J. Korean Math. Soc. **45** (2008), No. 4, pp. 1113–1133

REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH LIE ξ-PARALLEL NORMAL JACOBI OPERATOR

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ABSTRACT. In this paper we give some non-existence theorems for real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with Lie ξ-parallel normal Jacobi operator $\bar{R}_N$ and another geometric conditions.

0. Introduction

In the geometry of real hypersurfaces in complex space forms $M_n(c)$ or in quaternionic space forms $\mathbb{Q}_n(c)$ Kimura [7] (resp. Pérez [10]) has classified real hypersurfaces in $M_n(c)$ and (resp. in $\mathbb{Q}_n(c)$) with commuting Ricci tensor, that is, $S\phi = \phi S$, (resp. $S\phi_i = \phi_i S$, $i = 1, 2, 3$) where S and ϕ (resp. S and ϕ_i, $i = 1, 2, 3$) denote the Ricci tensor and the structure tensor of a real hypersurface M in $M_n(c)$ (resp. in $\mathbb{Q}_n(c)$).

In particular, Kimura and Maeda [8] have considered a real hypersurface M in a complex projective space $P_n(\mathbb{C})$ with Lie ξ-parallel Ricci tensor and classified that M is locally congruent to of type (A), a tube over a totally geodesic $P_k(\mathbb{C})$, of type (B), a tube over a complex quadric Q_{n-1}, $\cot^2 2r = n-2$, of type (C), a tube over $P_1(\mathbb{C}) \times P_{(n-1)/2}(\mathbb{C})$, $\cot^2 2r = \frac{1}{n-2}$ and n is odd, of type (D), a tube over a complex two-plane Grassmannian $G_2(\mathbb{C}^5)$, $\cot^2 2r = \frac{3}{5}$ and $n = 9$, of type (E), a tube over a Hermitian symmetric space $SO(10)/U(5)$, $\cot^2 2r = \frac{5}{9}$ and $n = 15$. Then it turns out that all of them mentioned above are Hopf hypersurfaces and have commuting Ricci tensors.

If the structure vector $\xi = -JN$ of a real hypersurface M in $P_n(\mathbb{C})$ is invariant by the shape operator, M is said to be a Hopf hypersurface, where J denotes a Kaehler structure of $P_n(\mathbb{C})$, N a unit normal vector of M in $P_n(\mathbb{C})$.

In a quaternionic projective space $\mathbb{Q}P^m$ Pérez and the second author [11] have classified real hypersurfaces in $\mathbb{Q}P^m$ with $\mathfrak{D}^\perp$-parallel curvature tensor $\nabla_{\xi_i} R = 0$, $i = 1, 2, 3$, where R denotes the curvature tensor of M in $\mathbb{Q}P^m$

Received January 2, 2007; Revised March 21, 2007.

2000 *Mathematics Subject Classification.* Primary 53C40; Secondary 53C15.

Key words and phrases. complex two-plane Grassmannians, real hypersurfaces, Lie ξ-parallel, normal Jacobi operator Hopf hypersurfaces.

This work was supported by Kyungpook National University Research Fund, 2007.

and $\mathfrak{D}^\perp$ a distribution defined by $\mathfrak{D}^\perp = \text{Span}\ \{\xi_1, \xi_2, \xi_3\}$. In such a case they are congruent to a tube of radius $\frac{\pi}{4}$ over a totally geodesic $\mathbb{Q}P^k$ in $\mathbb{Q}P^m$, $2 \leq k \leq m-2$.

The almost contact structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ mentioned above are defined by $\xi_i = -J_i N$, $i = 1, 2, 3$, where $\{J_1, J_2, J_3\}$ denote a quaternionic Kähler structure of $\mathbb{Q}P^m$ and N a unit normal field of M in $\mathbb{Q}P^m$.

In quaternionic space forms Berndt [2] has introduced the notion of normal Jacobi operator $\bar{R}_N = \bar{R}(X, N)N \in \text{End}\ T_xM$, $x \in M$ for real hypersurfaces M in a quaternionic projective space $\mathbb{Q}P^m$ or in a quaternionic hyperbolic space $\mathbb{Q}H^m$, where $\bar{R}$ denotes the curvature tensor of $\mathbb{Q}P^m$ and $\mathbb{Q}H^m$ respectively. He [2] has also shown that the curvature adaptedness, that is, the normal Jacobi operator $\bar{R}_N$ commutes with the shape operator A, is equivalent to the fact that the distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant by the shape operator A of M, where $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$.

Now let us consider a complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ which consists of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Then the situation for real hypersurfaces in $G_2(\mathbb{C}^{m+1})$ with parallel normal Jacobi operator is not so simple and will be quite different from the cases mentioned above.

Now in this paper we consider a real hypersurface M in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with Lie ξ-parallel normal Jacobi operator, $\mathcal{L}_\xi \bar{R}_N = 0$, where $\bar{R}$ and N respectively denotes the curvature tensor of the ambient space $G_2(\mathbb{C}^{m+2})$ and a unit normal vector of M in $G_2(\mathbb{C}^{m+2})$. The curvature tensor $\bar{R}(X, Y)Z$ for any vector fields X, Y and Z on $G_2(\mathbb{C}^{m+2})$ is explicitly defined in section 1. Then the normal Jacobi operator $\bar{R}_N$ for the unit normal vector N can be defined from the curvature tensor $\bar{R}(X, N)N$ by putting $Y = Z = N$.

The ambient space $G_2(\mathbb{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J (See Berndt [3]). So, in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometric conditions for real hypersurfaces that $[\xi] = \text{Span}\ \{\xi\}$ or $\mathfrak{D}^\perp = \text{Span}\ \{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. By using such kinds of conditions Berndt and the second author [4] have proved the following:

Theorem A. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

- *(A) M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*
- *(B) m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

If the structure vector field ξ of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is invariant by the shape operator, M is said to be a *Hopf hypersurface.* In such a case the integral curves of the structure vector field ξ are geodesics (See

Berndt and Suh [5]). Moreover, the flow generated by the integral curves of the structure vector field ξ for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is said to be *geodesic Reeb flow*. Moreover, we say that the Reeb vector field is Killing, that is, $\mathcal{L}_\xi g = 0$ for the Lie derivative along the direction of the structure vector field ξ, where g denotes the Riemannian metric induced from $G_2(\mathbb{C}^{m+2})$. Then this is equivalent to the fact that the structure tensor ϕ commutes with the shape operator A of M in $G_2(\mathbb{C}^{m+2})$. This condition also has the geometric meaning that the flow of Reeb vector field is isometric. Moreover, Berndt and the second author [5] have proved that real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with isometric flow is of a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Now by putting a unit normal vector N into the curvature tensor $\bar{R}$ of the ambient space $G_2(\mathbb{C}^{m+2})$, we calculate the normal Jacobi operator $\bar{R}_N$ in such a way that

$$\begin{aligned}\bar{R}(X,N)N &= X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\ &\quad - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)J_\nu(\phi X + \eta(X)N) - \eta_\nu(\phi X)(\phi_\nu\xi + \eta_\nu(\xi)N)\} \\ &= X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\ &\quad - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)(\phi_\nu\phi X - \eta(X)\xi_\nu) - \eta_\nu(\phi X)\phi_\nu\xi\}\end{aligned}$$

for any tangent vector field X on M in $G_2(\mathbb{C}^{m+2})$.

On the other hand, we introduce the following theorem due to Pérez and the present authors [6] as follows:

Theorem B. *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the normal Jacobi and the structure operators both commute with the shape operator, then M is congruent to one of the following:*

(*A*) *an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(*B*) *an open part of a tube around a totally geodesic and totally real $\mathbb{Q}P^n$, $m = 2n$, in $G_2(\mathbb{C}^{m+2})$.*

But related to the normal Jacobi operator $\bar{R}_N$, in this paper we want to give some non-existence theorems for real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ with Lie ξ-parallel normal Jacobi operator, that is, $\mathcal{L}_\xi \bar{R}_N = 0$ as follows:

Theorem 1. *There do not exist any real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying $\mathcal{L}_\xi \bar{R}_N = 0$ and $\xi \in \mathfrak{D}^\perp$.*

Theorem 2. *There do not exist any real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying $\mathcal{L}_\xi \bar{R}_N = 0$ and $\xi \in \mathfrak{D}$.*

On the other hand, we say that a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ has *commuting shape operator* on the distribution $\mathfrak{D}^\perp$ if the shape operator A of M commutes with the structure tensor ϕ on $\mathfrak{D}^\perp$, that is, $A\phi\xi_\nu = \phi A\xi_\nu$, $\nu = 1, 2, 3$.

Now in the final section, as an application of Theorems 1 and 2 we consider a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with Lie ξ-parallel and commuting shape operator on the distribution $\mathfrak{D}^\perp$. Then by virtue of Theorems 1 and 2 we assert the following:

Theorem 3. *There do not exist any Hopf real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\mathcal{L}_\xi \bar{R}_N = 0$ and commuting shape operator on the distribution $\mathfrak{D}^\perp$.*

1. Riemannian geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details we refer to [3], [4], and [5]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan-Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $Ad(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $Ad(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight. Since $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight we will assume $m \geq 2$ from now on. Note that the isomorphism $\mathrm{Spin}(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^6$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ is the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kähler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kähler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_1 is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $\mathrm{tr}(JJ_1) = 0$. This fact will be used frequently throughout this paper.

A canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{1.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

Let $p \in G_2(\mathbb{C}^{m+2})$ and W a subspace of $T_pG_2(\mathbb{C}^{m+2})$. We say that W is a quaternionic subspace of $T_pG_2(\mathbb{C}^{m+2})$ if $JW \subset W$ for all $J \in \mathfrak{J}_p$. And we say that W is a totally complex subspace of $T_pG_2(\mathbb{C}^{m+2})$ if there exists a one-dimensional subspace $\mathfrak{V}$ of $\mathfrak{J}_p$ such that $JW \subset W$ for all $J \in \mathfrak{V}$ and $JW \perp W$ for all $J \in \mathfrak{V}^{\perp} \subset \mathfrak{J}_p$. Here, the orthogonal complement of $\mathfrak{V}$ in $\mathfrak{J}_p$ is taken with respect to the bundle metric and orientation on $\mathfrak{J}$ for which any local oriented orthonormal frame field of $\mathfrak{J}$ is a canonical local basis of $\mathfrak{J}$. A quaternionic (resp. totally complex) submanifold of $G_2(\mathbb{C}^{m+2})$ is a submanifold all of whose tangent spaces are quaternionic (resp. totally complex) subspaces of the corresponding tangent spaces of $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\bar{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}
&\bar{R}(X,Y)Z \\
&= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\
&\quad - g(JX,Z)JY - 2g(JX,Y)JZ \\
&\quad + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} \\
&\quad + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},
\end{aligned} \tag{1.2}$$

where J_1, J_2, J_3 is any canonical local basis of $\mathfrak{J}$.

2. Some fundamental formulas for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

Now in this section we want to derive the normal Jacobi operator from the curvature tensor of complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ given in (1.2) and the equation of Gauss. Moreover, in this section we derive some basic formulae from the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (See [4], [5], [13], [14], and [15]).

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M,g). Let N be a local unit normal field of M and A the shape operator of M with respect to N. The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ induces on M an almost contact metric structure (ϕ,ξ,η,g). Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu,\xi_\nu,\eta_\nu,g)$ on M. Using the above expression for $\bar{R}$, the Codazzi equation becomes

$$\begin{aligned}
&(\nabla_X A)Y - (\nabla_Y A)X \\
&= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y-\eta_\nu(Y)\phi_\nu X-2g(\phi_\nu X,Y)\xi_\nu\}\\
&+\sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y-\eta_\nu(\phi Y)\phi_\nu\phi X\}\\
&+\sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y)-\eta(Y)\eta_\nu(\phi X)\}\xi_\nu.
\end{aligned}
$$

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

$$
\begin{aligned}
&\phi_{\nu+1}\xi_\nu=-\xi_{\nu+2},\quad \phi_\nu\xi_{\nu+1}=\xi_{\nu+2},\\
&\phi\xi_\nu=\phi_\nu\xi,\quad \eta_\nu(\phi X)=\eta(\phi_\nu X),\\
&\phi_\nu\phi_{\nu+1}X=\phi_{\nu+2}X+\eta_{\nu+1}(X)\xi_\nu,\\
&\phi_{\nu+1}\phi_\nu X=-\phi_{\nu+2}X+\eta_\nu(X)\xi_{\nu+1}.
\end{aligned}
\tag{2.1}
$$

Now let us put

$$
JX=\phi X+\eta(X)N,\quad J_\nu X=\phi_\nu X+\eta_\nu(X)N
\tag{2.2}
$$

for any tangent vector X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a normal vector of M in $G_2(\mathbb{C}^{m+2})$. Then from this and the formulas (1.1) and (2.1) we have that

$$
(\nabla_X\phi)Y=\eta(Y)AX-g(AX,Y)\xi,\quad \nabla_X\xi=\phi AX,
\tag{2.3}
$$

$$
\nabla_X\xi_\nu=q_{\nu+2}(X)\xi_{\nu+1}-q_{\nu+1}(X)\xi_{\nu+2}+\phi_\nu AX,
\tag{2.4}
$$

$$
\begin{aligned}
(\nabla_X\phi_\nu)Y=&-q_{\nu+1}(X)\phi_{\nu+2}Y+q_{\nu+2}(X)\phi_{\nu+1}Y+\eta_\nu(Y)AX\\
&-g(AX,Y)\xi_\nu.
\end{aligned}
\tag{2.5}
$$

Summing up these formulas, we find the following

$$
\begin{aligned}
\nabla_X(\phi_\nu\xi)&=\nabla_X(\phi\xi_\nu)\\
&=(\nabla_X\phi)\xi_\nu+\phi(\nabla_X\xi_\nu)\\
&=q_{\nu+2}(X)\phi_{\nu+1}\xi-q_{\nu+1}(X)\phi_{\nu+2}\xi+\phi_\nu\phi AX\\
&\quad-g(AX,\xi)\xi_\nu+\eta(\xi_\nu)AX.
\end{aligned}
\tag{2.6}
$$

Moreover, from $JJ_\nu=J_\nu J$, $\nu=1,2,3$, it follows that

$$
\phi\phi_\nu X=\phi_\nu\phi X+\eta_\nu(X)\xi-\eta(X)\xi_\nu.
\tag{2.7}
$$

3. Lie ξ-parallel normal Jacobi operator

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie ξ-parallel normal Jacobi operator, that is, $\mathfrak{L}_\xi \bar{R}_N = 0$. Then first of all, we write the normal Jacobi operator $\bar{R}_N$, which is given by

(3.1)
$$\begin{aligned}\bar{R}_N(X) &= \bar{R}(X,N)N \\ &= X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\ &\quad - \sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)J_\nu(\phi X + \eta(X)N) - \eta_\nu(\phi X)(\phi_\nu\xi + \eta_\nu(\xi)N)\Big\} \\ &= X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\xi_\nu \\ &\quad - \sum_{\nu=1}^{3}\Big\{\eta_\nu(\xi)(\phi_\nu\phi X - \eta(X)\xi_\nu) - \eta_\nu(\phi X)\phi_\nu\xi\Big\},\end{aligned}$$

where we have used the following

$$\begin{aligned} g(J_\nu JN, N) &= -g(JN, J_\nu N) = -g(\xi,\xi_\nu) = -\eta_\nu(\xi), \\ g(J_\nu JX, N) &= g(X, JJ_\nu N) = -g(X, J\xi_\nu) \\ &= -\,g(X, \phi\xi_\nu + \eta(\xi_\nu)N) = -g(X,\phi\xi_\nu), \end{aligned}$$

and

$$J_\nu JN = -J_\nu\xi = -\phi_\nu\xi - \eta_\nu(\xi)N.$$

Of course, by (2.7) we know that the normal Jacobi operator $\bar{R}_N$ could be symmetric endomorphism of T_xM, $x\in M$.

Now let us consider a Lie derivative of the normal Jacobi operator along the direction ξ. Then it is given by

$$\begin{aligned}(\mathcal{L}_\xi\bar{R}_N)X &= \mathcal{L}_\xi(\bar{R}_N X) - \bar{R}_N(\mathcal{L}_\xi X) \\ &= [\xi, \bar{R}_N X] - \bar{R}_N[\xi, X] \\ &= (\nabla_\xi\bar{R}_N)X - \phi A\bar{R}_N X + \bar{R}_N\phi AX,\end{aligned} \tag{3.2}$$

where the terms in the right side can be given respectively as follows:

$$\begin{aligned}(\nabla_\xi\bar{R}_N)X &= 3(\nabla_\xi\eta)(X)\xi + 3\eta(X)\nabla_\xi\xi + 3\sum_{\nu=1}^{3}(\nabla_\xi\eta_\nu)(X)\xi_\nu \\ &\quad + 3\sum_{\nu=1}^{3}\eta_\nu(X)\nabla_\xi\xi_\nu - \sum_{\nu=1}^{3}\Big[\xi(\eta_\nu(\xi))(\phi_\nu\phi X - \eta(X)\xi_\nu) \\ &\quad + \eta_\nu(\xi)\big\{(\nabla_\xi\phi_\nu\phi)X - (\nabla_\xi\eta)(X)\xi_\nu - \eta(X)\nabla_\xi\xi_\nu\big\} \\ &\quad - (\nabla_\xi\eta_\nu)(\phi X)\phi_\nu\xi - \eta_\nu((\nabla_\xi\phi)X)\phi_\nu\xi - \eta_\nu(\phi X)\nabla_\xi(\phi_\nu\xi)\Big],\end{aligned}$$

$$\begin{aligned}\phi A\bar{R}_N X &= \phi AX + 3\eta(X)\phi A\xi + 3\sum_{\nu=1}^{3}\eta_\nu(X)\phi A\xi_\nu \\ &\quad - \sum_{\nu=1}^{3}\Big[\eta_\nu(\xi)(\phi A\phi_\nu\phi X - \eta(X)\phi A\xi_\nu) - \eta_\nu(\phi X)\phi A\phi_\nu\xi\Big]\end{aligned}$$

and

$$\bar{R}_N\phi AX = \phi AX + 3\sum_{\nu=1}^{3}\eta_\nu(\phi AX)\xi_\nu - \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu\phi^2 AX - \eta_\nu(\phi^2 AX)\phi_\nu\xi\}.$$

Then by the formulas given in section 2, a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with Lie ξ-parallel of $\bar{R}_N$ along the direction of ξ and satisfies the following

(3.3)

$$\begin{aligned}
(\mathcal{L}_\xi\bar{R}_N)X &= (\nabla_\xi\bar{R}_N)X - \phi A\bar{R}_N X + \bar{R}_N\phi AX \\
&= 3g(\phi A\xi, X)\xi + 3\sum_{\nu=1}^{3}g(\phi_\nu A\xi, X)\xi_\nu + 3\sum_{\nu=1}^{3}\eta_\nu(X)\phi_\nu A\xi \\
&\quad - \sum_{\nu=1}^{3}\Big[\xi(\eta_\nu(\xi))(\phi_\nu\phi X - \eta(X)\xi_\nu) \\
&\quad + \eta_\nu(\xi)\big\{ - q_{\nu+1}(\xi)\phi_{\nu+2}\phi X + q_{\nu+2}(\xi)\phi_{\nu+1}\phi X \\
&\quad + \eta_\nu(\phi X)A\xi - g(A\xi, \phi X)\xi_\nu + \eta(X)\phi_\nu A\xi \\
&\quad - g(A\xi, X)\phi_\nu\xi - g(\phi A\xi, X)\xi_\nu \\
&\quad - \eta(X)(q_{\nu+2}(\xi)\xi_{\nu+1} - q_{\nu+1}(\xi)\xi_{\nu+2} + \phi_\nu A\xi)\big\} \\
&\quad - g(\phi_\nu A\xi, \phi X)\phi_\nu\xi - \eta(X)\eta_\nu(A\xi)\phi_\nu\xi + g(A\xi, X)\eta_\nu(\xi)\phi_\nu\xi \\
&\quad - \eta_\nu(\phi X)\big\{\eta_\nu(\xi)A\xi - g(A\xi, \xi)\xi_\nu + \phi_\nu\phi A\xi\big\}\Big] \\
&\quad - 3\sum_{\nu=1}^{3}\eta_\nu(X)\phi A\xi_\nu + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)(\phi A\phi_\nu\phi X - \eta(X)\phi A\xi_\nu) \\
&\quad - \eta_\nu(\phi X)\phi A\phi_\nu\xi\} + 3\sum_{\nu=1}^{3}\eta_\nu(\phi AX)\xi_\nu \\
&\quad + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu AX - \eta_\nu(AX)\phi_\nu\xi\} = 0,
\end{aligned}$$

where in the second equality we have used the following formulas

$$\begin{aligned}
&3\sum_{\nu=1}^{3}g(q_{\nu+2}(\xi)\xi_{\nu+1} - q_{\nu+1}(\xi)\xi_{\nu+2}, X)\xi_\nu \\
&+ 3\sum_{\nu=1}^{3}\eta_\nu(X)\big\{q_{\nu+2}(\xi)\xi_{\nu+1} - q_{\nu+1}(\xi)\xi_{\nu+2}\big\} = 0
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\nu=1}^{3}\{\eta_{\nu+1}(\phi X)q_{\nu+2}(\xi)\phi_\nu\xi - \eta_{\nu+2}(\phi X)q_{\nu+1}(\xi)\phi_\nu\xi \\
&- \eta_\nu(\phi X)q_{\nu+1}(\xi)\phi_{\nu+2}\xi + \eta_\nu(\phi X)q_{\nu+2}(\xi)\phi_{\nu+1}\xi\big\} = 0.
\end{aligned}$$

From this, by putting $X = \xi$ and using the formulas in Section 2 we have the following

$$\begin{aligned}
(\mathcal{L}_\xi\bar{R}_N)\xi &= 6\sum_{\nu=1}^{3}g(\phi_\nu A\xi, \xi)\xi_\nu + 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu A\xi \\
&\quad + \sum_{\nu=1}^{3}\Big[\xi(\eta_\nu(\xi))\xi_\nu + \eta_\nu(\xi)\big\{q_{\nu+2}(\xi)\xi_{\nu+1} - q_{\nu+1}(\xi)\xi_{\nu+2}\big\}\Big] \\
&\quad - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\phi A\xi_\nu = 0.
\end{aligned} \tag{3.4}$$

4. Lie ξ-parallel normal Jacobi operator for $\xi \in \mathfrak{D}^{\perp}$

In this section we want to give a complete proof of Theorem 1. In order to do this, we consider the case that $\xi \in \mathfrak{D}^{\perp}$. Accordingly, we may put $\xi = \xi_1$. Then (3.1) implies the following for any X on M

$$
\begin{aligned}
0 = {} & 3g(\phi A\xi, X)\xi + 3\sum_{\nu=1}^{3} g(\phi_\nu A\xi, X)\xi_\nu + 3\sum_{\nu=1}^{3} \eta_\nu(X)\phi_\nu A\xi \\
& + q_2(\xi)\phi_3\phi X - q_3(\xi)\phi_2\phi X + \eta(X)\{q_3(\xi)\xi_2 - q_2(\xi)\xi_3\} \\
& - g(\phi_2 A\xi, \phi X)\xi_3 - \eta(X)\eta_2(A\xi)\xi_3 + g(\phi_3 A\xi, \phi X)\xi_2 \\
& + \eta(X)\eta_3(A\xi)\xi_2 + \alpha\{\eta_2(X)\xi_3 - \eta_3(X)\xi_2\} \\
& + \eta_3(X)\phi_2\phi A\xi - \eta_2(X)\phi_3\phi A\xi \\
& - 3\sum_{\nu=1}^{3} \eta_\nu(X)\phi A\xi_\nu + \phi A\phi_1\phi X - \eta(X)\phi A\xi_1 \\
& + \eta_3(X)\phi A\xi_3 + \eta_2(X)\phi A\xi_2 + 3\{\eta_3(AX)\xi_2 - \eta_2(AX)\xi_3\} \\
& + \phi_1 AX + \eta_2(AX)\xi_3 - \eta_3(AX)\xi_2,
\end{aligned}
\tag{4.1}
$$

where α denotes $g(A\xi, \xi)$.

On the other hand, from $\nabla_X \xi_1 = \nabla_X \xi$ we know that

$$
q_2(\xi) = 2g(A\xi, \xi_2), \quad q_3(\xi) = 2g(A\xi, \xi_3). \tag{4.2}
$$

By putting $X = \xi_2$ in (4.1), we have

$$
\begin{aligned}
0 &= (\mathcal{L}_\xi \bar{R}_N)\xi_2 \\
&= 3g(A\xi, \xi_1)\xi_3 + 3\phi_2 A\xi + q_3(\xi)\xi_1 - \phi_3\phi A\xi - \phi A\xi_2 \\
&\quad + 2\{\eta_3(A\xi_2)\xi_2 - \eta_2(A\xi_2)\xi_3\} + \phi_1 A\xi_2.
\end{aligned}
$$

From this, taking an inner product with ξ_1, we have

$$
0 = 3g(A\xi, \xi_3) + q_3(\xi) + g(A\xi, \xi_3).
$$

Then from this, together with (4.2), it follows that

$$
q_3(\xi) = 0 \quad \text{and} \quad g(A\xi, \xi_3) = 0.
$$

Similarly, by putting $X = \xi_3$ in (4.1) we have

$$
\begin{aligned}
0 &= (\mathcal{L}_\xi \bar{R}_N)\xi_3 \\
&= -3g(A\xi, \xi)\xi_2 + 3\phi_3 A\xi - q_2(\xi)\xi_1 \\
&\quad + \phi_2\phi A\xi - \phi A\xi_3 + \phi_1 A\xi_3 \\
&\quad + 2g(A\xi_3, \xi_3)\xi_2 - 2g(A\xi_3, \xi_2)\xi_3.
\end{aligned}
$$

From this, by taking an inner product with ξ_1 and using (4.2) we have

$$
q_2(\xi) = 0 \quad \text{and} \quad g(A\xi, \xi_2) = 0.
$$

Then we may summarize such a fact as follows:

Lemma 4.1. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying Lie ξ-parallel normal Jacobi operator and $\xi \in \mathfrak{D}^\perp$. Then $A\xi = \alpha\xi + \beta U$, where U is a unit vector field orthogonal to ξ and belongs to $\mathfrak{D}$.*

From Lemma 4.1 we can prove the following

Lemma 4.2. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying Lie ξ-parallel normal Jacobi operator and $\xi \in \mathfrak{D}^\perp$. Then β identically vanishes, that is, the structure vector ξ is principal.*

Proof. By Lemma 4.1 we may put

$$A\xi = \alpha\xi + \beta U \tag{4.3}$$

for some unit normal U orthogonal to the structure vector ξ. Now let us construct an open set $\mathfrak{V}$ in such a way that $\mathfrak{V} = \{p \in M | \beta(p) \neq 0\}$. Then on such an open $\mathfrak{V}$ we proceed our assertion. Now substituting (4.3) into (4.1), we have the following

$$\begin{aligned} 0 = {} & 3\beta\phi_2 U - \beta\phi_3\phi U - \phi A\xi_2 \\ & + 2g(A\xi_2, \xi_3)\xi_2 - 2g(A\xi_2, \xi_2)\xi_3 + \phi_1 A\xi_2. \end{aligned}$$

From this, by taking an inner product with $\phi_2 U$ we have

$$\begin{aligned} 0 = {} & -3\beta g(\phi_2 U, \phi_2 U) - \beta g(\phi_3\phi U, \phi_2 U) - g(\phi A\xi_2, \phi_2 U) \\ & + g(\phi_1 A\xi_2, \phi_2 U) \\ = {} & 3\beta + \beta g(\phi U, \phi_3\phi_2 U) \\ = {} & 3\beta + \beta g(\phi U, -\phi_1 U + \eta_2(U)\xi_3) \\ = {} & 3\beta - \beta g(\phi U, \phi_1 U) \\ = {} & 2\beta, \end{aligned}$$

where in the second equality we have used $\nabla_{\xi_2}\xi = \nabla_{\xi_2}\xi_1$ and in the final equality we have used the formula $\nabla_\xi \xi = \nabla_\xi \xi_1$. But this is impossible on the open subset $\mathfrak{V}$. Accordingly, such an open $\mathfrak{V}$ can not exit on M. So we have our assertion. □

Lemma 4.3. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying Lie ξ-parallel normal Jacobi operator and $\xi \in \mathfrak{D}^\perp$. Then $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

Proof. Now we consider (4.1) when the structure vector ξ is principal. Then it follows that

$$\begin{aligned} 0 = {} & -2\eta_2(X)\phi A\xi_2 - 2\eta_3(X)\phi A\xi_3 + \phi A\phi_1\phi X \\ & + 2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 + \phi_1 AX. \end{aligned} \tag{4.4}$$

Now let us take an inner product (4.4) with ξ_2. Then it follows that

$$\begin{aligned} 0 = {} & -2\eta_2(X)g(A\xi_2, \xi_3) - 2\eta_3(X)g(A\xi_3, \xi_3) \\ & + g(\phi A\phi_1\phi X, \xi_2) + 2\eta_3(AX) + g(\phi_1 AX, \xi_2) \\ = {} & -2\eta_2(X)g(A\xi_2, \xi_3) - 2\eta_3(X)g(A\xi_3, \xi_3), \end{aligned} \tag{4.5}$$

where in the first equality we have used the following formula

$$\begin{aligned} g(\phi A\phi_1\phi X,\xi_2) &= -g(A\phi_1\phi X,\phi\xi_2) \\ &= g(A\phi_1\phi X,\xi_3) \\ &= g(A\phi\phi_1 X,\xi_3) \\ &= -g(\phi_1 X,\nabla_{\xi_3}\xi) \\ &= g(\nabla_{\xi_3}(\phi X),\xi) \\ &= g(\eta(X)A\xi_3 - g(A\xi_3,X)\xi,\xi) \\ &= -g(A\xi_3,X). \end{aligned}$$

From this, by putting $X=\xi_2$ and $X=\xi_3$ we have

$$g(A\xi_3,\xi_3) = g(A\xi_2,\xi_3) = 0.$$

On the other hand, by taking an inner product (4.4) with ξ_3 we have

$$2\eta_2(X)g(A\xi_2,\xi_2) + 2\eta_3(X)g(A\xi_3,\xi_2) = 0.$$

Then from this, by putting $X=\xi_2$ and $X=\xi_3$ we have respectively

$$g(A\xi_2,\xi_2) = g(A\xi_3,\xi_2) = 0.$$

Summing up these formulas, we conclude that $g(A\xi_i,\xi_j)=0$ for any i and j except $i=j=1$. Then we may put $A\xi_2 = X_2$ and $A\xi_3 = X_3$ for some X_2, $X_3 \in \mathfrak{D}$.

Now substituting these one into (4.4), we get the following

$$\begin{aligned} 0 &= g(\phi A\phi_1 X,\xi_2) + 2\eta_3(AX) + g(\phi_1 AX,\xi_2) \\ &= -g(A\phi_1 X,\phi\xi_2) + 2g(X_3,X) - g(AX,\xi_3) \\ &= g(\phi_1 X,X_3) + g(X_3,X) \end{aligned} \tag{4.6}$$

for any tangent vector field X on M. Then from this, by replacing X by $\phi_1 X$ we have

$$\begin{aligned} 0 &= g({\phi_1}^2 X,X_3) + g(X_3,\phi_1 X) \\ &= -g(X,X_3) + g(X_3,\phi_1 X). \end{aligned} \tag{4.7}$$

Then (4.6) and (4.7) gives X_2 and X_3 identically vanishing. That is, $A\xi_2=0$ and $A\xi_3=0$. Accordingly, we have our assertion in Lemma 4.2. □

Before going to give the proof of Theorem 1 in the introduction let us check that "What kind of model hypersurfaces given in Theorem A satisfy Lie ξ-parallel normal Jacobi operator." In other words, it will be an interesting problem to know whether there exist any real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying the condition $\mathcal{L}_\xi \bar{R}_N = 0$ for $\xi \in \mathfrak{D}^\perp$.

Then by virtue of Lemmas 4.1 and 4.2, we are able to recall a proposition given by Berndt and the second author [4] as follows:

For a tube of type A in Theorem A we have the following

Proposition A. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r),\ \beta = \sqrt{2}\cot(\sqrt{2}r),\ \lambda = -\sqrt{2}\tan(\sqrt{2}r),\ \mu = 0$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1,\ m(\beta) = 2,\ m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces we have

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1, \\ T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3, \\ T_\lambda &= \{X | X \perp \mathbb{H}\xi, JX = J_1 X\}, \\ T_\mu &= \{X | X \perp \mathbb{H}\xi, JX = -J_1 X\}, \end{aligned}$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{Q}\xi$ respectively denotes real, complex and quaternionic span of the structure vector ξ and $\mathbb{C}^\perp\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

Then in the proof of Lemma 4.3 we have asserted that $A\xi_2 = 0$ and $A\xi_3 = 0$. But the principal curvature $\beta = \sqrt{2}\cot(\sqrt{2}r)$ given in Proposition A is never vanishing for any $r \in (0, \frac{\pi}{4})$. So this makes a contradiction. Accordingly, we completed the proof of our Theorem 1.

5. Lie ξ-parallel normal Jacobi operator for $\xi \in \mathfrak{D}$

In this section, in order to prove our Theorem 2 in the introduction we will give several lemmas. Now we consider for the case that $\xi \in \mathfrak{D}$. Then using $\xi \in \mathfrak{D}$ in (3.3) we have the following

$$\begin{aligned} (\mathcal{L}_\xi \bar{R}_N)X &= (\nabla_\xi \bar{R}_N)X - \phi A \bar{R}_N X + \bar{R}_N \phi A X \\ &= 3g(\phi A\xi, X)\xi + 3\sum_{\nu=1}^{3} g(\phi_\nu A\xi, X)\xi_\nu \\ &\quad + 3\sum_{\nu=1}^{3} \eta_\nu(X)\phi_\nu A\xi \\ &\quad + \sum_{\nu=1}^{3} \Big[g(\phi_\nu A\xi, \phi X)\phi_\nu \xi + \eta(X)\eta_\nu(A\xi)\phi_\nu\xi \\ &\quad + \eta_\nu(\phi X)\{ -g(A\xi, \xi)\xi_\nu + \phi_\nu \phi A\xi \} \Big] \\ &\quad - 3\sum_{\nu=1}^{3} \eta_\nu(X)\phi A\xi_\nu - \sum_{\nu=1}^{3} \eta_\nu(\phi X)\phi A \phi_\nu \xi \\ &\quad + 3\sum_{\nu=1}^{3} \eta_\nu(\phi A X)\xi_\nu - \sum_{\nu=1}^{3} \eta_\nu(AX)\phi_\nu \xi = 0. \end{aligned} \tag{5.1}$$

Then we assert the following

Lemma 5.1. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying Lie ξ-parallel normal Jacobi operator and $\xi \in \mathfrak{D}$. Then the structure vector ξ is principal.*

Proof. Now let us put $X = \xi$ in (5.1) and use $\xi \in \mathfrak{D}$, we have

$$\begin{aligned} 0 &= 3\sum_{\nu=1}^{3} g(\phi_\nu A\xi, \xi)\xi_\nu + \sum_{\nu=1}^{3} \eta_\nu(A\xi)\phi_\nu \xi + 3\sum_{\nu=1}^{3} \eta_\nu(\phi A\xi)\xi_\nu \\ &\quad - \sum_{\nu=1}^{3} \eta_\nu(A\xi)\phi_\nu \xi \\ &= 6\sum_{\nu=1}^{3} g(\phi_\nu A\xi, \xi)\xi_\nu. \end{aligned}$$

From this we assert the following for any $\nu = 1, 2, 3$

$$g(A\xi, \phi_\nu \xi) = 0. \tag{5.2}$$

On the other hand, let us take an inner product (5.1) with the structure vector ξ and use the fact $\xi \in \mathfrak{D}$ and (5.2). Then it follows

$$\begin{aligned} 0 &= 3g(\phi A\xi, X) + 3\sum_{\nu=1}^{3} \eta_\nu(X) g(\phi_\nu A\xi, \xi) \\ &\quad + \sum_{\nu=1}^{3} \eta_\nu(\phi X) g(\phi_\nu \phi A\xi, \xi) \\ &= 3g(\phi A\xi, X) - \sum_{\nu=1}^{3} \eta_\nu(\phi X)\eta_\nu(A\xi). \end{aligned} \tag{5.3}$$

Now by putting $X = \phi\xi_\mu$ into (5.3) we have

$$g(A\xi, \xi_\mu) = 0 \tag{5.4}$$

for any $\mu = 1, 2, 3$. Then by virtue of (5.2) and (5.4) we may put

$$A\xi = \alpha\xi + X_0 \tag{5.5}$$

for some $X_0 \in \mathfrak{D}$ orthogonal to $\xi, \phi_1\xi, \phi_2\xi, \phi_3\xi$. Then by putting $X = \phi X_0$ in (5.3) we have $g(A\xi, X_0) = 0$. From this, together with (5.5), we have our assertion. □

Then by using Lemma 5.1 we want to verify $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$. In order to do this, first of all, we should verify the following

Lemma 5.2. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ satisfying Lie ξ-parallel normal Jacobi operator and $\xi \in \mathfrak{D}$. Then $g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0$.*

Proof. From the results of Lemma 5.1, we have the following

$$
\begin{aligned}
(\mathcal{L}_\xi \bar{R}_N)X = {} & 4\alpha\sum_{\nu=1}^{3} g(\phi_\nu\xi, X)\xi_\nu + 3\alpha\sum_{\nu=1}^{3}\eta_\nu(X)\phi_\nu\xi \\
& - \sum_{\nu=1}^{3}\Big[\xi(\eta_\nu(\xi))(\phi_\nu\phi X - \eta(X)\xi_\nu) \\
& + \eta_\nu(\xi)\{-q_{\nu+1}(\xi)\phi_{\nu+2}\phi X + q_{\nu+2}(\xi)\phi_{\nu+1}\phi X \\
& - \eta(X)(q_{\nu+2}(\xi)\xi_{\nu+1} - q_{\nu+1}(\xi)\xi_{\nu+2} + \alpha\phi_\nu\xi)\} \\
& - \alpha g(\phi_\nu\xi, \phi X)\phi_\nu\xi\Big] \\
& - 3\sum_{\nu=1}^{3}\eta_\nu(X)\phi A\xi_\nu \\
& + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)(\phi A\phi_\nu\phi X - \eta(X)\phi A\xi_\nu) \\
& - \eta_\nu(\phi X)\phi A\phi_\nu\xi\} + 3\sum_{\nu=1}^{3}\eta_\nu(\phi AX)\xi_\nu \\
& + \sum_{\nu=1}^{3}\{\eta_\nu(\xi)\phi_\nu AX - \eta_\nu(AX)\phi_\nu\xi\} = 0.
\end{aligned}
\tag{5.6}
$$

Since ξ is principal and $\xi\in\mathfrak{D}$, we have

$$
g(A\xi, \mathfrak{D}^\perp) = 0. \tag{5.7}
$$

From the formula (5.6) and $\xi\in\mathfrak{D}$, we have the following

$$
\begin{aligned}
(\mathcal{L}_\xi \bar{R}_N)X = {} & 4\alpha\sum_{\nu=1}^{3} g(\phi_\nu\xi, X)\xi_\nu + 4\alpha\sum_{\nu=1}^{3}\eta_\nu(X)\phi_\nu\xi \\
& - 3\sum_{\nu=1}^{3}\eta_\nu(X)\phi A\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(\phi X)\phi A\phi_\nu\xi \\
& + 3\sum_{\nu=1}^{3}\eta_\nu(\phi AX)\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(AX)\phi_\nu\xi = 0.
\end{aligned}
\tag{5.8}
$$

Now let us put $\mathfrak{D}_0(x) = \{X\in\mathfrak{D} | X\perp\xi\}$. From this, for $X\in\mathfrak{D}_0$, we have

$$
\begin{aligned}
0 = {} & 4\alpha\sum_{\nu=1}^{3} g(\phi_\nu\xi, X)\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(\phi X)\phi A\phi_\nu\xi \\
& + 3\sum_{\nu=1}^{3}\eta_\nu(\phi AX)\xi_\nu - \sum_{\nu=1}^{3}\eta_\nu(AX)\phi_\nu\xi.
\end{aligned}
\tag{5.9}
$$

Let us take an inner product the above equation with $\phi_i\xi$. Then we have

$$
0 = \sum_{\nu=1}^{3}\eta_\nu(\phi X)g(\phi A\phi_\nu\xi, \phi_i\xi) + g(AX, \xi_i). \tag{5.10}
$$

By the formula (5.10), for $X\in\mathfrak{D}_1$, we have

$$
g(AX, \xi_i) = 0, \quad i = 1, 2, 3, \tag{5.11}
$$

where the distribution $\mathfrak{D}_1$ is given by $\mathfrak{D}_1 = \{X \in \mathfrak{D}_0 | X \perp \phi_i \xi,\ i = 1, 2, 3\}$. On the other hand, by (2.3) and (2.4), we have the following

$$\begin{aligned} g(A\phi_i\xi, \xi_\mu) &= g(A\xi_\mu, \phi_i\xi) \\ &= g(A\xi_\mu, \phi\xi_i) \\ &= -g(\phi A\xi_\mu, \xi_i) \\ &= -g(\nabla_{\xi_\mu}\xi, \xi_i) \\ &= g(\xi, \nabla_{\xi_\mu}\xi_i) \\ &= g(\xi, \phi_i A\xi_\mu) \\ &= -g(A\phi_i\xi, \xi_\mu). \end{aligned}$$

From the above equation, we have

$$g(A\phi_i\xi, \xi_\mu) = 0 \tag{5.12}$$

for any $i, \mu = 1, 2, 3$. Hence, by (5.7), (5.11) and (5.12), we know that

$$g(A\mathfrak{D}, \mathfrak{D}^\perp) = 0.$$

□

Now by virtue of these Lemmas 5.1 and 5.2 we are able to use Theorem A due to Berndt and the second author [4]. That is, M is locally a tube over a totally geodesic and totally real quaternionic projective space $\mathbb{Q}P^n$, $m = 2n$. So for the geometrical structure for such a tube we recall the following proposition

Proposition B. *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r),\ \beta = 2\cot(2r),\ \gamma = 0,\ \lambda = \cot(r),\ \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1,\ m(\beta) = 3 = m(\gamma),\ m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi,\ T_\beta = \mathfrak{J}J\xi,\ T_\gamma = \mathfrak{J}\xi,\ T_\lambda,\ T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp,\ \mathfrak{J}T_\lambda = T_\lambda,\ \mathfrak{J}T_\mu = T_\mu,\ JT_\lambda = T_\mu.$$

Now let us construct a subdistribution $\mathfrak{D}_0$ in such a way that

$$[\xi] \oplus \mathfrak{D}_0 = \mathfrak{D},$$

where $[\xi]$ denotes a one-dimensional vector subspace spanned by the structure vector ξ. Then $\mathfrak{D}_0$ becomes $\mathfrak{D}_0 = \{X \in \mathfrak{D} | X \perp \xi\}$. Now we substitute any $X \in \mathfrak{D}_0$

in (5.17) and use $\xi \in \mathfrak{D}$ we have

$$(\mathcal{L}_\xi \bar{R}_N)X = 4\alpha \sum_{\nu=1}^{3} g(X, \phi_\nu \xi)\xi_\nu - \sum_{\nu=1}^{3} \eta_\nu(\phi X)\phi A \phi_\nu \xi + 3\sum_{\nu=1}^{3} \eta_\nu(\phi A X)\xi_\nu - \sum_{\nu=1}^{3} \eta_\nu(AX)\phi_\nu \xi.$$

From this, putting $X = \phi_\mu \xi$ and using $A\phi_\mu \xi = 0$, $\mu = 1, 2, 3$ in Proposition B, we have

$$(\mathcal{L}_\xi \bar{R}_N)\phi \xi_\mu = 4\alpha \xi_\mu.$$

But we have assumed that $\mathcal{L}_\xi \bar{R}_N = 0$. Then this gives $\alpha = 0$. But the constant principal curvature $\alpha = -2\tan(2r)$ in Proposition B never vanishing for $r \in (0, \frac{\pi}{4})$. This makes a contradiction for this case $\xi \in \mathfrak{D}$. So we complete the proof of Theorem 2 in the introduction.

6. Hopf hypersurfaces with ξ-parallel normal Jacobi operator

A real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be a *Hopf* if the structure vector ξ of M is principal. This means that $A\xi = \alpha\xi$, $\alpha = g(A\xi, \xi)$, for the shape operator A of M in $G_2(\mathbb{C}^{m+2})$. Of course, all of hypersurfaces in $G_2(\mathbb{C}^{m+2})$ mentioned in Theorem A are Hopf hypersurfaces. Moreover, by Propositions A and B we have known that the structure vector ξ for real hypersurfaces of type (A) and of type (B) in Theorem A belongs to the distribution $\mathfrak{D}^\perp$ and the distribution $\mathfrak{D}$ respectively.

In this section we consider a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie ξ-parallel normal Jacobi operator $\bar{R}_N$. Then it will be an interesting fact to check whether Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with Lie ξ-parallel normal Jacobi operator can exist or not.

In order to do this, we prove the following lemma which will be useful in the proof of our Theorem 3 given in the introduction.

Lemma 6.1. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator along the direction of ξ. Then the directional derivative of the principal curvature α is given by*

$$Y\alpha = -4\sum_{\nu=1}^{3} \eta_\nu(\xi)\eta_\nu(\phi Y)$$

for any vector field Y on M.

Proof. Now we assume that M is Hopf. So we may put $A\xi = \alpha\xi$. Then the formula (3.4) implies that

$$\alpha \sum_{\nu=1}^{3} \eta_\nu(\xi)\phi_\nu \xi = \sum_{\nu=1}^{3} \eta_\nu(\xi)\phi A \xi_\nu. \tag{6.1}$$

Now let us consider a vector U defined in such a way that

$$U = \sum_{\nu=1}^{3} \eta_\nu(\xi)\xi_\nu. \tag{6.2}$$

If we put $\xi = X_1 + X_2$ for some vector X_1 in the distribution $\mathfrak{D}^\perp$ and some vector X_2 in $\mathfrak{D}$, then we know that X_1 becomes the vector U. Now hereafter,

unless otherwise stated, let us decompose the structure vector ξ by $\xi = U + X_2$. Then (6.1) can be written as follows

$$\phi AU = \alpha\phi U. \tag{6.3}$$

Now differentiating (6.2) covariantly and using the formulas given in Section 2, we have

$$\begin{aligned}\nabla_X U &= \sum_{\nu=1}^{3}\{g(\nabla_X\xi_\nu,\xi)\xi_\nu + g(\xi_\nu,\nabla_X\xi)\xi_\nu + \eta_\nu(\xi)\nabla_X\xi_\nu\}\\ &= 2\sum_{\nu=1}^{3} g(\xi_\nu,\phi AX)\xi_\nu + \sum_{\nu=1}^{3}\eta_\nu(\xi)\phi_\nu AX.\end{aligned}$$

On the other hand, by applying the structure tensor ϕ to (6.1) we know the following

$$AU = \alpha U \quad \text{and} \quad AX_2 = \alpha X_2. \tag{6.4}$$

Now differentiating the first formula of (6.4) and using the above formula, we have the following

$$(\nabla_X A)U + A\nabla_X U = (X\alpha)U + \alpha\nabla_X U.$$

Then it follows that

$$\begin{aligned}&g(U,(\nabla_X A)Y)\\ &= g((\nabla_X A)U,Y)\\ &= (X\alpha)g(U,Y) + \alpha g(\nabla_X U,Y) - g(A\nabla_X U,Y)\\ &= (X\alpha)\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(Y) + \alpha\{2\sum_{\nu=1}^{3} g(\xi_\nu,\phi AX)\eta_\nu(Y)\\ &\quad + \sum_{\nu=1}^{3}\eta_\nu(\xi)g(\phi_\nu AX,Y)\}\\ &\quad - g(2\sum_{\nu=1}^{3} g(\xi_\nu,\phi AX)A\xi_\nu + \sum_{\nu=1}^{3}\eta_\nu(\xi)A\phi_\nu AX,Y).\end{aligned} \tag{6.5}$$

From this, let us take a skew-symmetric part of (6.5), then by virtue of the equation of Codazzi the left side becomes

$$\begin{aligned}&g((\nabla_X A)Y - (\nabla_Y A)X,U)\\ &= \sum_{\nu=1}^{3}\{\eta(X)g(\phi Y,\xi_\nu)\eta_\nu(\xi) - \eta(Y)g(\phi X,\xi_\nu)\eta_\nu(\xi)\}\\ &\quad - 2\sum_{\nu=1}^{3} g(\phi X,Y)\eta_\nu(\xi)^2 - 2\sum_{\nu=1}^{3} g(\phi_\nu X,Y)\eta_\nu(\xi)\\ &\quad + 2\sum_{\nu=1}^{3}\big[\eta_\nu(X)\{-\eta_{\nu+2}(Y)\eta_{\nu+1}(\xi) + \eta_{\nu+1}(Y)\eta_{\nu+2}(\xi)\}\\ &\quad + \eta_\nu(\phi X)\{-\eta_{\nu+2}(\phi Y)\eta_{\nu+1}(\xi) + \eta_{\nu+1}(\phi Y)\eta_{\nu+2}(\xi)\}\big]\\ &\quad + \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\eta_\nu(\xi),\end{aligned} \tag{6.6}$$

where we have used the following

$$\begin{aligned} g(\phi_\nu\phi Y, U) &= -g(\phi Y, \phi_\nu U) \\ &= -g(\phi Y, \eta_{\nu+1}(\xi)\phi_\nu\xi_{\nu+1} + \eta_{\nu+2}(\xi)\phi_\nu\xi_{\nu+2}) \\ &= -\eta_{\nu+2}(\phi Y)\eta_{\nu+1}(\xi) + \eta_{\nu+1}(\phi Y)\eta_{\nu+2}(\xi) \end{aligned}$$

and

$$g(\phi_\nu Y, U) = -\eta_{\nu+2}(Y)\eta_{\nu+1}(\xi) + \eta_{\nu+1}(Y)\eta_{\nu+2}(\xi).$$

Moreover, the skew-symmetric part in the right side of (6.5) becomes

$$\begin{aligned} &(X\alpha)\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(Y) - (Y\alpha)\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(X) \\ &\quad + 2\alpha\sum_{\nu=1}^{3}\{g(\xi_\nu, \phi AX)\eta_\nu(Y) - g(\xi_\nu, \phi AY)\eta_\nu(X)\} \\ &\quad + \alpha\sum_{\nu=1}^{3}\eta_\nu(\xi)g((\phi_\nu A + A\phi_\nu)X, Y) - 2\sum_{\nu=1}^{3}\{g(\xi_\nu, \phi AX)g(A\xi_\nu, Y) \\ &\quad - g(\xi_\nu, \phi AY)g(A\xi_\nu, X)\} - 2\sum_{\nu=1}^{3}\eta_\nu(\xi)g(A\phi_\nu AX, Y). \end{aligned}$$

Then by putting $X = \xi$ into the both sides of the above formulas and using $A\xi = \alpha\xi$, we have

(6.7)

$$\begin{aligned} 4\sum_{\nu=1}^{3}\eta_\nu(\phi Y)\eta_\nu(\xi) &= (\xi\alpha)\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(Y) - (Y\alpha)\sum_{\nu=1}^{3}\eta_\nu(\xi)^2 \\ &\quad + \alpha^2\sum_{\nu=1}^{3}\eta_\nu(\xi)g(\phi_\nu\xi, Y) + \alpha\sum_{\nu=1}^{3}g(\xi_\nu, \phi AY)\eta_\nu(\xi). \end{aligned}$$

On the other hand, if we differentiate $A\xi = \alpha\xi$ and take an inner product with ξ, then the Codazzi equation gives the following

$$\begin{aligned} &-2g(\phi X, Y) + 2\sum_{\nu=1}^{3}\{\eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi)\} \\ &= g((\nabla_X A)Y - (\nabla_Y A)X, \xi) \\ &= g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X) \\ &= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y). \end{aligned}$$

From this, if we put $X = \xi$, then

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y). \tag{6.8}$$

Then by putting $Y = X_2$ in (6.8) we have

$$X_2\alpha = \|X_2\|^2(\xi\alpha), \tag{6.9}$$

where we have used

$$\eta_\nu(\phi X_2) = -g(\phi_\nu\xi, X_2) = -g(\phi_\nu U + \phi_\nu X_2, X_2) = 0. \tag{6.10}$$

Now we put $Y = X_2$ in (6.7), and use (6.8), (6.10) and $AX_2 = \alpha X_2$ in the obtained equation. Then it follows that

$$(X_2\alpha)(1 - \sum_{\nu=1}^{3}\eta_\nu(\xi)^2) = (\xi\alpha)\eta(X_2). \tag{6.11}$$

Then from (6.9) and (6.11) we have

$$\|X_2\|^2(1-\sum\nolimits_{\nu=1}^{3}\eta_\nu(\xi)^2)(\xi\alpha)=(\xi\alpha)\eta(X_2)=(\xi\alpha)\|X_2\|^2,$$

which gives that

$$(\sum\nolimits_{\nu=1}^{3}\eta_\nu(\xi)^2)\|X_2\|^2(\xi\alpha)=0. \tag{6.12}$$

From this, together with the decomposition of the structure vector ξ in the assumption, we have $\xi\alpha=0$. Then (6.8) completes the proof of Lemma 6.1. □

Now let us show that the structure vector ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$ when a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ has commuting shape operator, that is $A\phi=\phi A$ on the distribution $\mathfrak{D}^\perp$. In order to do this we also assumed that the structure vector ξ is decomposed into two distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp$. That is, ξ is decomposed into $\xi=U+X_2$

Now, by using $\xi\alpha=0$ in (6.7) and (6.8), we have

$$(Y\alpha)(\sum\nolimits_{\nu=1}^{3}\eta_\nu(\xi)^2-1)=\alpha^2g(\phi U,Y)-\alpha g(\phi U,AY). \tag{6.13}$$

Moreover, from (6.8) together with $\xi\alpha=0$ we have

$$Y\alpha=4g(\phi U,Y) \tag{6.14}$$

for any tangent vector field Y on M. So (6.14) gives $Y\alpha=0$ for any Y orthogonal to ϕU. Then from this together with (6.13) we have

$$\alpha g(A\phi U,Y)=0 \tag{6.15}$$

for any Y orthogonal to ϕU.

For the case where $\alpha=0$, by (6.14) we can make a contradiction, because $\phi U=-\phi X_2$ never vanishing under the decomposition. So we assume that the function $\alpha\neq 0$. Then (6.15) gives that $g(A\phi U,Y)=0$ for any Y orthogonal to ϕU. So we may put

$$A\phi U=\beta\phi U. \tag{6.16}$$

Now by putting $Y=\phi U$ in (6.13) and (6.14), and using (6.16), we have

$$\begin{aligned}-4\|\phi U\|^2\|X_2\|^2 &= -(\phi U\alpha)\|X_2\|^2\\ &=(\alpha^2-\alpha\beta)\|\phi U\|^2\\ &=\alpha(\alpha-\beta)\|\phi U\|^2.\end{aligned}$$

This gives

$$\alpha(\alpha-\beta)=-4\|X_2\|^2. \tag{6.17}$$

But we have asserted that M has commuting shape operator on the distribution $\mathfrak{D}^\perp$. This means that $\phi AU=A\phi U=\alpha\phi U$ for $U=\sum_{\nu=1}^{3}\eta_\nu(\xi)\xi_\nu\in\mathfrak{D}^\perp$. From this together with (6.17), we can make a contradiction. Then summing up these process and Lemma 6.1 we can assert the following

Lemma 6.2. *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator along the direction of ξ. If M has commuting shape operator on the distribution $\mathfrak{D}^\perp$, then the structure vector ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Accordingly, by Lemma 6.2 and together with Theorem 1 and Theorem 2 for each case $\xi \in \mathfrak{D}^\perp$ and $\xi \in \mathfrak{D}$ respectively, we give the complete proof of our Theorem 3 mentioned in the introduction.

Remark 6.1. A tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ in Theorem A has commuting shape operator on the distribution $\mathfrak{D}^\perp$. Of course, it is Hopf. But, in section 4 we have asserted that such a hypersurface can not satisfy $\mathcal{L}_\xi \bar{R}_N = 0$.

Remark 6.2. A tube over a totally real totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$ has not commuting shape operator on the distribution $\mathfrak{D}^\perp$. In section 5 we have also proved that such a hypersurface is Hopf but can not satisfy $\mathcal{L}_\xi \bar{R}_N = 0$.

Acknowledgments. This work was done while the first author was post doctoral fellow at Department of Mathematics, Kyungpook National University in Korea during the year 2007. The present authors would like to express their sincere gratitude to the referee for his careful reading of our manuscript and useful suggestions to develop the first version of this paper.

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