystal basis (L, B) , where

$$L = \mathbb{A}_0 w_0 \oplus \mathbb{A}_0 w_1 \oplus \mathbb{A}_0 w_2, \quad B = \{ \overline{w_k} \mid k = 0, 1, 2 \}.$$

and the crystal graph is given by

$$\overline{w_0} \xrightarrow{i} \overline{w_1} \xrightarrow{i} \overline{w_2}.$$

If Y_k is an i -component of type II_l ($l \geq 1$), then we associate the 4-dimensional representation

$$V_k = W_l = \mathbb{Q}(q)w_0 \oplus \mathbb{Q}(q)w_1 \oplus \mathbb{Q}(q)w_2 \oplus \mathbb{Q}(q)u,$$

where the $U_{(i)}$ -module action is given by

$$(5.18) \quad \begin{aligned} K_i w_0 &= q_i^2 w_0, \quad K_i w_1 = w_1, \quad K_i w_2 = q_i^{-2} w_2, \quad K_i u = u, \\ e_i w_0 &= 0, \quad e_i w_1 = q_i^{-1}(1 - (-q_i^2)^{l+1})w_0, \\ e_i w_2 &= w_1 + q_i(1 - (-q_i^2)^l)u, \quad e_i u = w_0, \\ f_i w_0 &= w_1 + q_i(1 - (-q_i^2)^l)u, \quad f_i w_1 = q_i^{-1}(1 - (-q_i^2)^{l+1})w_2, \\ f_i w_2 &= 0, \quad f_i u = w_2. \end{aligned}$$

We identify the i -component Y_k with a basis element of W_l as follows :

$$\begin{aligned} w_0 &\leftrightarrow \begin{array}{|c|} \hline \overbrace{\quad \quad \quad}^l \\ \hline \end{array} & w_1 &\leftrightarrow \begin{array}{|c|} \hline \overline{\quad \quad \quad} \\ \hline \end{array} \\ w_2 &\leftrightarrow \begin{array}{|c|} \hline \overline{\quad \quad \quad} \\ \hline \end{array} & u &\leftrightarrow \begin{array}{|c|} \hline \overline{\quad \quad \quad} \\ \hline \end{array}. \end{aligned}$$

The $U_{(i)}$ -module W_l is decomposed as $W_l \cong V(2) \oplus V(0)$, where

$$\begin{aligned} V(2) &\cong \mathbb{Q}(q)w_0 \oplus \mathbb{Q}(q)(w_1 + q_i(1 - (-q_i^2)^l)u) \oplus \mathbb{Q}(q)w_2, \\ V(0) &\cong \mathbb{Q}(q)(u - \frac{q_i}{(1 - (-q_i^2)^{l+1})}w_1). \end{aligned}$$

Hence, the crystal basis (L, B) of W_l is given by

$$\begin{aligned} L &= \mathbb{A}_0 w_0 \oplus \mathbb{A}_0 (w_1 + q_i(1 - (-q_i^2)^l)u) \oplus \mathbb{A}_0 w_2 \\ &\quad \oplus \mathbb{A}_0 (u - \frac{q_i}{(1 - (-q_i^2)^{l+1})}w_1), \\ B &= \{ \bar{w}_0, \bar{w}_1, \bar{w}_2, \bar{u} \} \end{aligned}$$

with the crystal graph

$$\begin{array}{ccc} \bar{w}_0 & \xrightarrow{i} & \bar{w}_1 \\ & \downarrow i & \\ \bar{u} & & \bar{w}_2. \end{array}$$

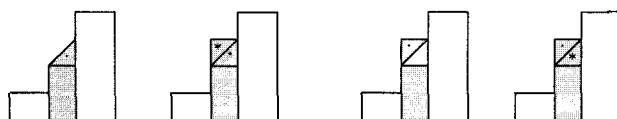
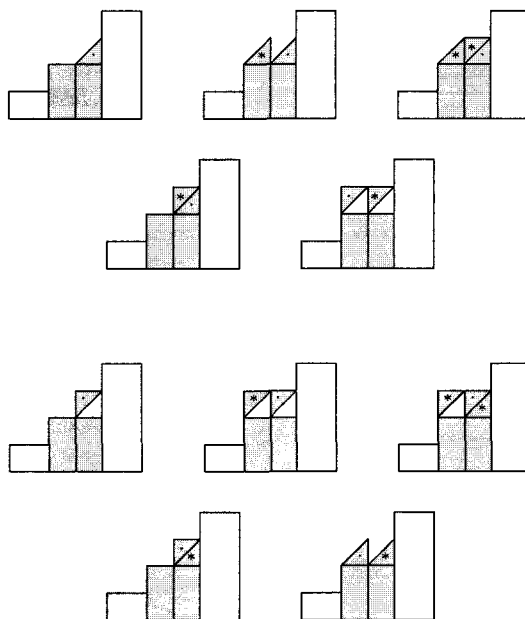
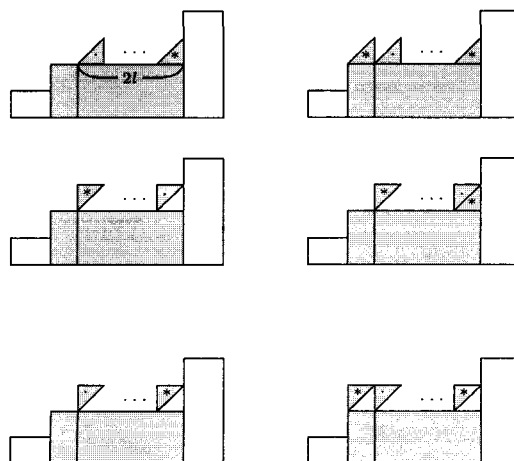
Example 5.9. In Example 5.8, we have

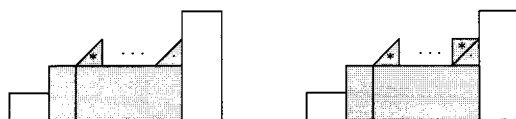
$$\begin{aligned} Y = & \dots \begin{array}{|c|c|c|c|c|c|c|c|} \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline & & 0 & 0 \\ \hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 0 & 0 \\ \hline 0 & 0 \\ \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 1 & 1 \\ \hline 0 & 0 \\ \hline 0 & 0 \\ \hline 0 & 0 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline 2 \\ \hline 1 \\ \hline 0 \\ \hline 0 \\ \hline 1 \\ \hline 2 \\ \hline 1 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \\ & \in V \otimes W_3 \otimes U \otimes W_0. \end{aligned}$$

Case 3. Suppose that the i -blocks are of type III. For convenience, we denote by $*$ the color i , and denote by \cdot the color of the blocks placed on the opposite side of the i -blocks. In this case, the (virtually) admissible i -slots and the (virtually) removable i -blocks in Y appear in one of the following situations:

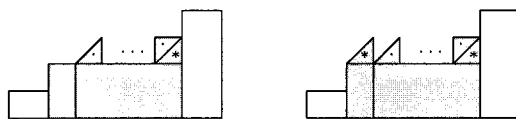
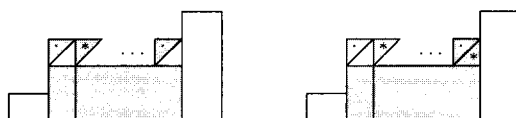
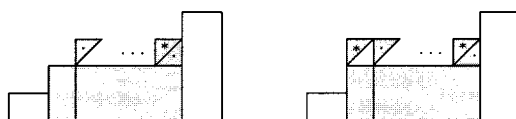
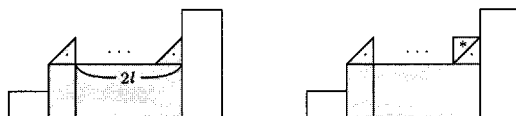
$$(III_\infty) \quad \begin{array}{c} \dots \quad \nearrow \quad \nearrow \quad \nearrow \quad \nearrow \quad \nearrow \quad \nearrow \quad \nearrow \end{array} \quad \begin{array}{c} \dots \quad \nearrow \quad \nearrow \quad \nearrow \quad \nearrow \quad \nearrow \quad \nearrow \quad \nearrow \end{array}$$

$$(III_0) \quad \begin{array}{c} \begin{array}{|c|c|c|} \hline & * & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline & \cdot & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline & * & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline & \cdot & \\ \hline \end{array} \end{array}$$

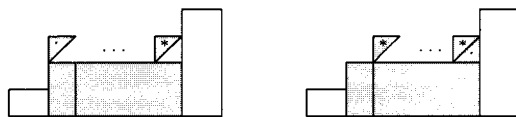
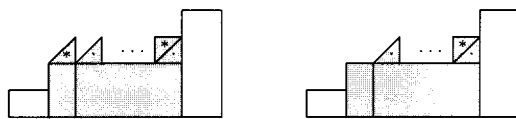
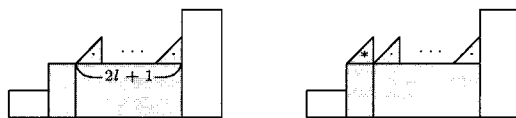
(III₁)(III_{2l}⁺) ($l \geq 1$)

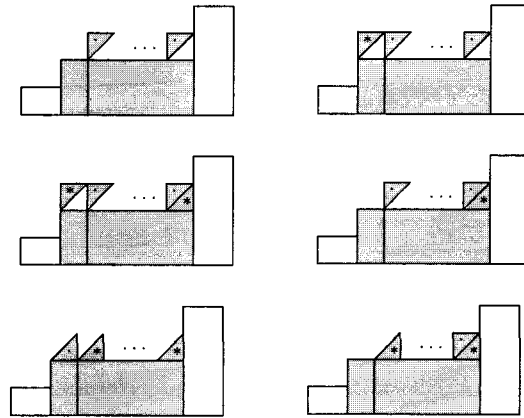


$(III_{2l}^-) \ (l \geq 1)$



$(III_{2l+1}) \ (l \geq 1)$





In case III_∞ , let y_N be the column containing the admissible i -slot in the ground-state wall or the removable i -block that touches the ground-state wall as indicated in the above figure. We denote by Y_0 the part of Y consisting of y_N and the blocks in the ground-state wall lying in the left of y_N , and call it an i -component of type III_∞ .

In case $\text{III}_0, \text{III}_1, \text{III}_{2l}^\pm$ and III_{2l+1} ($l \geq 1$), the whole shaded part containing (virtually) admissible i -slots and (virtually) removable i -blocks will be called an i -component of type $\text{III}_0, \text{III}_1, \text{III}_{2l}^\pm$ and III_{2l+1} ($l \geq 1$), respectively.

Let Y_1 be the left-most i -component of type $\text{III}_0, \text{III}_1, \text{III}_{2l}^\pm$ and III_{2l+1} . If there is no i -component of type III_∞ , we denote by Y_0 the part of Y consisting of blocks lying in the left of Y_1 , and call it a *trivial i -component*. The parts of Y lying between two i -components of type $\text{III}_\infty, \text{III}_0, \text{III}_1, \text{III}_{2l}^\pm$ and III_{2l+1} will also be called the *trivial i -components*.

In this way, we obtain a unique decomposition $Y = (Y_0, Y_1, \dots, Y_r)$ of Y , where each Y_k is an i -component of type $\text{III}_\infty, \text{III}_0, \text{III}_1, \text{III}_{2l}^\pm, \text{III}_{2l+1}$ or a trivial i -component.

Example 5.10. Let $\mathfrak{g} = B_3^{(1)}$, $\Lambda = \Lambda_0$ and $i = 0$. If

$$Y = \begin{array}{c} \begin{array}{ccccccc} & & & & & & 1 \\ & & & & & 0 & 1 \\ & & & & 2 & 2 & 2 \\ & & & 3 & 3 & 3 \\ & & 3 & 3 & 3 \\ & 2 & 2 & 2 & 2 \\ & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \\ \begin{array}{cccccccc} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array} \end{array},$$

then we have $Y = (Y_0, Y_1, Y_2, Y_3)$, where

$$Y_0 = \dots \begin{array}{|c|c|c|c|} \hline \diagup & \diagup & \diagup & \diagup \\ \hline 0 & 1 & 0 & 1 \\ \hline \end{array} : i\text{-component of type III}_\infty,$$

$$Y_1 = \begin{array}{|c|c|c|c|} \hline & 0 & 1 & 0 \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline 2 & 2 & 2 & 2 \\ \hline 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ \hline \end{array} : i\text{-component of type III}_3,$$

$$Y_2 = \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline 0 \\ \hline 2 \\ \hline 3 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline 0 \\ \hline \end{array} : \text{trivial } i\text{-component, and}$$

$$Y_3 = \begin{array}{|c|c|c|} \hline 1 & 0 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \hline 3 & 3 & 3 \\ \hline 2 & 2 & 2 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \hline 3 & 3 & 3 \\ \hline 2 & 2 & 2 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} : i\text{-component of type III}_2^-.$$

Let Y be a proper Young wall with the decomposition $Y = (Y_0, Y_1, \dots, Y_r)$ into i -components. To each Y_k , we associate a $U_{(i)}$ -module V_k as follows. Then we will view Y as $Y_0 \otimes Y_1 \otimes \dots \otimes Y_r$ inside $V_0 \otimes V_1 \otimes \dots \otimes V_r$.

If Y_k is a trivial i -component, then we associate the trivial representation $V_k = U = \mathbb{Q}(q)u$, and we identify Y_k with u . If Y_k is an i -component of type III_∞ or III_0 , then we associate the 2-dimensional representation $V_k = V = \mathbb{Q}(q)v_0 \oplus \mathbb{Q}(q)v_1$, where the $U_{(i)}$ -module action is given by (5.16). We identify Y_k with a basis element of V as follows:

$$\begin{array}{ccc} v_0 \leftrightarrow \dots \begin{array}{|c|c|c|c|} \hline \diagup & \diagup & \diagup & \diagup \\ \hline \end{array} & v_1 \leftrightarrow \dots \begin{array}{|c|c|c|c|} \hline \diagup & \diagup & \diagup & \diagup \\ \hline \end{array} & \\ \\ v_0 \leftrightarrow \begin{array}{|c|} \hline * \\ \hline \end{array} & v_1 \leftrightarrow \begin{array}{|c|} \hline * \\ \hline \end{array} & v_0 \leftrightarrow \begin{array}{|c|} \hline * \\ \hline \end{array} & v_1 \leftrightarrow \begin{array}{|c|} \hline * \\ \hline \end{array} \end{array}$$

$$v_0 \leftrightarrow \begin{array}{|c|} \hline \diagup \\ \hline \end{array} \quad v_1 \leftrightarrow \begin{array}{|c|} \hline \diagup * \\ \hline \end{array} \quad v_0 \leftrightarrow \begin{array}{|c|} \hline \diagup \\ \hline \end{array} \quad v_1 \leftrightarrow \begin{array}{|c|} \hline \diagup * \\ \hline \end{array}$$

If Y_k is an i -component of type III_1 , then we associate the 5-dimensional representation

$$V_k = W_1 = \mathbb{Q}(q)w_0 \oplus \mathbb{Q}(q)w_1 \oplus \mathbb{Q}(q)w_2 \oplus \mathbb{Q}(q)u \oplus \mathbb{Q}(q)u',$$

where the $U_{(i)}$ -module action is given by

$$(5.19) \quad \begin{aligned} K_i w_0 &= q_i^2 w_0, \quad e_i w_0 = 0, \quad f_i w_0 = w_1 + q_i u, \\ K_i w_1 &= w_1, \quad e_i w_1 = q_i^{-1} w_0, \quad f_i w_1 = q_i^{-1} w_2, \\ K_i w_2 &= q_i^{-2} w_2, \quad e_i w_2 = w_1 + q_i u, \quad f_i w_2 = 0, \\ K_i u &= u, \quad e_i u = w_0, \quad f_i u = w_2, \\ K_i u' &= u', \quad e_i u' = -q_i w_0, \quad f_i u' = -q_i w_2. \end{aligned}$$

We identify the i -component Y_k with a basis element of W_1 as follows:

$$\begin{aligned} w_0 &\leftrightarrow \begin{array}{|c|} \hline \diagup \\ \hline \end{array} & w_1 &\leftrightarrow \begin{array}{|c|} \hline \diagup * \\ \hline \end{array} & w_2 &\leftrightarrow \begin{array}{|c|} \hline \diagup * \\ \hline \end{array} \\ u &\leftrightarrow \begin{array}{|c|} \hline \diagup * \\ \hline \end{array} & u' &\leftrightarrow \begin{array}{|c|} \hline \diagup * \\ \hline \end{array} \\ w_0 &\leftrightarrow \begin{array}{|c|} \hline \diagup \\ \hline \end{array} & w_1 &\leftrightarrow \begin{array}{|c|} \hline \diagup * \\ \hline \end{array} & w_2 &\leftrightarrow \begin{array}{|c|} \hline \diagup * \\ \hline \end{array} \\ u &\leftrightarrow \begin{array}{|c|} \hline \diagup * \\ \hline \end{array} & u' &\leftrightarrow \begin{array}{|c|} \hline \diagup * \\ \hline \end{array} \end{aligned}$$

The $U_{(i)}$ -module W_1 is decomposed as $W_1 \cong V(2) \oplus V(0) \oplus V(0)$, where

$$(5.20) \quad \begin{aligned} V(2) &\cong \mathbb{Q}(q)w_0 \oplus \mathbb{Q}(q)(w_1 + q_i u) \oplus \mathbb{Q}(q)w_2, \\ V(0) &\cong \mathbb{Q}(q)(u - q_i w_1) \cong \mathbb{Q}(q)(u' + q_i^2 w_1). \end{aligned}$$

Hence, the crystal basis (L, B) of W_1 is given by

$$(5.21) \quad \begin{aligned} L &= \mathbb{A}_0 w_0 \oplus \mathbb{A}_0 (w_1 + q_i u) \oplus \mathbb{A}_0 w_2 \\ &\quad \oplus \mathbb{A}_0 (u - q_i w_1) \oplus \mathbb{A}_0 (u' + q_i^2 w_1), \\ B &= \{ \overline{w_0}, \overline{w_1}, \overline{w_2}, \overline{u}, \overline{u'} \} \end{aligned}$$

with the crystal graph

$$\overline{w_0} \xrightarrow{i} \overline{w_1} \xrightarrow{i} \overline{w_2} \quad \overline{u} \quad \overline{u'}.$$

If Y_k is an i -component of type III_{2l}^+ , then we associate the 4-dimensional representation

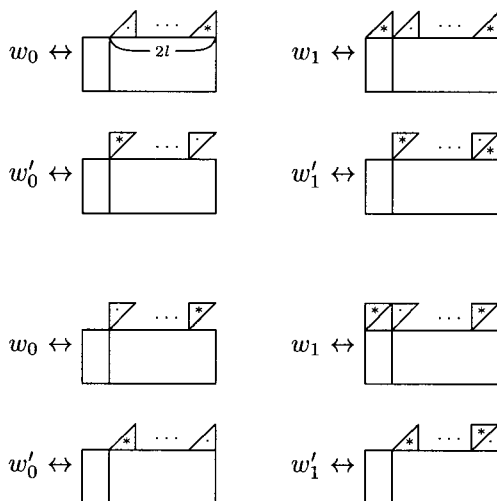
$$V_k = W_{2l}^\pm = \mathbb{Q}(q)w_0 \oplus \mathbb{Q}(q)w_1 \oplus \mathbb{Q}(q)w'_0 \oplus \mathbb{Q}(q)w'_1,$$

where the $U_{(i)}$ -module action is given by

$$(5.22) \quad \begin{aligned} K_i w_0 &= q_i w_0, \quad K_i w_1 = q_i^{-1} w_1, \\ K_i w'_0 &= q_i w'_0, \quad K_i w'_1 = q_i^{-1} w'_1, \\ e_i w_0 &= 0, \quad e_i w_1 = w_0 \pm q_i^{2l} w'_0, \\ e_i w'_0 &= 0, \quad e_i w'_1 = w'_0, \\ f_i w_0 &= w_1 \mp q_i^{2l} w'_1, \quad f_i w_1 = 0, \\ f_i w'_0 &= w'_1, \quad f_i w'_1 = 0. \end{aligned}$$

We identify the i -component Y_k with a basis element of W_{2l}^\pm as follows :

W_{2l}^+



W_{2l}^-



$$\begin{array}{cc}
 w'_0 \leftrightarrow \begin{array}{|c|c|} \hline \diagup \quad \dots \quad \diagup \\ \hline \end{array} & w'_1 \leftrightarrow \begin{array}{|c|c|} \hline \diagup \quad \diagup \quad \dots \quad \diagup \\ \hline \end{array} \\
 \\
 w_0 \leftrightarrow \begin{array}{|c|c|} \hline \diagup \quad \diagup \quad \dots \quad \diagup \\ \hline \end{array} & w_1 \leftrightarrow \begin{array}{|c|c|} \hline \diagup \quad \diagup \quad \dots \quad \diagup \\ \hline \end{array} \\
 \\
 w'_0 \leftrightarrow \begin{array}{|c|c|} \hline \diagup \quad \dots \quad \diagup \\ \hline \end{array} & w'_1 \leftrightarrow \begin{array}{|c|c|} \hline \diagup \quad \diagup \quad \dots \quad \diagup \\ \hline \end{array} .
 \end{array}$$

The $U_{(i)}$ -module W_{2l}^\pm is decomposed as $W_{2l}^\pm \cong V(1) \oplus V(1)$, where

$$\begin{aligned}
 (5.23) \quad V(2) &\cong \mathbb{Q}(q)w_0 \oplus \mathbb{Q}(q)(w_1 \mp q_i^{2l}w'_1), \\
 &\cong \mathbb{Q}(q)w'_0 \oplus \mathbb{Q}(q)w'_1.
 \end{aligned}$$

Hence, the crystal basis (L, B) of W_{2l}^\pm is given by

$$\begin{aligned}
 (5.24) \quad L &= \mathbb{A}_0 w_0 \oplus \mathbb{A}_0 (w_1 \mp q_i^{2l}w'_1) \oplus \mathbb{A}_0 w'_0 \oplus \mathbb{A}_0 w'_1, \\
 B &= \{ \overline{w_0}, \overline{w_1}, \overline{w'_0}, \overline{w'_1} \}
 \end{aligned}$$

with the crystal graph

$$\overline{w_0} \xrightarrow{i} \overline{w_1} \quad \overline{w'_0} \xrightarrow{i} \overline{w'_1}.$$

If Y_k is an i -component of type III_{2l+1} , then we associate the 6-dimensional representation

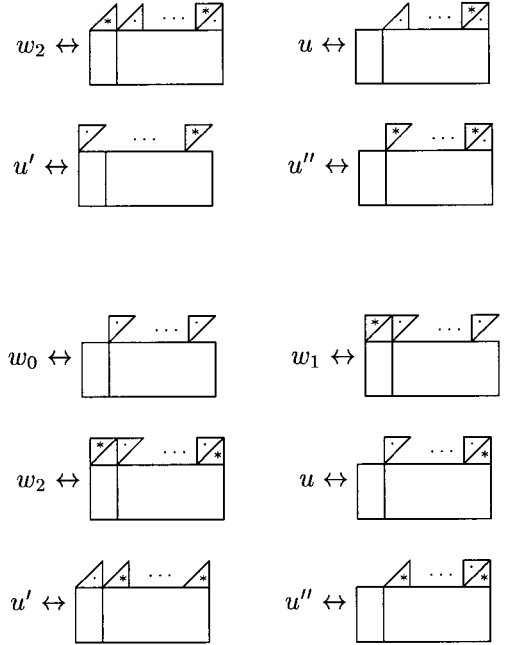
$$V_k = W_{2l+1} = \mathbb{Q}(q)w_0 \oplus \mathbb{Q}(q)w_1 \oplus \mathbb{Q}(q)w_2 \oplus \mathbb{Q}(q)u \oplus \mathbb{Q}(q)u' \oplus \mathbb{Q}(q)u'',$$

where the $U_{(i)}$ -module action is given by

$$\begin{aligned}
 (5.25) \quad K_i w_0 &= q_i^2 w_0, \quad e_i w_0 = 0, \quad f_i w_0 = w_1 + q_i u + q_i^{2l+1} u'', \\
 K_i w_1 &= w_1, \quad e_i w_1 = q_i^{-1} w_0, \quad f_i w_1 = q_i^{-1} w_2, \\
 K_i w_2 &= q_i^{-2} w_2, \quad e_i w_2 = w_1 + q_i u + q_i^{2l+1} u'', \quad f_i w_2 = 0, \\
 K_i u &= u, \quad e_i u = w_0, \quad f_i u = w_2, \\
 K_i u' &= u', \quad e_i u' = -q_i^{2l+1} w_0, \quad f_i u' = -q_i^{2l+1} w_2, \\
 K_i u'' &= u'', \quad e_i u'' = 0, \quad f_i u'' = 0.
 \end{aligned}$$

We identify the i -component Y_k with a basis element of W_{2l+1} as follows:

$$w_0 \leftrightarrow \begin{array}{|c|c|} \hline \diagup \quad \dots \quad \diagup \\ \hline \end{array} \quad w_1 \leftrightarrow \begin{array}{|c|c|} \hline \diagup \quad \diagup \quad \dots \quad \diagup \\ \hline \end{array}$$



The $U_{(i)}$ -module W_{2l+1} is decomposed as $W_{2l+1} \cong V(2) \oplus V(0)^{\oplus 3}$, where

$$(5.26) \quad \begin{aligned} V(2) &\cong \mathbb{Q}(q)w_0 \oplus \mathbb{Q}(q)(w_1 + q_i u + q_i^{2l+1} u'') \oplus \mathbb{Q}(q)w_2, \\ V(0) &\cong \mathbb{Q}(q)(u - q_i w_1) \cong \mathbb{Q}(q)(u' + q_i^{2l+2} w_1) \cong \mathbb{Q}(q)u''. \end{aligned}$$

Hence, the crystal basis (L, B) of W_{2l+1} is given by

$$(5.27) \quad \begin{aligned} L &= \mathbb{A}_0 w_0 \oplus \mathbb{A}_0 (w_1 + q_i u + q_i^{2l+1} u'') \oplus \mathbb{A}_0 w_2 \\ &\quad \oplus \mathbb{A}_0 (u - q_i w_1) \oplus \mathbb{A}_0 (u' + q_i^{2l+2} w_1) \oplus \mathbb{A}_0 u'', \\ B &= \{ \overline{w_0}, \overline{w_1}, \overline{w_2}, \overline{u}, \overline{u'}, \overline{u''} \} \end{aligned}$$

with the crystal graph

$$\overline{w_0} \xrightarrow{i} \overline{w_1} \xrightarrow{i} \overline{w_2} \quad \overline{u} \quad \overline{u'} \quad \overline{u''}.$$

Example 5.11. In Example 5.10, we have

$$Y = \dots \begin{array}{|c|c|} \hline 0 & 1 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline & 0 & 1 & 0 \\ \hline 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 \\ \hline 2 & 2 & 2 & 2 \\ \hline 1 & 0 & 1 & 0 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline 0 \\ \hline 2 \\ \hline 3 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline 0 \\ \hline \end{array} \otimes \begin{array}{|c|c|c|} \hline 1 & 0 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \hline 3 & 3 & 3 \\ \hline 2 & 2 & 2 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline 2 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \hline 3 & 3 & 3 \\ \hline 2 & 2 & 2 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} \in V \otimes W_3 \otimes U \otimes W_2^-.$$

Remark 5.12. Let Y be a proper Young wall in $\mathcal{Z}(\Lambda)$ with the decomposition $Y = (Y_0, Y_1, \dots, Y_r)$ into i -components. For each $0 \leq k \leq r$, let V_k be the $U_{(i)}$ -module associated with Y_k , whose crystal basis is (L_k, B_k) . We identify Y_k with a basis element of V_k . Then Y_k can also be viewed as a crystal element in B_k and hence, $\varphi_i(Y_k)$ and $\varepsilon_i(Y_k)$ are well-defined. On the other hand, as a part of Y , we defined $\varphi_i(Y_k)$ (resp. $\varepsilon_i(Y_k)$) to be the number of $+$'s (resp. $-$'s) in the i -signature of Y_k . It is easy to verify that these two definitions give the same values for the i -component Y_k .

Let Y be a proper Young wall in $\mathcal{Z}(\Lambda)$ with the decomposition $Y = (Y_0, Y_1, \dots, Y_r)$ into i -components, and let V_k be the $U_{(i)}$ -module associated with Y_k ($0 \leq k \leq r$). Recall that Y_k is identified with a basis element of V_k . Set

$$(5.28) \quad V_Y = V_0 \otimes V_1 \otimes \dots \otimes V_r.$$

We define a $\mathbb{Q}(q)$ -linear map $\theta_Y : V_Y \rightarrow \mathcal{F}(\Lambda)$ by

$$(5.29) \quad \theta_Y(Y'_0 \otimes Y'_1 \otimes \dots \otimes Y'_r) = Y' = (Y'_0, Y'_1, \dots, Y'_r),$$

where $Y'_0 \otimes Y'_1 \otimes \dots \otimes Y'_r$ runs over the basis element of V_Y . Then it is easy to see that θ_Y is injective and Y is contained in $\text{Im } \theta_Y$.

We are now ready to prove the relation (5.15).

Lemma 5.13. *The linear map θ_Y is a $U_{(i)}$ -module homomorphism. In particular, we have*

$$(e_i f_i - f_i e_i)Y = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} Y.$$

Proof. By definition of $U_{(i)}$ -action on V_Y and $U_q(\mathfrak{g})$ -action on $\mathcal{F}(\Lambda)$, it is rather tedious but straightforward to verify that

$$(5.30) \quad \theta_Y(x \cdot v) = x \cdot \theta_Y(v) \quad \text{for all } x \in U_{(i)} \text{ and } v \in V_Y.$$

We will prove only when $x = K_i$. Let $v = Y'_0 \otimes Y'_1 \otimes \dots \otimes Y'_r \in V_Y$ and put $Y' = \theta_Y(v) = (Y'_0, Y'_1, \dots, Y'_r)$. Then we see from the action of K_i on $Y'_k \in V_k$

that $K_i v = q_i^l v$ where $l = \sum_{k=0}^r (\varphi_i(Y'_k) - \varepsilon_i(Y'_k))$. By Remark 5.12, we have $l = \varphi_i(Y') - \varepsilon_i(Y') = \text{wt}(Y')(h_i)$, which implies $\theta_Y(K_i v) = K_i \theta_Y(v)$.

Hence

$$\begin{aligned} (e_i f_i - f_i e_i)Y &= \theta_Y((e_i f_i - f_i e_i)(Y_0 \otimes Y_1 \otimes \cdots \otimes Y_r)) \\ (5.31) \quad &= \theta_Y\left(\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}(Y_0 \otimes Y_1 \otimes \cdots \otimes Y_r)\right) = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}Y. \end{aligned}$$

□

Example 5.14. Let us illustrate (5.30). Let Y be the proper Young wall in Example 5.10, and let $v = Y_0 \otimes Y_1 \otimes Y_2 \otimes Y_3$ be the element in V_Y associated to the decomposition of Y into 0-component given in Example 5.11 such that $\theta_Y(v) = Y$. Note that Y_0 corresponds to v_0 in V , Y_1 to u'' in W_3 , Y_2 to u in U , and Y_3 to w_0 in W_2^- . Since $f_0 u = f_0 u'' = 0$, we have

$$\begin{aligned} f_0 v &= f_0(v_0 \otimes u'' \otimes u \otimes w_0) \\ &= (f_0 v_0) \otimes u'' \otimes u \otimes w_0 + q v_0 \otimes u'' \otimes u \otimes (f_0 w_0) \\ &= v_1 \otimes u'' \otimes u \otimes w_0 + v_0 \otimes u'' \otimes u \otimes w_1 + q^3 v_0 \otimes u'' \otimes u \otimes w'_1. \end{aligned}$$

Hence, $\theta_Y(f_0 v) = f_0 \theta_Y(v) = f_0 Y$.

Lemma 5.15. Let Y be a proper Young wall in $\mathcal{Z}(\Lambda)$. For $i, j \in I$ with $i \neq j$, we have

$$(5.32) \quad (e_i f_j - f_j e_i)Y = 0.$$

Proof. The proof is quite lengthy and is based on a case-by-case check. We give a sketch of the verification and leave the details to the reader.

For $i \in I$, we define

$\text{adm}_i(Y)$ = the set of all (virtually) admissible i -slots in Y ,
 $\text{rmv}_i(Y)$ = the set of all (virtually) removable i -blocks in Y .

Consider

$$\begin{aligned} e_i f_j Y &= \sum_{\substack{b \in \text{adm}_j(Y) \\ c \in \text{rmv}_i(Y \swarrow b)}} q_j^{L_j(b; Y)} q_i^{-R_i(c; Y \swarrow b)} ((Y \swarrow b) \nearrow c), \\ (5.33) \quad f_j e_i Y &= \sum_{\substack{c \in \text{rmv}_i(Y) \\ b \in \text{adm}_j(Y \nearrow c)}} q_j^{L_j(b; Y \nearrow c)} q_i^{-R_i(c; Y)} ((Y \nearrow c) \swarrow b). \end{aligned}$$

Let \square be the type of the i -blocks and let \square' be the type of the j -blocks. Suppose $(\square, \square') \neq (\text{III}, \text{III})$ and observe that

$$\begin{aligned} (5.34) \quad b \in \text{adm}_j(Y), c \in \text{rmv}_i(Y \swarrow b) \\ \text{if and only if } c \in \text{rmv}_i(Y), b \in \text{adm}_j(Y \nearrow c). \end{aligned}$$

In this case, we have $((Y \swarrow b) \nearrow c) = ((Y \nearrow c) \swarrow b)$. Furthermore, if $(\alpha_i|\alpha_j) = 0$ (equivalently, i -blocks and j -blocks are not adjacent in each column of Y) or b is located to the left of c , then it is clear that

$$(5.35) \quad L_j(b; Y) = L_j(b; Y \nearrow c), \quad R_i(c; Y \swarrow b) = R_i(c; Y),$$

which implies that the corresponding two summands in $(e_i f_j - f_j e_i)Y$ are cancelled out. Therefore, we have only to consider the case when $(\alpha_i|\alpha_j) \neq 0$ and b is located to the right of c and show that

$$(5.36) \quad q_j^{L_j(b; Y)} q_i^{-R_i(c; Y \swarrow b)} = q_j^{L_j(b; Y \nearrow c)} q_i^{-R_i(c; Y)}.$$

On the other hand, we have the following :

(i) if $(\alpha_i|\alpha_i) = (\alpha_j|\alpha_j)$, i.e., $s_i = s_j$, then we have

$$(5.37) \quad \begin{aligned} L_j(b; Y) &= L_j(b; Y \nearrow c) + 1, \\ R_i(c; Y \swarrow b) &= R_i(c; Y) + 1. \end{aligned}$$

(ii) if $(\alpha_i|\alpha_i) > (\alpha_j|\alpha_j)$, i.e., $s_i = 2s_j$, then we have

$$(5.38) \quad \begin{aligned} L_j(b; Y) &= L_j(b; Y \nearrow c) + 2, \\ R_i(c; Y \swarrow b) &= R_i(c; Y) + 1. \end{aligned}$$

(iii) if $(\alpha_i|\alpha_i) < (\alpha_j|\alpha_j)$, i.e., $2s_i = s_j$, then we have

$$(5.39) \quad \begin{aligned} L_j(b; Y) &= L_j(b; Y \nearrow c) + 1, \\ R_i(c; Y \swarrow b) &= R_i(c; Y) + 2. \end{aligned}$$

For example, consider (5.37). Let $Y = (y_k)_{k=0}^\infty$ be the proper Young wall in Example 5.10, and let $i = 2$ and $j = 0$ with $s_0 = s_2 = 2$. Let b be the admissible 0-slot in y_0 and c the removable 2-block in y_7 . Note that $L_0(b; Y) = 1$. By removing c from Y , 0-block at the top of y_6 becomes removable and $L_0(b; Y \nearrow c) = 0$. Also, by adding a 0-block at the top of y_0 , an admissible 2-slot is created at the same column and we have $R_2(c; Y \swarrow b) = 1$ while $R_2(c; Y) = 0$. Hence (5.37) is satisfied. In general, one can check the other relations in a similar way.

Now, it follows that

$$(5.40) \quad s_j L_j(b; Y) - s_i R_i(c; Y \swarrow b) = s_j L_j(b; Y \nearrow c) - s_i R_i(c; Y),$$

which proves (5.36).

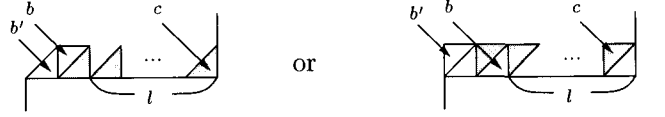
Now suppose that $\square = \square' = \text{III}$. Note that $(\alpha_i|\alpha_i) = (\alpha_j|\alpha_j)$ and $(\alpha_i|\alpha_j) = 0$. For $b \in \text{adm}_j(Y)$ and $c \in \text{rmv}_i(Y \swarrow b)$, set

$$W_{i,j}^+(b, c) = q_j^{L_j(b; Y)} q_i^{-R_i(c; Y \swarrow b)} ((Y \swarrow b) \nearrow c),$$

and for $c \in \text{rmv}_i(Y)$ and $b \in \text{adm}_j(Y \nearrow c)$, set

$$W_{i,j}^-(b, c) = q_j^{L_j(b; Y \nearrow c)} q_i^{-R_i(c; Y)} ((Y \nearrow c) \swarrow b).$$

First, consider the following cases.

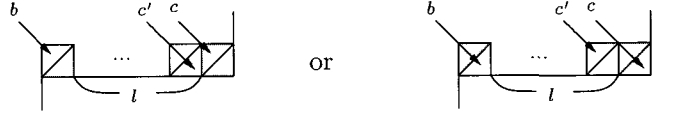
Case 1.

where $l = 2m$ ($m \geq 0$), $b, b' \in \text{adm}_j(Y)$ and $c \in \text{rmv}_i(Y \swarrow b) \cap \text{rmv}_i(Y \swarrow b')$. In this case, we have

$$\begin{aligned} ((Y \swarrow b') \nearrow c) &= (-q_i)((Y \swarrow b) \nearrow c), \\ L_j(b'; Y) &= L_j(b; Y) - 1, \quad R_i(c; Y \swarrow b') = R_i(c; Y \swarrow b), \end{aligned}$$

which yields

$$(5.41) \quad W_{i,j}^+(b, c) = -W_{i,j}^+(b', c).$$

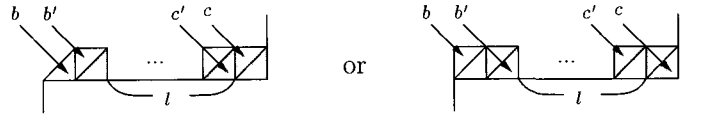
Case 2.

where $l = 2m$ ($m \geq 0$), $b \in \text{adm}_j(Y \nearrow c) \cap \text{adm}_j(Y \nearrow c')$ and $c, c' \in \text{rmv}_i(Y)$. In this case, we have

$$\begin{aligned} ((Y \nearrow c) \swarrow b) &= (-q_i)((Y \nearrow c') \swarrow b), \\ L_j(b; Y \nearrow c) &= L_j(b; Y \nearrow c'), \quad R_i(c; Y) = R_i(c'; Y) + 1, \end{aligned}$$

which yields

$$(5.42) \quad W_{i,j}^-(b, c) = -W_{i,j}^-(b, c')$$

Case 3.

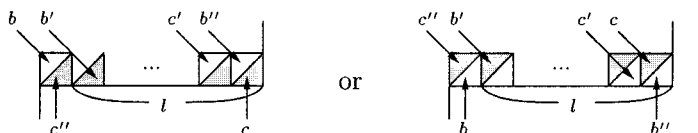
where $l = 2m$ ($m \geq 0$),

$$b \in \text{adm}_j(Y) \cap \text{adm}_j(Y \nearrow c), \quad b' \in \text{adm}_j(Y \nearrow c),$$

$$\text{and} \quad c \in \text{rmv}_i(Y) \cap \text{rmv}_i(Y \swarrow b), \quad c' \in \text{rmv}_i(Y \swarrow b).$$

In this case, we have

$$\begin{aligned} W_{i,j}^+(b, c) &= W_{i,j}^-(b, c), \\ W_{i,j}^+(b, c') &= W_{i,j}^-(b', c). \end{aligned}$$

Case 4.

where $l = 2m$ ($m \geq 0$),

$$b \in \text{adm}_j(Y) \cap \text{adm}_j(Y \nearrow c), \quad b' \in \text{adm}_j(Y \nearrow c),$$

$$b'' \in \text{adm}_j(Y) \cap \text{adm}_j(Y \nearrow c''),$$

$$c \in \text{rmv}_i(Y) \cap \text{rmv}_i(Y \swarrow b), \quad c' \in \text{rmv}_i(Y \swarrow b),$$

and $c'' \in \text{rmv}_i(Y) \cap \text{rmv}_i(Y \swarrow b'').$

Similarly, we have

$$W_{i,j}^+(b, c) = W_{i,j}^-(b, c),$$

$$W_{i,j}^+(b, c') = W_{i,j}^-(b', c),$$

$$W_{i,j}^+(b'', c'') = W_{i,j}^-(b'', c'').$$

For the other cases, it is easy to verify that (5.34) holds and $W_{i,j}^+(b, c) = W_{i,j}^-(b, c)$. Therefore, $(e_i f_j - f_j e_i)Y = 0$, which completes the proof of the lemma. \square

Proof of Theorem 5.5. By Lemma 5.13, Lemma 5.15 and Proposition B.1 in [13], the $U_q(\mathfrak{g})$ -action on $\mathcal{F}(\Lambda)$ satisfies all the relations in (1.4). Therefore, $\mathcal{F}(\Lambda)$ becomes a $U_q(\mathfrak{g})$ -module in the category \mathcal{O}_{int} . \square

Remark 5.16. If $\mathfrak{g} = A_n^{(1)}$, the Fock space $\mathcal{F}(\Lambda)$ is equal to the Fock space constructed by Misra and Miwa [19] where $\mathcal{Z}(\Lambda)$ is the set of *Young diagrams* and $\mathcal{Y}(\Lambda)$ is the set of *n-reduced Young diagrams*. In [13], Kashiwara, Miwa, Petersen and Yung gave a more abstract construction of the Fock space representations of quantum affine algebras. More precisely, for a level l perfect representation V of $U'_q(\mathfrak{g})$, they first defined the *q-deformed wedge space* $\bigwedge^r(V^{\text{aff}})$ where V^{aff} denotes the affinization of V . Then they defined the Fock space to be the inductive limit of *q-deformed wedge spaces*. The set of *normally ordered wedges* (defined by the energy function) form a $\mathbb{Q}(q)$ -basis of the Fock space. For the level 1 case, the Fock space representation constructed in [13] is isomorphic to $U_q(\mathfrak{g})$ -module $\mathcal{F}(\Lambda)$ constructed in this paper. Moreover, there is a bijection between the set of normally ordered wedges and the set of proper Young walls $\mathcal{Z}(\Lambda)$ except for the case $\mathfrak{g} = D_{n+1}^{(2)}$. We expect that one can also construct the higher level Fock space representations of quantum affine algebras using combinatorics of Young walls.

6. Crystal basis of $\mathcal{F}(\Lambda)$

Let $\mathcal{L}(\Lambda) = \bigoplus_{Y \in \mathcal{Z}(\Lambda)} \mathbb{A}_0 Y$. We will show that $(\mathcal{L}(\Lambda), \mathcal{Z}(\Lambda))$ is a crystal basis of $\mathcal{F}(\Lambda)$. In particular, the crystal of $\mathcal{F}(\Lambda)$ is isomorphic to the $U_q(\mathfrak{g})$ -crystal $\mathcal{Z}(\Lambda)$ defined by the abstract Kashiwara operators \tilde{E}_i and \tilde{F}_i ($i \in I$).

Observe that the pair $(\mathcal{L}(\Lambda), \mathcal{Z}(\Lambda))$ satisfies the first four conditions in Definition 2.1. For the rest three conditions, the main step is to show that the Kashiwara operators \tilde{e}_i and \tilde{f}_i ($i \in I$) on $\mathcal{Z}(\Lambda)$ induced by the $U_q(\mathfrak{g})$ -module action on $\mathcal{F}(\Lambda)$ coincide with the abstract Kashiwara operators \tilde{E}_i and \tilde{F}_i ($i \in I$) on $\mathcal{Z}(\Lambda)$ defined in Section 4. The proof of this step relies on the crystal basis theory for $U_q(\mathfrak{sl}_2)$ -modules and the tensor product rule.

Theorem 6.1. *The pair $(\mathcal{L}(\Lambda), \mathcal{Z}(\Lambda))$ is a crystal basis of the Fock space representation $\mathcal{F}(\Lambda)$. Moreover, the crystal of $\mathcal{F}(\Lambda)$ is isomorphic to the $U_q(\mathfrak{g})$ -crystal $\mathcal{Z}(\Lambda)$ given in Section 4.*

Proof. We will show that the pair $(\mathcal{L}(\Lambda), \mathcal{Z}(\Lambda))$ satisfies the conditions (v), (vi) and (vii) in Definition 2.1.

Fix $i \in I$. Let Y be a proper Young wall in $\mathcal{Z}(\Lambda)$ with the i -component decomposition (Y_0, Y_1, \dots, Y_r) , and let $V_Y = V_0 \otimes \cdots \otimes V_r$ be the $U_{(i)}$ -module constructed in Section 5. By Lemma 5.13, $\theta_Y : V_Y \rightarrow \mathcal{F}(\Lambda)$ is an injective $U_{(i)}$ -module homomorphism. Since (L_k, B_k) is a crystal basis of V_k ($0 \leq k \leq r$), V_Y has a crystal basis (L_Y, B_Y) given by

$$(6.1) \quad L_Y = L_0 \otimes \cdots \otimes L_r, \quad B_Y = B_0 \otimes \cdots \otimes B_r.$$

Furthermore, θ_Y satisfies

$$(6.2) \quad \theta_Y(L_Y) \subset \mathcal{L}(\Lambda), \quad \overline{\theta_Y}(B_Y) \subset \mathcal{Z}(\Lambda),$$

where $\overline{\theta_Y} : L_Y/qL_Y \rightarrow \mathcal{L}(\Lambda)/q\mathcal{L}(\Lambda)$ is the injective \mathbb{Q} -linear map induced from θ_Y .

Let \tilde{e}_i and \tilde{f}_i be the Kashiwara operators induced from the $U_{(i)}$ -module structure on V_Y and $\mathcal{F}(\Lambda)$. Since \tilde{e}_i and \tilde{f}_i commute with θ_Y and Y is contained in $\theta_Y(L_Y)$, we have $\tilde{e}_i Y, \tilde{f}_i Y \in \mathcal{L}(\Lambda)$. Hence, the condition (v) is satisfied.

Since \tilde{e}_i and \tilde{f}_i commute with $\overline{\theta_Y}$, we also have $\tilde{e}_i Y, \tilde{f}_i Y \in \mathcal{Z}(\Lambda) \cup \{0\}$ mod $q\mathcal{L}(\Lambda)$. Furthermore, if $\tilde{e}_i Y \neq 0$ (resp. $\tilde{f}_i Y \neq 0$) (mod $q\mathcal{L}(\Lambda)$), then $\tilde{e}_i \tilde{f}_i Y = Y$ (resp. $\tilde{f}_i \tilde{e}_i Y = Y$) (mod $q\mathcal{L}(\Lambda)$), which implies that the condition (vi) and (vii) are satisfied. Hence, $(\mathcal{L}(\Lambda), \mathcal{Z}(\Lambda))$ is a crystal basis of $\mathcal{F}(\Lambda)$.

It remains to show that

$$(6.3) \quad \tilde{e}_i Y = \tilde{E}_i Y, \quad \tilde{f}_i Y = \tilde{F}_i Y \quad (\text{mod } q\mathcal{L}(\Lambda)).$$

It follows from Remark 5.12 that the i -signature of $Y_0 \otimes \cdots \otimes Y_r$ (see Section 2) is equal to the i -signature of Y (see Section 4). Then by the tensor product rule and the definitions of \tilde{E}_i and \tilde{F}_i , we have

$$(6.4) \quad \begin{aligned} \overline{\theta_Y}(\tilde{e}_i(Y_0 \otimes \cdots \otimes Y_r)) &= \tilde{E}_i Y, \\ \overline{\theta_Y}(\tilde{f}_i(Y_0 \otimes \cdots \otimes Y_r)) &= \tilde{F}_i Y. \end{aligned}$$

Therefore, we obtain (6.3) and conclude that the crystal of $\mathcal{F}(\Lambda)$ is isomorphic to the $U_q(\mathfrak{g})$ -crystal $\mathcal{Z}(\Lambda)$. \square

Using Theorem 6.1, one can decompose the Fock space $\mathcal{F}(\Lambda)$ into a direct sum of irreducible highest weight modules over $U_q(\mathfrak{g})$ by locating the maximal vectors in the crystal $\mathcal{Z}(\Lambda)$.

Corollary 6.2.

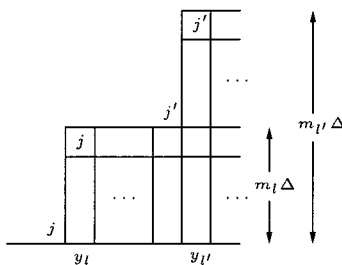
$$(6.5) \quad \mathcal{F}(\Lambda) = \begin{cases} \bigoplus_{m \geq 0} V(\Lambda - m\delta)^{\oplus p(m)} & \text{if } \mathfrak{g} \neq D_{n+1}^{(2)}, \\ \bigoplus_{m \geq 0} V(\Lambda - 2m\delta)^{\oplus p(m)} & \text{if } \mathfrak{g} = D_{n+1}^{(2)}, \end{cases}$$

where $p(m)$ denotes the number of partitions of m .

Proof. We will show that the weight of each maximal vector in $\mathcal{Z}(\Lambda)$ is of the form $\Lambda - m\delta$ (resp. $\Lambda - 2m\delta$) if $\mathfrak{g} \neq D_{n+1}^{(2)}$ (resp. $\mathfrak{g} = D_{n+1}^{(2)}$) for some $m \geq 0$, and that there exists a bijection between the set of partitions of m ($m \geq 0$) and the set of maximal vectors in $\mathcal{Z}(\Lambda)$ with weight $\Lambda - m\delta$ (resp. $\Lambda - 2m\delta$) if $\mathfrak{g} \neq D_{n+1}^{(2)}$ (resp. $\mathfrak{g} = D_{n+1}^{(2)}$). Let $Y = (y_k)_{k=0}^\infty \in \mathcal{Z}(\Lambda)$ be a maximal vector, i.e., $\tilde{e}_i Y = \tilde{E}_i Y = 0$ for all $i \in I$. Suppose that Y is the ground-state wall Y_Λ . Since $\text{wt}(Y_\Lambda) = \Lambda$ and $\mathcal{Z}(\Lambda)_\Lambda = \{Y_\Lambda\}$, the multiplicity of $V(\Lambda)$ in $\mathcal{F}(\Lambda)$ is 1. From now on, we assume that $Y \neq Y_\Lambda$. Let l be the maximum such that $|y_l| \neq 0$. Suppose that $Y' = (y_k)_{k=l+1}^\infty \in \mathcal{Z}(\Lambda_j)$ for some $j \in I$. Denote by \square the type of the j -block. Let Δ be the volume of the δ -column.

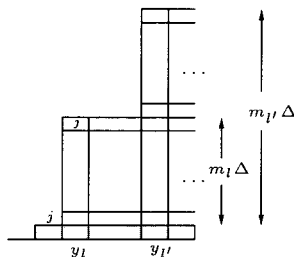
Case 1. $\square = \text{I}$

This case occurs only when $\mathfrak{g} = A_n^{(1)}$. Since y_{l+1} is j -admissible and Y is a maximal vector, there is a removable j -block on top of y_l , and hence y_l is obtained by adding some δ -columns to the ground-state wall, or $|y_l| = m_l \Delta$ for some $m_l \geq 1$. If $l \neq 0$, let l' be the maximum such that $l' < l$ and $|y_{l'}| > |y_l|$. Note that $y_{l'+1}$ is j' -admissible for some $j' \in I$. Since Y is a maximal vector, there exists an j' -block on top of $y_{l'}$, which implies that $|y_{l'}| = m_{l'} \Delta$ for some $m_{l'} \geq 1$. By repeating the above argument column by column from left to right, we conclude that for $0 \leq k \leq l$, $|y_k| = m_k \Delta$ for some $m_k \geq 1$. Moreover, (m_0, m_1, \dots, m_l) forms a partition and $\text{wt}(Y) = \Lambda - (\sum_{k=0}^l m_k) \delta$.

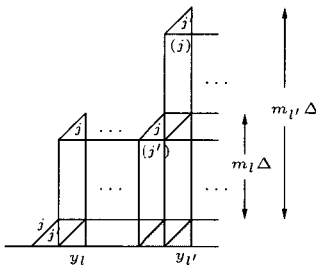


Case 2. $\square = \text{II}$

We see from the pattern for $\mathcal{Z}(\Lambda)$ that $\Lambda = \Lambda_j$. By the maximality of Y , the j -signature of y_{l+1} is $+$ and the j -signature of y_l is $-$ or $-+$. Hence y_l is obtained by adding some δ -columns to the ground-state wall. If $l \neq 0$, let l' be the maximum such that $l' < l$ and $|y_{l'}| > |y_l|$. Also by the maximality of Y , the j -signature of $y_{l'}$ is $-$ or $-+$, which means that $y_{l'}$ is obtained by adding some δ -columns to the ground state wall. Repeating the above argument from left to right, we conclude that for $0 \leq k \leq l$, the total volume of the blocks added on the k th column is $m_k \Delta$ for some $m_k \geq 1$. Hence, (m_0, \dots, m_l) forms a partition, and $\text{wt}(Y) = \Lambda - (\sum_{k=0}^l m_k) \delta$ if $\mathfrak{g} \neq D_{n+1}^{(2)}$, $\text{wt}(Y) = \Lambda - 2(\sum_{k=0}^l m_k) \delta$ if $\mathfrak{g} = D_{n+1}^{(2)}$.

**Case 3.** $\square = \text{III}$

Let j' be the color of the type III block, with which the j -block forms a unit cube. By the maximality of Y , we observe that the y_{l+1} is j -admissible and y_l is j -removable but not j' -removable. Hence y_l is obtained by adding some δ -columns to the ground-state wall. If $l \neq 0$, let l' be the maximum such that $l' < l$ and $|y_{l'}| > |y_l|$. If $y_{l'+1}$ is j -admissible, then by the maximality of Y , $y_{l'}$ is j -removable but not j' -removable. On the other hand, if $y_{l'+1}$ is j' -admissible, then by the maximality of Y , $y_{l'}$ is j' -removable but not j -removable. As in Case 1 and 2, by repeating the above argument, we conclude that for $0 \leq k \leq l$, the total volume of the blocks added on the k th column is $m_k \Delta$ for some $m_k \geq 1$. Hence, (m_0, m_1, \dots, m_l) forms a partition and $\text{wt}(Y) = \Lambda - (\sum_{k=0}^l m_k) \delta$.



Conversely, for a given partition $(m_k)_{k=0}^\infty$ of a nonnegative integer m , there exists a unique proper Young wall $Y = (y_k)_{k=0}^\infty$ in $\mathcal{Z}(\Lambda)$ such that y_k is obtained by adding m_k many δ -columns to the k th column of the ground state wall Y_Λ (hence the total volume of the blocks added to the k th column is $m_k\Delta$). It is easy to check that Y is a maximal vector with $\text{wt}(Y) = \Lambda - m\delta$ (resp. $\Lambda - 2m\delta$) if $\mathfrak{g} \neq D_{n+1}^{(2)}$ (resp. $\mathfrak{g} = D_{n+1}^{(2)}$). \square

7. Generalized Lascoux-Leclerc-Thibon algorithm

In this section, we generalize Lascoux-Leclerc-Thibon algorithm ([14]) to obtain an effective algorithm for constructing the global basis $\mathcal{G}(\Lambda)$ of the basic representation $V(\Lambda)$ of $U_q(\mathfrak{g})$. Observe that $V(\Lambda)$ is realized as the $U_q(\mathfrak{g})$ -submodule of $\mathcal{F}(\Lambda)$ generated by the ground state wall Y_Λ . Also recall that the crystal $B(\Lambda)$ of $V(\Lambda)$ is isomorphic to the $U_q(\mathfrak{g})$ -crystal $\mathcal{Y}(\Lambda)$ consisting of reduced proper Young walls. Thus our goal is the following: for each reduced proper Young wall $Y \in \mathcal{Y}(\Lambda)$, we would like to give an algorithm for computing the corresponding global basis element $G(Y)$ as a linear combination of proper Young walls in $\mathcal{Z}(\Lambda)$.

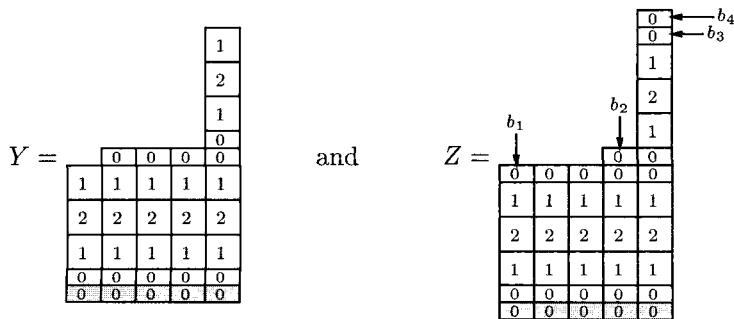
For this purpose, we first investigate the action of divided powers $f_i^{(r)}$ ($i \in I, r \geq 1$) on the proper Young walls. Let Y be a proper Young wall in $\mathcal{Z}(\Lambda)$ (not necessarily reduced), and write

$$(7.1) \quad f_i^{(r)} Y = \sum_{\substack{Z \in \mathcal{Z}(\Lambda) \\ \text{wt}(Z) = \text{wt}(Y) - r\alpha_i}} Q_{Y,Z}(q) Z,$$

where $Q_{Y,Z}(q) \in \mathbb{Q}(q)$. For each $Z = (z_k)_{k=0}^\infty \in \mathcal{Z}(\Lambda)$ with $Q_{Y,Z}(q) \neq 0$, there exists a unique sequence of proper Young walls $Y = Y_0, Y_1, \dots, Y_r = Z$ such that

- (i) $\lambda_{k+1} Y_{k+1} = Y_k \swarrow b_{k+1}$ for some $\lambda_{k+1} \in \mathbb{Z}[q, q^{-1}]$ and a (virtually) admissible i -slot b_{k+1} of Y_k ,
- (ii) b_{k+1} is placed on top of b_k or to the right of b_k .

Example 7.1. Let $\mathfrak{g} = A_4^{(2)}$, $\Lambda = \Lambda_0$, $i = 0$, and consider



Then, we have a sequence of proper Young walls $Y = Y_0, Y_1, Y_2, Y_3, Y_4 = Z$, where

$$Y_1 = \begin{array}{|c|c|c|c|c|c|} \hline & & & & & 1 \\ \hline & & & & & 2 \\ \hline & & & & & 1 \\ \hline & & & & 0 & \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & \\ \hline 2 & 2 & 2 & 2 & 2 & \\ \hline 1 & 1 & 1 & 1 & 1 & \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}, \quad Y_2 = \begin{array}{|c|c|c|c|c|c|} \hline & & & & & 1 \\ \hline & & & & & 2 \\ \hline & & & & & 1 \\ \hline & & & 0 & 0 & \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & \\ \hline 2 & 2 & 2 & 2 & 2 & \\ \hline 1 & 1 & 1 & 1 & 1 & \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}, \quad Y_3 = \begin{array}{|c|c|c|c|c|c|} \hline & & & & & 0 \\ \hline & & & & & 1 \\ \hline & & & & & 2 \\ \hline & & & & & 1 \\ \hline & & & 0 & 0 & \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & \\ \hline 2 & 2 & 2 & 2 & 2 & \\ \hline 1 & 1 & 1 & 1 & 1 & \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

and $\lambda_1 = 1$, $\lambda_2 = q^{-1}(1 - q^8)$, $\lambda_3 = 1$, $\lambda_4 = [2] = (q + q^{-1})$.

Let $Q_{Y_k, Y_{k+1}}(q)$ be the coefficient of Y_{k+1} in the expression of $f_i Y_k$. Define

$$(7.2) \quad Q_{Y, Z}^\circ(q) = \prod_{k=0}^{r-1} Q_{Y_k, Y_{k+1}}(q) \in \mathbb{Z}[q, q^{-1}].$$

Note that each b_k ($1 \leq k \leq r$) can be viewed as an i -block (not necessarily removable) in Z . When b_k 's are of type II, we define

$$(7.3) \quad \begin{aligned} J_1 &= \{k \mid b_{k-1} \text{ lies beneath } b_k\}, \\ J_2 &= \{k \mid \text{there exists an } i\text{-block } (\neq b_{k-1}) \text{ beneath } b_k \text{ in } Z\}, \\ J_3 &= \{k \mid \text{there exists no } i\text{-block on top of and beneath } b_k \text{ in } Z\}, \\ S &= \{k \in J_2 \mid b_k \text{ and } b_{k-1} \text{ lie in the same } i\text{-component of } Z\}, \\ T &= \{k \in J_3 \mid k+1 \in S\}. \end{aligned}$$

Set $l = |J_1|$, $m = |J_2|$ and $n = |J_3|$. For each $k \in S$, let $\mu_k = q\lambda_k$. Note that $2l + m + n = r$. Then we have

Lemma 7.2. *Let Y be a proper Young wall in $\mathcal{Z}(\Lambda)$, and suppose that $Q_{Y, Z}(q) \neq 0$ for some $Z \in \mathcal{Z}(\Lambda)$ with $\text{wt}(Z) = \text{wt}(Y) - r\alpha_i$. Then we have*

$$(7.4) \quad Q_{Y, Z}(q) = \begin{cases} Q_{Y, Z}^\circ(q) q_i^{\binom{r}{2}} & \text{if the } i\text{-blocks are of type I or III,} \\ Q_{Y, Z}^\circ(q) \frac{q^{\sigma(l, m, n)}}{[2]^l \prod_{k \in S} \mu_k} & \text{if the } i\text{-blocks are of type II,} \end{cases}$$

where $\sigma(l, m, n) = 4\binom{l}{2} + \binom{m}{2} + \binom{n}{2} + 2l(m+n) + mn$.

In particular, we have $Q_{Y, Z}(q) \in \mathbb{Z}[q, q^{-1}]$.

Example 7.3. In Example 7.1, we have $J_1 = \{4\}$, $J_2 = \{2\}$ and $J_3 = \{1\}$. Also, we have $S = \{2\}$, $T = \{1\}$. Since $Q_{Y, Z}^\circ(q) = q^{-1}(1 - q^8)[2]$, $\mu_2 = q\lambda_2 =$

$(1 - q^8)$ and $\sigma(1, 1, 1) = 5$, it follows that

$$Q_{Y,Z}(q) = Q_{Y,Z}^\circ(q) \times \frac{q^5}{[2](1 - q^8)} = q^4.$$

Hence, the coefficient of Z in $f_0^{(4)}Y$ is q^4 .

Proof of Lemma 7.2. Case 1. Suppose that the i -blocks are of type I or III. We will use induction on r . For $r = 1$, it is clear. Suppose that (7.4) holds for $r - 1$. Let Z_k be the proper Young wall in $\mathcal{Z}(\Lambda)$ ($1 \leq k \leq r$) such that b_k is also a (virtually) admissible i -slot of Z_k and $Z_k \searrow b_k = Z$ (up to scalar multiplication). By definition of $Q_{Y,Z}(q)$, we have

$$[r]_i Q_{Y,Z}(q) = \sum_{k=1}^r Q_{Y,Z_k}(q) Q_{Z_k,Z}(q).$$

Note that

$$\begin{aligned} Q_{Z_k,Z}(q) &= Q_{Z_k,Z}^\circ(q), \\ Q_{Y,Z_k}^\circ(q) Q_{Z_k,Z}^\circ(q) &= Q_{Y,Z}^\circ(q) q_i^{2(r-k)}. \end{aligned}$$

By induction hypothesis, we have

$$\begin{aligned} [r]_i Q_{Y,Z}(q) &= \sum_{k=1}^r Q_{Y,Z_k}^\circ(q) Q_{Z_k,Z}^\circ(q) q_i^{\binom{r-1}{2}} \\ &= \sum_{k=1}^r Q_{Y,Z}^\circ(q) q_i^{2(r-k) + \binom{r-1}{2}} \\ &= \sum_{k=1}^r Q_{Y,Z}^\circ(q) q_i^{2(r-k) - r + 1 + \binom{r}{2}} = Q_{Y,Z}^\circ(q) q_i^{\binom{r}{2}} [r]_i. \end{aligned}$$

This completes the induction argument.

Case 2. Suppose that i -blocks are of type II. We will also use induction on r . For $r = 1$, it is clear. Suppose that (7.4) holds for $r - 1$ ($r \geq 2$). Set $J = J_1 \sqcup J_2 \sqcup J_3$. For $k \in J$, let Z_k be the unique proper Young wall in $\mathcal{Z}(\Lambda)$ such that b_k is also an admissible i -slot of Z_k and $Z_k \searrow b_k = \lambda_k Z$ (λ_k was already given in the definition of b_k). We have

$$(7.5) \quad \lambda_k = \begin{cases} [2] & \text{if } k \in J_1, \\ q^{-1}(1 - (-q^2)^{l_k+1}) & \text{if } k \in J_2 \text{ and } k \geq 2, \\ 1 & \text{otherwise} \end{cases}$$

for some $l_k \geq 0$, and $\mu_k = (1 - (-q^2)^{l_k-1})$ for $k \in S$. As in Case 1, we have

$$(7.6) \quad [r]_i Q_{Y,Z}(q) = \sum_{k \in J} Q_{Y,Z_k}(q) Q_{Z_k,Z}(q),$$

and

$$Q_{Z_k, Z}(q) = Q_{Z_k, Z}^\circ(q),$$

$$Q_{Y, Z_k}^\circ(q) Q_{Z_k, Z}^\circ(q) = \begin{cases} Q_{Y, Z}^\circ(q) q^{2(r-k)} \frac{(1-(-q^2)^{l_{k+1}})}{(1-(-q^2)^{l_{k+1}+1})} & \text{if } k \in T, \\ Q_{Y, Z}^\circ(q) q^{2(r-k)} & \text{otherwise.} \end{cases}$$

By induction hypothesis, we have

$$\begin{aligned} [r]_i Q_{Y, Z}(q) &= Q_{Y, Z}^\circ(q) \times \\ &\left(\sum_{k \in J_1} \frac{q^{\sigma(l-1, m, n+1)+2(r-k)}}{[2]^{l-1} \prod_{k' \in S} \mu_{k'}} + \sum_{k \in J_2 \setminus S} \frac{q^{\sigma(l, m-1, n)+2(r-k)}}{[2]^l \prod_{k' \in S} \mu_{k'}} \right. \\ &\quad + \sum_{k \in J_3 \setminus T} \frac{q^{\sigma(l, m, n-1)+2(r-k)}}{[2]^l \prod_{k' \in S} \mu_{k'}} + \sum_{k \in S} \frac{q^{\sigma(l, m-1, n)+2(r-k)}}{[2]^l \prod_{k' \in S \setminus \{k\}} \mu_{k'}} \\ &\quad \left. + \sum_{k \in T} \frac{q^{\sigma(l, m, n-1)+2(r-k)}}{[2]^l \prod_{k' \in S \setminus \{k+1\}} \mu_{k'}} \frac{(1-(-q^2)^{l_{k+1}})}{(1-(-q^2)^{l_{k+1}+1})} \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\sum_{k \in S} \frac{q^{\sigma(l, m-1, n)+2(r-k)}}{[2]^l \prod_{k' \in S \setminus \{k\}} \mu_{k'}} + \sum_{k \in T} \frac{q^{\sigma(l, m, n-1)+2(r-k)}}{[2]^l \prod_{k' \in S \setminus \{k+1\}} \mu_{k'}} \frac{(1-(-q^2)^{l_{k+1}})}{(1-(-q^2)^{l_{k+1}+1})} \\ &= \frac{1}{[2]^l \prod_{k' \in S} \mu_{k'}} \sum_{k \in T} \left(q^{\sigma(l, m-1, n)+2(r-k-1)} (1-(-q^2)^{l_{k+1}+1}) \right. \\ &\quad \left. + q^{\sigma(l, m, n-1)+2(r-k)} (1-(-q^2)^{l_{k+1}}) \right) \\ &= \frac{1}{[2]^l \prod_{k' \in S} \mu_{k'}} \sum_{k \in T} \left(q^{\sigma(l, m-1, n)+2(r-k-1)} + q^{\sigma(l, m, n-1)+2(r-k)} \right). \end{aligned}$$

Since

$$\begin{aligned} \sigma(l-1, m, n+1) &= \sigma(l, m, n) - r + 2 \\ \sigma(l, m-1, n) &= \sigma(l, m, n-1) = \sigma(l, m, n) - r + 1, \end{aligned}$$

we have

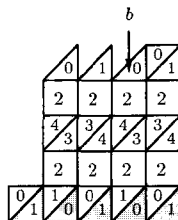
$$\begin{aligned} &[r] Q_{Y, Z}(q) \\ &= Q_{Y, Z}^\circ(q) \frac{q^{\sigma(l, m, n)}}{[2]^l \prod_{k' \in S} \mu_{k'}} \left(\sum_{k \in J_1} q^{-r+1+2(r-k+1)} + \sum_{k \in J} q^{-r+1+2(r-k)} \right) \\ &= Q_{Y, Z}^\circ(q) \frac{q^{\sigma(l, m, n)}}{[2]^l \prod_{k' \in S} \mu_{k'}} \left(\sum_{k=1}^r q^{-(r-1)+2(r-k)} \right) = Q_{Y, Z}^\circ(q) \frac{q^{\sigma(l, m, n)}}{[2]^l \prod_{k' \in S} \mu_{k'}} [r], \end{aligned}$$

which completes our induction argument.

Finally, since $[2]^l \prod_{k \in S} \mu_k$ divides $Q_{Y, Z}^\circ(q)$, we have $Q_{Y, Z}(q) \in \mathbb{Z}[q, q^{-1}]$. \square

Let Y be a proper Young wall in $\mathcal{Z}(\Lambda)$ and let b be a block in Y . Suppose that b lies in the k th column of Y . The *coordinate of b* is defined to be (k, l) where l is the number of unit cubes lying below b . Note that two different blocks of type II or III have the same coordinate if they are parts of a unit cube. Also, each coordinate corresponds to a unit cube in a given pattern.

Example 7.4. The block b in the following figure has the coordinate $(1, 4)$.

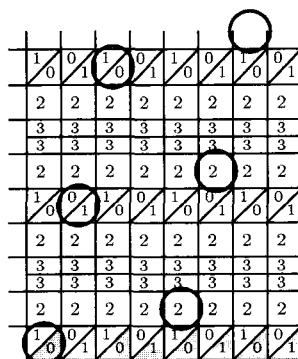


Let $c = (k, l)$ ($k, l \geq 0$) be a coordinate of a block in the pattern for $\mathcal{Z}(\Lambda)$. We define the *ladder at c* to be the finite sequence of coordinates as follows;

$$c = (k, l), (k-1, l+\Delta'), (k-2, l+2\Delta'), \dots, (0, l+k\Delta'),$$

where $\Delta' = \Delta - 1$ (resp. Δ) if $\mathfrak{g} = A_n^{(1)}$ (resp. $\mathfrak{g} \neq A_n^{(1)}$) (This is a generalization of the ladder for Young diagrams. See [5]).

Example 7.5. If $\mathfrak{g} = B_3^{(1)}$ and $\Lambda = \Lambda_0$, then $\Delta = 4$. There are two ladders in the following figure. The left one is the ladder at $(7, 0)$, and the right one is the ladder at $(3, 1)$.



Let Y be a reduced proper Young wall in $\mathcal{Y}(\Lambda)$ and let y_l be the left-most column in Y with $|y_l| \neq 0$. Take a block b lying at the top of y_l and let i be its color. (If the b is of type III and there is another block of type III on top of y_l , we take the block at the front.) Suppose that the coordinate of b is $c = (k, l)$ and let L_c be the ladder at c . We denote by $Y \cap L_c$ the i -blocks in Y whose

coordinates are in L_c . (In fact, L_c is the left-most ladder that has a nontrivial intersection with Y .) We define \bar{Y} to be the proper Young wall that is obtained from Y by removing all the i -blocks in $Y \cap L_c$. Then it is easy to see that \bar{Y} is also reduced. That is, we remove all the i -blocks along the left-most ladder to obtain another reduced proper Young wall. This process will play a crucial role in constructing the global basis $\mathcal{G}(\Lambda)$ of $V(\Lambda)$.

Example 7.6. (a) If $\mathfrak{g} = A_5^{(2)}$ and

$$Y = \begin{array}{cccc} & & & 2 \\ & & 3 & 3 \\ & \textcircled{2} & 2 & 2 \\ & 0 & 1 & 0 \\ & 2 & 2 & 2 \\ & 3 & 3 & 3 \\ \textcircled{2} & 2 & 2 & 2 \\ 1 & 0 & 1 & 0 \end{array}, \quad \text{then } \bar{Y} = \begin{array}{cccc} & & & 2 \\ & & 3 & 3 \\ & 2 & 2 & \\ & 0 & 1 & 0 \\ & 2 & 2 & 2 \\ & 3 & 3 & 3 \\ & 2 & 2 & 2 \\ 1 & 0 & 1 & 0 \end{array}.$$

(b) If $\mathfrak{g} = A_4^{(2)}$ and

$$Y = \begin{array}{cccc} & \textcircled{0} & 0 & 0 \\ & 1 & 1 & 1 \\ & 2 & 2 & 2 \\ & 1 & 1 & 1 \\ \textcircled{0} & 0 & 0 & 0 \\ \textcircled{0} & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ \textcircled{0} & 0 & 0 & 0 \\ \textcircled{0} & 0 & 0 & 0 \end{array}, \quad \text{then } \bar{Y} = \begin{array}{cccc} & & 0 & 0 \\ & 1 & 1 & 1 \\ & 2 & 2 & 2 \\ & 1 & 1 & 1 \\ & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}.$$

(c) If $\mathfrak{g} = B_3^{(1)}$ and

$$Y = \begin{array}{cccc} & \textcircled{0} & 1 & \\ & 2 & 2 & \\ & 3 & 3 & \\ & 3 & 3 & \\ & 2 & 2 & \\ \textcircled{0} & 1 & 0 & 1 \\ & 2 & 2 & 2 \\ & 3 & 3 & 3 \\ & 3 & 3 & 3 \\ & 2 & 2 & 2 \\ \textcircled{1} & 0 & 1 & 0 \end{array}, \quad \text{then } \bar{Y} = \begin{array}{cccc} & & 0 & 1 \\ & 2 & 2 & \\ & 3 & 3 & \\ & 3 & 3 & \\ & 2 & 2 & \\ & 0 & 1 & 0 \\ & 2 & 2 & 2 \\ & 3 & 3 & 3 \\ & 3 & 3 & 3 \\ & 2 & 2 & 2 \\ & 0 & 1 & 0 \end{array}.$$

Proposition 7.7. *Let Y be a reduced proper Young wall in $\mathcal{Y}(\Lambda)$. Suppose that $\text{wt}(\bar{Y}) = \text{wt}(Y) + r\alpha_i$ for some $i \in I$ and $r \geq 1$. Then we have*

$$f_i^{(r)}\bar{Y} = Y + \sum_{\substack{Z \in \mathcal{Z}(\Lambda) \\ \text{wt}(Z) = \text{wt}(Y) \\ Z \neq Y}} Q_{\bar{Y},Z}(q)Z.$$

That is, we have $Q_{\bar{Y},Y}(q) = 1$.

Proof. By definition of \bar{Y} , there exists a unique sequence of proper Young walls $Y_0 = \bar{Y}, \dots, Y_r = Y$ such that for $1 \leq k \leq r$

- (i) $Y_k = Y_{k-1} \swarrow b_k$ (up to scalar multiplication) for some admissible i -slot b_k of Y_{k-1} ,
- (ii) there exists no admissible i -slot located to the left of b_k .

In other words, $\{b_k \mid 1 \leq k \leq r\}$ are added to \bar{Y} from left to right and from bottom to top with no admissible i -slot to the left of each b_k .

Suppose that b_k 's are of type I or III. Then it is easy to see that

$$Q_{\bar{Y},Y}^\circ(q) = \prod_{k=0}^{r-1} Q_{Y_k, Y_{k+1}}(q) = q_i^{-\binom{r}{2}},$$

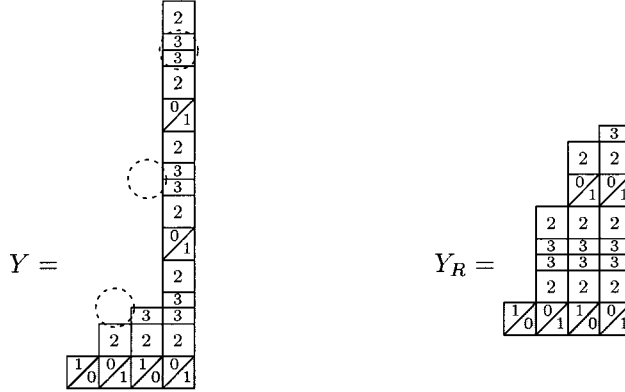
which implies that $Q_{\bar{Y},Y}(q) = 1$ by Lemma 7.2.

Suppose that b_k 's are of type II. Let J_1, J_2, J_3 be the sets given in (7.3). Set $l = |J_1|$, $m = |J_2|$, and $n = |J_3|$. By definition of \bar{Y} , $m \leq 1$ and if $m = 1$, then $J_2 = \{1\}$ and b_1 is placed on the column which is part of the ground-state wall. Also, (ii) implies that $n \leq 1$ and that if $n = 1$, then $J_3 = \{r\}$. Note that $S = T = \emptyset$. Thus we have

$$Q_{\bar{Y},Y}^\circ(q) = [2]_{q_i}^l q_i^{-4\binom{l}{2} - \binom{m}{2} - \binom{n}{2} - 2l(m+n) - mn},$$

which implies that $Q_{\bar{Y},Y}(q) = 1$ by Lemma 7.2. \square

Let Y be a proper Young wall in $\mathcal{Z}(\Lambda)$. Let L be a ladder and suppose that there exists at least one i -block in $Y \cap L$ for some $i \in I$ except the block in the ground state wall. Then we rearrange these i -blocks from the bottom along L , that is, we slide these i -blocks downwards along L as far as possible. Repeat this procedure ladder by ladder until no block can be moved downward along a ladder. Then we obtain another proper Young wall Y^R , which we call the *reduced form of Y* . By definition, Y^R is a reduced proper Young wall. Moreover, we have $|Y^R| \geq |Y|$ and the equality holds if and only if Y is reduced.

Example 7.8.

where a dotted line indicates a ladder containing 3-blocks in y_0 .

Lemma 7.9 (cf. [5]). *Let Y be a reduced proper Young wall in $\mathcal{Y}(\Lambda)$ and let Z be a proper Young wall in $\mathcal{Z}(\Lambda)$ such that $|\bar{Y}| \supseteq |Z^R|$. Suppose that $\text{wt}(\bar{Y}) = \text{wt}(Y) + r\alpha_i$ for some $i \in I$ and $r \geq 1$. Then, for each $W \in \mathcal{Z}(\Lambda)$ appearing in the expression of $f_i^{(r)}Z$, we have*

- (a) $|Y| \geq |W^R|$.
- (b) If $|Y| = |W|$, then $|\bar{Y}| = |Z|$ and Z is a reduced proper Young wall.
- (c) If $|Y| = |W|$ and $\bar{Y} = Z$, then $Y = W$.

Proof. (a) Let L be the left-most ladder which has a nontrivial intersection with Y . Denote by $y_p, y_{p-1}, \dots, y_{p-s}$ ($s \geq 0$) the first $s+1$ columns in Y which meet L , and denote by $\bar{y}_p, \bar{y}_{p-1}, \dots, \bar{y}_{p-s}$ ($s \geq 0$) the corresponding columns in \bar{Y} . Since $Z^R = (z_k^R)_{k=0}^\infty$ is reduced, we have

$$(7.7) \quad |\bar{y}_{p-t}| \geq |z_{p-t}^R| \quad \text{for } 0 \leq t \leq s.$$

Since $W = (w_k)_{k=0}^\infty$ (and hence $W^R = (w_k^R)_{k=0}^\infty$) is given by adding r many i -blocks on Z , we also have

$$(7.8) \quad |y_{p-t}| \geq |w_{p-t}^R| \quad \text{for } 0 \leq t \leq s.$$

Therefore, if $s' \leq s$, then

$$(7.9) \quad \sum_{t=0}^{s'} |y_{p-t}| \geq \sum_{t=0}^{s'} |w_{p-t}^R|,$$

and if $s < s' \leq p$, then

$$(7.10) \quad \sum_{t=0}^{s'} |y_{p-t}| = \sum_{t=0}^{s'} |\bar{y}_{p-t}| + r \geq \sum_{t=0}^{s'} |z_{p-t}^R| + r \geq \sum_{t=0}^{s'} |w_{p-t}^R|.$$

Hence we conclude $|Y| \supseteq |W^R|$.

(b) Suppose that $|Y| = |W|$. Since $|\bar{y}_p| \geq |z_p|$ and w_p is obtained by adding some i -blocks on z_p , we have $\bar{y}_p = z_p$ and $y_p = w_p$. Suppose that for $0 \leq u \leq t < s$,

$$(7.11) \quad \bar{y}_{p-u} = z_{p-u}, \quad y_{p-u} = w_{p-u}.$$

Note that

$$(7.12) \quad \sum_{u=0}^{t+1} |\bar{y}_{p-u}| \geq \sum_{u=0}^{t+1} |z_{p-u}^R| \geq \sum_{u=0}^{t+1} |z_{p-u}|.$$

By our hypothesis, we have $|\bar{y}_{p-t-1}| \geq |z_{p-t-1}|$. Since $|y_{p-t-1}| = |w_{p-t-1}|$ and w_{p-t-1} is obtained by adding some i -blocks on z_{p-t-1} , we have $\bar{y}_{p-t-1} = z_{p-t-1}$ and $y_{p-t-1} = w_{p-t-1}$. By induction, $\bar{y}_{p-u} = z_{p-u}$ and $y_{p-u} = w_{p-u}$ for $0 \leq u \leq s$. Since all the i -blocks are added on $(\bar{y}_k)_{k=p-s}^\infty$ and $(w_k)_{k=p-s}^\infty$, we have $(\bar{y}_k)_{k=0}^{p-s-1} = (y_k)_{k=0}^{p-s-1}$ and $(w_k)_{k=0}^{p-s-1} = (z_k)_{k=0}^{p-s-1}$, which implies $|y_k| = |\bar{y}_k| = |w_k| = |z_k|$ for all $0 \leq k \leq p-s-1$. Hence, $|\bar{Y}| = |Z| = |Z^R|$ and Z is reduced.

(c) follows directly from the proof of (b) \square

Let Y be a reduced proper Young wall in $\mathcal{Y}(\Lambda)$. There exists a unique sequence of reduced proper Young walls $\{Y_k\}_{k=0}^N$ such that $Y_0 = Y$, $Y_1 = \bar{Y}_0$, \dots , $Y_{k+1} = \bar{Y}_k$, \dots , $Y_N = \bar{Y}_{N-1} = Y_\Lambda$. Suppose that $Y_k = \bar{Y}_{k-1}$ is obtained by removing r_k many i_k -blocks from Y_{k-1} ($1 \leq k \leq N$). We define

$$(7.13) \quad A(Y) = f_{i_1}^{(r_1)} \cdots f_{i_N}^{(r_N)} Y_\Lambda \in V(\Lambda)_\mathbb{A}.$$

Example 7.10. If $\mathfrak{g} = A_4^{(2)}$, $\Lambda = \Lambda_0$ and

$$Y = \begin{array}{|c|} \hline 0 \\ \hline 1 \\ \hline 2 \\ \hline 1 \\ \hline 1 \\ \hline 1 & 1 \\ \hline 2 & 2 \\ \hline 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}, \text{ then we have } A(Y) = f_0^{(4)} f_1^{(2)} f_2^{(2)} f_1^{(2)} f_0^{(3)} f_1 f_2 f_1 f_0 Y_{\Lambda_0}.$$

By definition, $\overline{A(Y)} = A(Y)$. Write

$$(7.14) \quad A(Y) = \sum_{Z \in \mathcal{Z}(\Lambda)} A_{Y,Z}(q) Z,$$

where $A_{Y,Z}(q) \in \mathbb{Q}(q)$. Then, the coefficients $A_{Y,Z}(q)$ satisfy the following properties.

Proposition 7.11. *Let Y be a reduced proper Young wall in $\mathcal{Y}(\Lambda)$. Then, for a proper Young wall $Z \in \mathcal{Z}(\Lambda)$, we have*

- (a) $A_{Y,Z}(q) \in \mathbb{Z}[q, q^{-1}]$,
- (b) $A_{Y,Z}(q) = 0$ unless $|Y| \geq |Z^R|$ and $\text{wt}(Y) = \text{wt}(Z)$,
- (c) if $A_{Y,Z}(q) \neq 0$ and $|Y| = |Z|$, then $Y = Z$ and $A_{Y,Y}(q) = 1$.

Proof. We will use induction on l , the number of blocks in Y which have been added to Y_Λ . If $l = 1$, it is clear. Suppose that $l > 1$, and (a)–(c) hold for $l' < l$. If $A(Y) = f_{i_1}^{(r_1)} \cdots f_{i_N}^{(r_N)} Y_\Lambda$ for some $N \geq 1$, then we have

$$\begin{aligned}
 (7.15) \quad A(Y) &= f_{i_1}^{(r_1)} A(\bar{Y}) = \sum_{|\bar{Y}| \geq |Z^R|} A_{\bar{Y},Z}(q) f_{i_1}^{(r_1)} Z \\
 &= \sum_{|\bar{Y}| \geq |Z^R|} A_{\bar{Y},Z}(q) \left(\sum_{|Y| \geq |W^R|} Q_{Z,W}(q) W \right) \quad \text{by Lemma 7.9 (a)} \\
 &= \sum_{|Y| \geq |W^R|} \left(\sum_{|\bar{Y}| \geq |Z^R|} A_{\bar{Y},Z}(q) Q_{Z,W}(q) \right) W.
 \end{aligned}$$

By induction hypothesis and Lemma 7.2, we have

$$(7.16) \quad A_{Y,W}(q) = \sum_{|\bar{Y}| \geq |Z^R|} A_{\bar{Y},Z}(q) Q_{Z,W}(q) \in \mathbb{Z}[q, q^{-1}],$$

and $A_{Y,W}(q) = 0$ unless $|Y| \geq |W^R|$ and $\text{wt}(Y) = \text{wt}(W)$.

If $A_{Y,W}(q) \neq 0$ and $|Y| = |W|$, then Lemma 7.9 (b) implies that $|Z| = |\bar{Y}|$ for $A_{\bar{Y},Z}(q) \neq 0$. Hence, $Z = \bar{Y}$ by induction hypothesis. Finally, we have $Y = W$ by Lemma 7.9 (c), and hence $A_{Y,Y}(q) = A_{\bar{Y},\bar{Y}}(q) Q_{\bar{Y},Y}(q) = 1$ by Proposition 7.7, which completes the induction argument. \square

For proper Young walls $Y = (y_k)_{k=0}^\infty$ and $Z = (z_k)_{k=0}^\infty$, we define $|Y| > |Z|$ if there exists $k \geq 0$ such that $|y_k| > |z_k|$ and $|y_l| = |z_l|$ for all $l > k$. Thus we have a total ordering on the set of partitions. Note that $|Y| \geq |Z|$ implies $|Y| \geq |Z|$. Now we define a total ordering $>$ on the set $\mathcal{Z}(\Lambda)$ of proper Young walls as follows. First, we fix an arbitrary total ordering \succ on the set of proper Young walls with the same associated partition. Then we define

$$(7.17) \quad Y > Z \quad \text{if and only if} \quad \begin{aligned} & \text{(i) } |Y| > |Z| \quad \text{or} \quad \text{(ii) } |Y| = |Z| \text{ and } Y \succ Z. \end{aligned}$$

For example, if $|Y| \geq |Z|$ and $|Y| \neq |Z|$, then we have $Y > Z$.

Let Y be a reduced proper Young wall in $\mathcal{Y}(\Lambda)$. By Proposition 7.11, we may write

$$A(Y) = Y + \sum_{Y > Z} A_{Y,Z}(q) Z.$$

It follows that the set $\mathcal{A}(\Lambda) = \{A(Y) \mid Y \in \mathcal{Y}(\Lambda)\}$ is linearly independent over $\mathbb{Q}(q)$. Since $\dim V(\Lambda)_\lambda = |\mathcal{Y}(\Lambda)_\lambda|$ for $\lambda \leq \Lambda$, we conclude that $\mathcal{A}(\Lambda)$ is a $\mathbb{Q}(q)$ -basis of $V(\Lambda)$.

Let $\mathcal{G}(\Lambda) = \{G(Y) \mid Y \in \mathcal{Y}(\Lambda)\}$ be the global basis of $V(\Lambda)_\mathbb{A}$. Then for each reduced proper Young wall $Y \in \mathcal{Y}(\Lambda)$, we may write

$$(7.18) \quad G(Y) = \sum_{Z \in \mathcal{Z}(\Lambda)} G_{Y,Z}(q) Z \in V(\Lambda)_\mathbb{A} \cap \mathcal{L}(\Lambda)$$

for some $G_{Y,Z}(q) \in \mathbb{A}_0$. Since $\mathcal{G}(\Lambda)$ is an \mathbb{A} -basis of $V(\Lambda)_\mathbb{A}$, $G(Y)$ can be expressed as an \mathbb{A} -linear combination of the vectors $f_{i_1}^{(r_1)} \cdots f_{i_N}^{(r_N)} Y_\Lambda$. By Lemma 7.2, it is easy to see that $G_{Y,Z}(q) \in \mathbb{Q}[q, q^{-1}]$. Moreover, since $G(Y) \equiv Y \pmod{q\mathcal{L}(\Lambda)}$, the coefficients $G_{Y,Z}(q)$ satisfy the following properties:

- (i) $G_{Y,Z}(q) \in \mathbb{Q}[q]$,
- (ii) $G_{Y,Z}(q) \in q\mathbb{Q}[q]$ unless $Y = Z$,
- (iii) $G_{Y,Y}(q) = 1$.

On the other hand, since $\mathcal{G}(\Lambda)$ and $\mathcal{A}(\Lambda)$ are both $\mathbb{Q}(q)$ -basis of $V(\Lambda)$, there exists the transition matrix $H = (H_{Y,W}(q))_{Y,W \in \mathcal{Y}(\Lambda)}$ such that

$$(7.19) \quad G(Y) = \sum_{W \in \mathcal{Y}(\Lambda)} H_{Y,W}(q) A(W),$$

where the indices are decreasing with respect to the total ordering $>$ on $\mathcal{Y}(\Lambda)$. Since $\overline{G(Y)} = G(Y)$ and $\overline{A(W)} = A(W)$, we have $H_{Y,W}(q) = H_{Y,W}(q^{-1})$ for all $Y, W \in \mathcal{Y}(\Lambda)$. The following proposition provides the last ingredients for our algorithm.

Proposition 7.12. (a) *The coefficients $H_{Y,W}(q)$ satisfy the following properties:*

- (i) $H_{Y,W}(q) \in \mathbb{Q}[q, q^{-1}]$.
- (ii) $H_{Y,W}(q) = 0$ unless $Y \geq W$ and $\text{wt}(Y) = \text{wt}(W)$.
- (iii) $H_{Y,Y}(q) = 1$.

(b) *The set $\mathcal{A}(\Lambda)$ is an \mathbb{A} -basis of $V(\Lambda)_\mathbb{A}$.*

Proof. (a) Consider the following square matrices indexed by $\mathcal{Y}(\Lambda)$

$$(7.20) \quad G = (G_{Y,Z}(q)), \quad A = (A_{W,Z}(q)),$$

where the indices are given by the total ordering $>$ in a decreasing manner. Then by (7.19), $G = HA$. Since A is an upper triangular matrix whose diagonal entries are all 1, we conclude that A is invertible and the entries of A^{-1} are in $\mathbb{Q}[q, q^{-1}]$. It follows that $H = GA^{-1}$ and $H_{Y,W}(q) \in \mathbb{Q}[q, q^{-1}]$ for all $Y, W \in \mathcal{Y}(\Lambda)$. This proves (i).

Next, let W be the reduced proper Young wall in $\mathcal{Y}(\Lambda)$ that is maximal with respect to the total ordering $>$ on $\mathcal{Y}(\Lambda)$ among the ones with $H_{Y,W}(q) \neq 0$. By the maximality of W and Proposition 7.11 (b), we have $G_{Y,W}(q) = H_{Y,W}(q)A_{W,W}(q) = H_{Y,W}(q)$. Since $G_{Y,W}(q) \in \mathbb{Q}[q]$ and $G_{Y,W}(q) = H_{Y,W}(q) =$

$H_{Y,W}(q^{-1}) = G_{Y,W}(q^{-1})$, $G_{Y,W}(q)$ must be a constant. It follows that $Y = W$ and $H_{Y,Y}(q) = G_{Y,Y}(q) = 1$. This proves (ii) and (iii).

(b) By (i) and (7.19), every element of $\mathcal{G}(\Lambda)$ can be expressed as an \mathbb{A} -linear combination of the elements in $\mathcal{A}(\Lambda)$. Hence, $\mathcal{A}(\Lambda)$ is an \mathbb{A} -basis of $V(\Lambda)_{\mathbb{A}}$. \square

Observe that, by Proposition 7.12 (a), H is invertible and H^{-1} is also an upper triangular matrix whose diagonal entries are all 1. Hence for each reduced proper Young wall $Y \in \mathcal{Y}(\Lambda)_{\lambda}$ ($\lambda \leq \Lambda$), $A(Y)$ can be expressed uniquely as

$$(7.21) \quad A(Y) = G(Y) + \sum_{\substack{Z \in \mathcal{Y}(\Lambda)_{\lambda} \\ Y > Z}} H'_{Y,Z}(q) G(Z)$$

for some $H'_{Y,Z}(q) \in \mathbb{Q}[q, q^{-1}]$ such that $H'_{Y,Z}(q) = H'_{Y,Z}(q^{-1})$.

Now, we are ready to give a generalized version of Lascoux-Leclerc-Thibon algorithm for constructing the global basis element $G(Y)$ (cf. [14]).

Fix a weight $\lambda \leq \Lambda$ of $V(\Lambda)$, and we list all the reduced proper Young walls in $\mathcal{Y}(\Lambda)_{\lambda}$ using the total ordering $>$:

$$Y_1 > Y_2 > \cdots > Y_l.$$

We will construct the basis element $G(Y_k)$ ($1 \leq k \leq l$) in a recursive way.

First, by (7.21), we have $G(Y_l) = A(Y_l)$ because Y_l is the minimal element. Suppose that we have computed $G(Y_{k+1}), \dots, G(Y_l)$. Then, by (7.21), there exist uniquely determined coefficients $\gamma_s(q) \in \mathbb{Q}[q, q^{-1}]$ ($k < s \leq l$) with $\gamma_s(q) = \gamma_s(q^{-1})$ such that

$$(7.22) \quad \begin{aligned} G(Y_k) &= A(Y_k) - \gamma_{k+1}(q)G(Y_{k+1}) \\ &\quad - \gamma_{k+2}(q)G(Y_{k+2}) - \cdots - \gamma_l(q)G(Y_l). \end{aligned}$$

Since $G(Y_k) \equiv Y_k \pmod{q\mathcal{L}(\Lambda)}$ and $\gamma_s(q) = \gamma_s(q^{-1})$, $\gamma_s(q)$ ($k < s \leq l$) are determined recursively as follows:

(G.1) if $A_{Y_k, Y_{k+1}}(q) = \sum_{i=-r}^{r'} a_i q^i$, then $\gamma_{k+1}(q) = \sum_{i=1}^r a_{-i}(q^i + q^{-i}) + a_0$.

(G.2) if the coefficient of Y_s ($s > k+1$) in $A(Y_k) - \sum_{p=k+1}^{s-1} \gamma_p(q)G(Y_p)$ is given by $\sum_{i=-r}^{r'} a_i q^i$, then $\gamma_s(q) = \sum_{i=1}^r a_{-i}(q^i + q^{-i}) + a_0$.

Using this procedure, one can construct $G(Y_k)$ ($k = 1, \dots, l$).

To summarize, we obtain the *generalized Lascoux-Leclerc-Thibon algorithm*:

Theorem 7.13. *Let Y be a reduced proper Young wall in $\mathcal{Y}(\Lambda)$. Then the corresponding global basis element $G(Y)$ can be constructed recursively using the algorithm given in (7.22), (G.1) and (G.2). Moreover $G(Y)$ has the form*

$$(7.23) \quad G(Y) = Y + \sum_{\substack{Z \in \mathcal{Z}(\Lambda) \\ |Y| \triangleright |Z|}} G_{Y,Z}(q)Z,$$

where $G_{Y,Z}(q) \in q\mathbb{Z}[q]$ for $Y \neq Z$.

By the construction of $G(Y)$ and Proposition 7.11 (b), we have

$$(7.24) \quad G_{Y,Z}(q) = 0 \quad \text{unless } |Y| \triangleright |Z^R|.$$

Hence, we can also apply the modified algorithm introduced in [18] as follows.

Let $Y \in \mathcal{Y}(\Lambda)$ be a reduced proper Young wall and suppose that $G(\bar{Y})$ and $G(Y')$ ($Y > Y'$) have been constructed. Set $C(Y) = f_i^{(r)}G(\bar{Y})$ where $\text{wt}(Y) = \text{wt}(\bar{Y}) - r\alpha_i$. Note that $C(Y) = \overline{C(Y')}$. By Lemma 7.9 (a) and Proposition 7.7, we have

$$(7.25) \quad C(Y) = Y + \sum_{|Y| \triangleright |Z^R|} C_{Y,Z}(q)Z$$

for some $C_{Y,Z}(q) \in \mathbb{Q}[q, q^{-1}]$. Hence, for each $Y' \in \mathcal{Y}(\Lambda)$, there exists uniquely determined coefficients $\zeta_{Y,Y'}(q) \in \mathbb{Q}[q, q^{-1}]$ with $\zeta_{Y,Y'}(q) = \zeta_{Y,Y'}(q^{-1})$ such that

$$(7.26) \quad G(Y) = C(Y) - \sum_{Y > Y'} \zeta_{Y,Y'}(q) G(Y').$$

Since $G(Y) \equiv Y \pmod{q\mathcal{L}(\Lambda)}$ and $\zeta_{Y,Y'}(q) = \zeta_{Y,Y'}(q^{-1})$, the coefficients $\zeta_{Y,Y'}(q)$ are determined recursively as follows:

(G'.1) if Y' is the maximal one such that $Y > Y'$ and $C_{Y,Y'}(q) = \sum_{i=-r}^r a_i q^i$, then $\zeta_{Y,Y'}(q) = \sum_{i=-r}^r a_{-i}(q^i + q^{-i}) + a_0$.

(G'.2) if the coefficient of Y' in $C(Y) - \sum_{Y' > Z > Y'} \zeta_{Y,Z}(q)G(Z)$ is given by $\sum_{i=-r}^r a_i q^i$, then $\zeta_{Y,Y'}(q) = \sum_{i=1}^r a_{-i}(q^i + q^{-i}) + a_0$.

To summarize, we obtain the *modified generalized LLT algorithm*:

Corollary 7.14. *Let Y be a reduced proper Young wall in $\mathcal{Y}(\Lambda)$. Then the corresponding global basis element $G(Y)$ can be constructed recursively using the algorithm given in (7.26), (G'.1) and (G'.2).*

In the following, we illustrate this result with several examples.

Example 7.15. Suppose that $\mathfrak{g} = A_4^{(2)}$. Note that $q_0 = q$, $q_1 = q^2$, and $q_2 = q^4$.

(a) Let Y be one of the following reduced proper Young walls:

Then Y is the smallest one among the reduced proper Young walls with the same weight with respect to the total ordering. Therefore, we have $A(Y) = G(Y)$ by (7.21) and

$$G\left(\begin{array}{|c|c|c|} \hline & & 0 \\ \hline & & 1 \\ \hline & & 2 \\ \hline 1 & 1 & \\ \hline 0 & 0 & \\ \hline 0 & 0 & \\ \hline \end{array}\right) = f_1 f_0^{(2)} f_1 f_2 f_1 f_0 Y_\Lambda = \begin{array}{|c|c|c|} \hline & & 0 \\ \hline & & 1 \\ \hline & & 2 \\ \hline 1 & 1 & \\ \hline 0 & 0 & \\ \hline 0 & 0 & \\ \hline \end{array} + q^2 \begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline 0 \\ \hline 1 \\ \hline 2 \\ \hline 1 \\ \hline 0 \\ \hline 0 \\ \hline \end{array},$$

$$G\left(\begin{array}{|c|c|c|} \hline & & 0 \\ \hline 1 & 1 & \\ \hline 2 & 2 & \\ \hline 1 & 1 & \\ \hline 0 & 0 & \\ \hline 0 & 0 & \\ \hline \end{array}\right) = f_1 f_2 f_1 f_0^{(2)} f_1 f_2 f_1 f_0 Y_\Lambda = \begin{array}{|c|c|c|} \hline & & 0 \\ \hline 1 & 1 & \\ \hline 2 & 2 & \\ \hline 1 & 1 & \\ \hline 0 & 0 & \\ \hline 0 & 0 & \\ \hline \end{array} + q^2 \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 1 \\ \hline 0 \\ \hline 0 \\ \hline 1 \\ \hline 2 \\ \hline 1 \\ \hline 0 \\ \hline 0 \\ \hline \end{array},$$

$$G\left(\begin{array}{|c|c|c|} \hline & & 0 \\ \hline & & 0 \\ \hline & & 1 \\ \hline & & 2 \\ \hline & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}\right) = f_0 f_1 f_0^{(3)} f_1 f_2 f_1 f_0 Y_\Lambda = \begin{array}{|c|c|c|} \hline & & 0 \\ \hline & & 0 \\ \hline & & 1 \\ \hline & & 2 \\ \hline & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}$$

(b) Observe that

$$\begin{aligned} & A\left(\begin{array}{|c|c|c|} \hline & & 0 \\ \hline & & 0 \\ \hline & 1 & 1 \\ \hline & 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array}\right) \\ &= f_1 f_0 f_1 f_2 f_1 f_0^{(3)} f_1 f_2 f_1 f_0 Y_\Lambda \\ &= \begin{array}{|c|c|c|} \hline & & 0 \\ \hline & & 0 \\ \hline & 1 & 1 \\ \hline & 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} + q^2 \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & & 0 \\ \hline & & 0 \\ \hline & 1 & 1 \\ \hline & 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} + (1 + q^4) \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & & 0 \\ \hline & & 0 \\ \hline & 1 & 1 \\ \hline & 2 & 2 \\ \hline 1 & 1 & \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \end{aligned}$$

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On the other hand,

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 \right) = A\left(
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 \end{array}
 \right) \\
 = f_0 f_1 f_2 f_1^{(2)} f_0^{(3)} f_1 f_2 f_1 f_0 Y_\Lambda \\
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 \begin{array}{c}
 \begin{array}{|c|c|c|c|c|}
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 + q^2
 \begin{array}{|c|c|c|c|c|}
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 + q
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 + q^3
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 0 \\
 \hline
 0 \\
 \hline
 1 \\
 \hline
 2 \\
 \hline
 1 & 1 \\
 \hline
 0 & 0 \\
 \hline
 0 & 0 \\
 \hline
 \end{array}
 \end{array}
 .$$

Therefore,

$$\begin{array}{c}
 G\left(
 \begin{array}{|c|c|c|c|c|}
 \hline
 0 \\
 \hline
 0 \\
 \hline
 1 & 1 \\
 \hline
 2 & 2 \\
 \hline
 1 & 1 & 1 \\
 \hline
 0 & 0 & 0 \\
 \hline
 0 & 0 & 0 \\
 \hline
 \end{array}
 \right) = A\left(
 \begin{array}{|c|c|c|c|c|}
 \hline
 0 \\
 \hline
 0 \\
 \hline
 1 & 1 \\
 \hline
 2 & 2 \\
 \hline
 1 & 1 & 1 \\
 \hline
 0 & 0 & 0 \\
 \hline
 0 & 0 & 0 \\
 \hline
 \end{array}
 \right) - G\left(
 \begin{array}{|c|c|c|c|c|}
 \hline
 1 \\
 \hline
 0 \\
 \hline
 0 \\
 \hline
 1 & 1 \\
 \hline
 2 & 2 \\
 \hline
 1 & 1 \\
 \hline
 0 & 0 & 0 \\
 \hline
 0 & 0 & 0 \\
 \hline
 \end{array}
 \right)$$

$$\begin{array}{|c|c|c|} \hline & & 0 \\ \hline & & 0 \\ \hline 1 & 1 & \\ \hline 2 & 2 & \\ \hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} + q^2 \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & & 0 \\ \hline & & 0 \\ \hline & & 1 \\ \hline 2 & 2 & \\ \hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} + q^4 \begin{array}{|c|c|c|} \hline & & 1 \\ \hline & & 0 \\ \hline & & 0 \\ \hline & & 0 \\ \hline 1 & 1 & \\ \hline 2 & 2 & \\ \hline 1 & 1 & \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} .$$

Example 7.16. Suppose that $\mathfrak{g} = B_3^{(1)}$. Note that $q_0 = q_1 = q_2 = q^2$ and $q_3 = q$.

(a) We have

$$G\left(\begin{array}{|c|c|c|} \hline & 0 & 1 \\ \hline 2 & 2 & \\ \hline 3 & 3 & 3 \\ \hline 2 & 2 & 2 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} \right) = A\left(\begin{array}{|c|c|c|} \hline & 0 & 1 \\ \hline 2 & 2 & \\ \hline 3 & 3 & 3 \\ \hline 2 & 2 & 2 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} \right)$$

$$= f_3 f_2 f_0^{(2)} f_2 f_3^{(2)} f_2 f_1^{(2)} f_2 f_3^{(2)} f_2 f_0 Y_{\Lambda_0}$$

$$= \begin{array}{|c|c|c|} \hline & 0 & 1 \\ \hline 2 & 2 & \\ \hline 3 & 3 & 3 \\ \hline 2 & 2 & 2 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} + q^2 \begin{array}{|c|c|c|} \hline & 0 & 1 \\ \hline 2 & 2 & \\ \hline 3 & 3 & 3 \\ \hline 2 & 2 & 2 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} + q^4 \begin{array}{|c|c|c|} \hline & & 3 \\ \hline & 2 & \\ \hline & 0 & 1 \\ \hline 2 & 2 & \\ \hline 3 & 3 & \\ \hline 3 & 3 & \\ \hline 2 & 2 & \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} + q^4 \begin{array}{|c|c|c|} \hline & & 3 \\ \hline & 2 & \\ \hline & 0 & 1 \\ \hline 2 & 2 & \\ \hline 3 & 3 & \\ \hline 3 & 3 & \\ \hline 2 & 2 & \\ \hline 1 & 0 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array} ,$$

$$G\left(\begin{array}{|c|c|c|} \hline 3 & 3 & 3 \\ \hline 2 & 2 & 2 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} \right) = A\left(\begin{array}{|c|c|c|} \hline 3 & 3 & 3 \\ \hline 2 & 2 & 2 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} \right)$$

$$= f_3 f_2 f_0 f_3 f_2 f_1 f_3 f_2 f_0 Y_{\Lambda_0}$$

$$= \begin{array}{|c|c|c|} \hline 3 & 3 & 3 \\ \hline 2 & 2 & 2 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} + q(1 - q^4) \begin{array}{|c|c|c|} \hline & & 3 \\ \hline 2 & 2 & 2 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} + q(1 + q^2) \begin{array}{|c|c|c|} \hline & & 2 \\ \hline & & 3 \\ \hline & & 3 \\ \hline 2 & 2 & \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} .$$

$$A(\begin{array}{|c|c|c|} \hline & & \begin{array}{c} 0 \\ \hline 1 \end{array} \\ \hline & \begin{array}{c} 2 \\ \hline 3 \\ \hline 3 \end{array} & \begin{array}{c} 2 \\ \hline 3 \\ \hline 3 \end{array} \\ \hline \begin{array}{c} 2 \\ \hline 0 \end{array} & \begin{array}{c} 2 \\ \hline 1 \end{array} & \begin{array}{c} 2 \\ \hline 0 \end{array} \\ \hline \begin{array}{c} 0 \\ \hline 1 \end{array} & \begin{array}{c} 1 \\ \hline 0 \end{array} & \begin{array}{c} 0 \\ \hline 1 \end{array} \\ \hline \end{array})$$

$$\begin{array}{c}
\begin{array}{|c|c|c|} \hline & 0 & 1 \\ \hline 2 & & 2 \\ \hline 3 & 3 & \\ \hline 2 & 2 & 2 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array}
+ q^2
\begin{array}{|c|c|c|} \hline & 2 & \\ \hline 0 & & 1 \\ \hline & 2 & \\ \hline 3 & 3 & \\ \hline 3 & 3 & \\ \hline 2 & 2 & 2 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array}
+ (1 + q^4)
\begin{array}{|c|c|c|} \hline & 2 & \\ \hline & 0 & 1 \\ \hline 2 & 2 & \\ \hline 3 & 3 & \\ \hline 3 & 3 & \\ \hline 2 & 2 & \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array}
\\
+ q^2
\begin{array}{|c|c|c|} \hline 2 & & \\ \hline 3 & & \\ \hline 3 & & \\ \hline 2 & & \\ \hline 0 & & 1 \\ \hline 2 & & \\ \hline 3 & & \\ \hline 3 & & \\ \hline 2 & 2 & \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array}
+ q^2
\begin{array}{|c|c|c|} \hline & 2 & \\ \hline 0 & & 1 \\ \hline 2 & 2 & \\ \hline 3 & 3 & \\ \hline 3 & 3 & \\ \hline 2 & 2 & \\ \hline 1 & 0 & 1 \\ \hline \end{array}
+ q^4
\begin{array}{|c|c|c|} \hline & 0 & \\ \hline & 2 & \\ \hline & 3 & \\ \hline & 3 & \\ \hline & 2 & \\ \hline 0 & & 1 \\ \hline 2 & & \\ \hline & 3 & \\ \hline & 3 & \\ \hline 2 & 2 & \\ \hline 1 & 0 & 1 \\ \hline \end{array}
\end{array}$$

$$G\left(\begin{array}{c} 2 \\ 0/1 \\ 2 \\ 2 \\ 3/3 \\ 3/3 \\ 2 \\ 2 \\ 0/1 \\ 1/0 \\ 0/1 \end{array} \right) = A\left(\begin{array}{c} 2 \\ 0/1 \\ 2 \\ 2 \\ 3/3 \\ 3/3 \\ 2 \\ 2 \\ 0/1 \\ 1/0 \\ 0/1 \end{array} \right)$$

$$= f_0 f_2 f_3^{(2)} f_2^{(2)} f_1^{(2)} f_0 f_2 f_3^{(2)} f_2 f_0 Y_{\Lambda_0}$$

$$= \begin{array}{|c|c|c|} \hline & & 2 \\ \hline & 0 & 1 \\ \hline 2 & 2 & \\ \hline 3 & 3 & \\ \hline 3 & 3 & \\ \hline 2 & 2 & \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} + q^2 \begin{array}{|c|c|c|} \hline 2 \\ \hline 3 \\ \hline 3 \\ \hline 2 \\ \hline 0 & 1 \\ \hline 2 \\ \hline 2 \\ \hline 3 \\ \hline 3 \\ \hline 2 & 2 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} + q^2 \begin{array}{|c|c|c|} \hline & & 2 \\ \hline & 0 & 1 \\ \hline 2 & 2 & \\ \hline 3 & 3 & \\ \hline 3 & 3 & \\ \hline 2 & 2 & \\ \hline 1 & 0 & 1 \\ \hline \end{array} + q^4 \begin{array}{|c|c|c|} \hline 0 \\ \hline 2 \\ \hline 3 \\ \hline 3 \\ \hline 2 \\ \hline 0 & 1 \\ \hline 2 \\ \hline 3 \\ \hline 3 \\ \hline 2 & 2 \\ \hline 1 & 0 & 1 \\ \hline \end{array}.$$

Hence,

$$G\left(\begin{array}{|c|c|c|} \hline & & 0 \\ \hline & & 1 \\ \hline 2 & 2 & \\ \hline 3 & 3 & \\ \hline 3 & 3 & \\ \hline 2 & 2 & 2 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array}\right) = A\left(\begin{array}{|c|c|c|} \hline & & 0 \\ \hline & & 1 \\ \hline 2 & 2 & \\ \hline 3 & 3 & \\ \hline 3 & 3 & \\ \hline 2 & 2 & 2 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array}\right) - G\left(\begin{array}{|c|c|c|} \hline & & 2 \\ \hline & 0 & 1 \\ \hline 2 & 2 & \\ \hline 3 & 3 & \\ \hline 3 & 3 & \\ \hline 2 & 2 & \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array}\right)$$

$$= \begin{array}{|c|c|c|} \hline & & 0 \\ \hline & & 1 \\ \hline 2 & 2 & \\ \hline 3 & 3 & \\ \hline 3 & 3 & \\ \hline 2 & 2 & 2 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} + q^2 \begin{array}{|c|c|c|} \hline & & 2 \\ \hline & 0 & 1 \\ \hline 3 & 3 & \\ \hline 3 & 3 & \\ \hline 2 & 2 & 2 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} + q^4 \begin{array}{|c|c|c|} \hline & & 2 \\ \hline & 0 & 1 \\ \hline 2 & 2 & \\ \hline 3 & 3 & \\ \hline 3 & 3 & \\ \hline 2 & 2 & \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array}.$$

8. Appendix

In this section, we list the patterns for building the walls which are given in [7].

(a) $A_n^{(1)}$ ($n \geq 1$),

On Y_{A_i} :

					0	1
					n	0
					$n-1$	n
					\vdots	\vdots
0	1	2	3	\cdots	i	$i+1$
n	0	1	2	\cdots	$i-1$	i

(b) $A_{2n-1}^{(2)}$ ($n \geq 3$),On Y_{Λ_0} :

	2	2	2	2
1	0	1	0	1
0	1	0	1	0
2	2	2	2	2
\vdots	\vdots	\vdots	\vdots	\vdots
n	n	n	n	n
\vdots	\vdots	\vdots	\vdots	\vdots
2	2	2	2	2
1	0	1	0	1
0	1	0	1	0

On Y_{Λ_1} :

	2	2	2	2
0	1	0	1	0
1	0	1	0	1
2	2	2	2	2
\vdots	\vdots	\vdots	\vdots	\vdots
n	n	n	n	n
\vdots	\vdots	\vdots	\vdots	\vdots
2	2	2	2	2
0	1	0	1	0
1	0	1	0	1

(c) $D_n^{(1)}$ ($n \geq 4$),On Y_{Λ_0} :

	2	2	2	2
1	0	1	0	1
0	1	0	1	0
2	2	2	2	2
\vdots	\vdots	\vdots	\vdots	\vdots
$n-2$	$n-2$	$n-2$	$n-2$	$n-2$
n	$n-1$	n	$n-1$	n
$n-1$	n	$n-1$	n	$n-1$
$n-2$	$n-2$	$n-2$	$n-2$	$n-2$
\vdots	\vdots	\vdots	\vdots	\vdots
2	2	2	2	2
1	0	1	0	1
0	1	0	1	0

On Y_{Λ_1} :

	2	2	2	2
0	1	0	1	0
1	0	1	0	1
2	2	2	2	2
\vdots	\vdots	\vdots	\vdots	\vdots
$n-2$	$n-2$	$n-2$	$n-2$	$n-2$
n	$n-1$	n	$n-1$	n
$n-1$	n	$n-1$	n	$n-1$
$n-2$	$n-2$	$n-2$	$n-2$	$n-2$
\vdots	\vdots	\vdots	\vdots	\vdots
2	2	2	2	2
0	1	0	1	0
1	0	1	0	1

On $Y_{\Lambda_{n-1}}$:

$n-2$	$n-2$	$n-2$	$n-2$
n	$n-1$	n	$n-1$
$n-1$	n	$n-1$	n
$n-2$	$n-2$	$n-2$	$n-2$
\vdots	\vdots	\vdots	\vdots
2	2	2	2
1	0	1	0
0	1	0	1
2	2	2	2
\vdots	\vdots	\vdots	\vdots
$n-2$	$n-2$	$n-2$	$n-2$
n	$n-1$	n	$n-1$
$n-1$	n	$n-1$	n

On Y_{Λ_n} :

$n-2$	$n-2$	$n-2$	$n-2$
$n-1$	n	$n-1$	n
n	$n-1$	n	$n-1$
$n-2$	$n-2$	$n-2$	$n-2$
\vdots	\vdots	\vdots	\vdots
2	2	2	2
1	0	1	0
0	1	0	1
2	2	2	2
\vdots	\vdots	\vdots	\vdots
$n-2$	$n-2$	$n-2$	$n-2$
$n-1$	n	$n-1$	n
n	$n-1$	n	$n-1$

(d) $A_{2n}^{(2)}$ ($n \geq 1$)On Y_{Λ_0} :

1	1	1	1
0	0	0	0
0	0	0	0
1	1	1	1
\vdots	\vdots	\vdots	\vdots
n	n	n	n
\vdots	\vdots	\vdots	\vdots
1	1	1	1
0	0	0	0
0	0	0	0

(e) $D_{n+1}^{(2)}$ ($n \geq 2$),On Y_{Λ_0} :

1	1	1	1
0	0	0	0
0	0	0	0
1	1	1	1
\vdots	\vdots	\vdots	\vdots
$n-1$	$n-1$	$n-1$	$n-1$
n	n	n	n
n	n	n	n
$n-1$	$n-1$	$n-1$	$n-1$
\vdots	\vdots	\vdots	\vdots
1	1	1	1
0	0	0	0
0	0	0	0

On Y_{Λ_n} :

$n-1$	$n-1$	$n-1$	$n-1$
n	n	n	n
n	n	n	n
$n-1$	$n-1$	$n-1$	$n-1$
\vdots	\vdots	\vdots	\vdots
1	1	1	1
0	0	0	0
0	0	0	0
1	1	1	1
\vdots	\vdots	\vdots	\vdots
$n-1$	$n-1$	$n-1$	$n-1$
n	n	n	n
n	n	n	n

(f) $B_n^{(1)}$ ($n \geq 3$),On Y_{Λ_0} :

2	2	2	2
1	0	1	0
0	1	0	1
2	2	2	2
\vdots	\vdots	\vdots	\vdots
$n-1$	$n-1$	$n-1$	$n-1$
n	n	n	n
n	n	n	n
$n-1$	$n-1$	$n-1$	$n-1$
\vdots	\vdots	\vdots	\vdots
2	2	2	2
1	0	1	0
0	1	0	1

On Y_{Λ_1} :

2	2	2	2
0	1	0	1
1	0	1	0
2	2	2	2
\vdots	\vdots	\vdots	\vdots
$n-1$	$n-1$	$n-1$	$n-1$
n	n	n	n
n	n	n	n
$n-1$	$n-1$	$n-1$	$n-1$
\vdots	\vdots	\vdots	\vdots
2	2	2	2
0	1	0	1
1	0	1	0

On Y_{Λ_n} :

$n-1$	$n-1$	$n-1$	$n-1$
n	n	n	n
n	n	n	n
$n-1$	$n-1$	$n-1$	$n-1$
\vdots	\vdots	\vdots	\vdots
2	2	2	2
$\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$	$\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}$	$\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}$
2	2	2	2
\vdots	\vdots	\vdots	\vdots
$n-1$	$n-1$	$n-1$	$n-1$
n	n	n	n
n	n	n	n

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