

FACE PAIRING MAPS OF FORD DOMAINS FOR CUSPED HYPERBOLIC 3-MANIFOLDS

SUNGBOK HONG AND JUNGSOO KIM

ABSTRACT. We will describe a way to construct Ford domains of cusped hyperbolic 3-manifolds on maximal cusp diagrams and compute fundamental groups using face pairing maps as generators and Cannon-Floyd-Parry's edge cycles as relations. We also describe explicitly a cutting and pasting alteration to reduce the number of faces on the bottom region of Ford domains. We expect that our analysis of Ford domains will be useful on other future research.

1. Introduction

A hyperbolic 3-manifold is a Riemannian 3-manifold of constant sectional curvature -1 . We will restrict our attention to a complete hyperbolic 3-manifold with one torus cusp. Such a manifold M can be described as \mathbb{H}^3/Γ where Γ is a discrete torsion free subgroup of $PSL_2(\mathbb{C})$. Since M has a torus cusp, Γ has a rank 2 subgroup generated by parabolic elements. The subgroup generated by parabolic elements fixing a point p in S_∞^2 will be denoted by Γ_p . Given a cusp c which lifts to a fixed point p in S_∞^2 , a cusp neighborhood in M is simply the projection to M of a horoball centered at p . A cusp neighborhood in M lifts to a collection of horoballs in \mathbb{H}^3 with disjoint interior as is shown in figure 1. Let those horoballs expand until they collide each other as in figure 2. We can make the height of the horosphere at infinity be 1. The image of interior of those horoballs gives a maximal cusp neighborhood. Topologically a cusp neighborhood is homeomorphic to $T^2 \times [0, \infty)$. Again expand maximal horoballs further until they fill up the whole hyperbolic space \mathbb{H}^3 . We may regard horospheres as balloons which gently expand.

Now we can get a surface of collision with the unbounded horoball (horoball at infinity) and this surface divides \mathbb{H}^3 into two parts. Let F be the part containing the unbounded horoball. The canonical choice of Ford domain is F/Γ_∞ and M can be obtained by pairing faces of F/Γ_∞ . (See [5]). In the case of figure eight knot complement, Γ_∞ is generated by two parabolic elements

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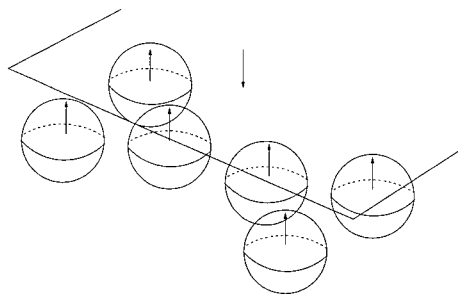
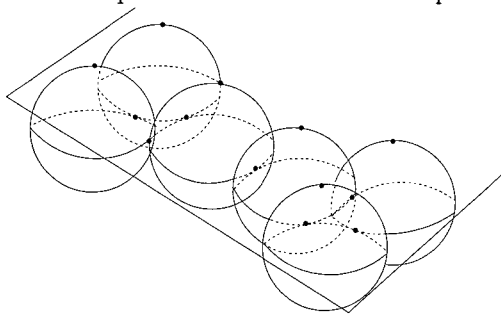
FIGURE 1. Expand the horoballs in \mathbb{H}^3 equivariantly

FIGURE 2. Horoballs project to a maximal cusp neighborhood.

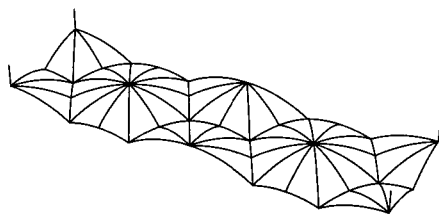


FIGURE 3. Ford domain of figure-eight knot complement.

whose euclidean translation lengths on the horosphere at infinity are 1 and $2\sqrt{3}$ and the angle made by the two translations is $\frac{\pi}{2}$. (See more details of maximal cusp diagram in p.19 of [1]). In figure 3, we may see the shape of F/Γ_∞ for the figure eight knot complement with front-back and left-right sides identified by parabolic elements fixing ∞ .

A face pairing $\epsilon = \{\epsilon_i\}$ of the 3-ball B^3 sews together the 2-cells of a cellulation of the boundary 2-sphere S^2 of B^3 isomorphically in pairs, every 2-cell being identified with a different 2-cell. For a face f of S^2 let ϵ_i denote the cellular homeomorphism by which ϵ_i identifies f with $\epsilon_i(f)$. Let \sim be the

equivalence relation on the set of edges of S^2 generated by $e \sim \epsilon_i(e)$ when e is an edge of f . We call the resulting equivalence classes edge cycles.

If an edge e of the face f is identified with an edge e' by a face pairing map ϵ_i then it will be denoted by $e \xrightarrow{\epsilon_i} e'$.

Given a face pairing ϵ , let $M = B^3/\epsilon$ and let M_0 be the open manifold obtained by deleting the vertices from M . If M is already a manifold, then M and M_0 have the same fundamental group. The following classical result which is quoted from section 2 of [3] will be used to compute fundamental groups of cusped hyperbolic 3-manifolds.

Proposition 1.1. *The open manifold M_0 has fundamental group by the generators and relations*

$$\langle x_1, x_2, \dots \mid W_1, W_2, \dots \rangle$$

where the generators x_1, x_2, \dots correspond to the face-pairing maps and the relations W_1, W_2, \dots are the words arising from edge cycles.

We use ideal triangulations to find face-pairing of bottom faces of Ford domains. But if ideal triangulations were given from the outset then we can compute fundamental groups of cusped hyperbolic 3-manifolds more conveniently by using face pairing of ideal tetrahedra. We will describe explicitly how to obtain a Ford domain on maximal cusp diagram for the sibling of figure eight knot complement in section 4 and we explain a cutting and pasting method to reduce the number of faces on the bottom region of Ford domains in section 5.

2. Face-pairing maps

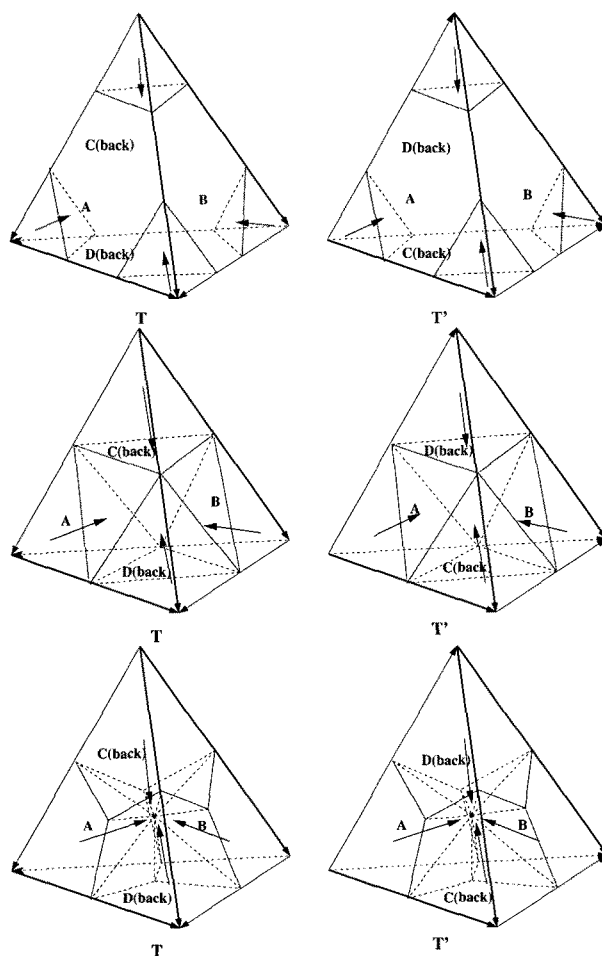
In Section 2 and Section 3 we consider only the case of figure-8 knot complement M . It is known that M is obtained by gluing boundary faces of two ideal tetrahedra. We may reconstruct F/Γ_∞ of figure 3 from an ideal triangulation of M .

First expand a cusp neighborhood until the whole manifold M is filled. This procedure can be seen in figure 4.

Cut along the collision locus and flatten out then we have 8 triangles in the interior of ideal tetrahedra (4 triangles in each tetrahedron). Each of 8 triangles has a barycentric subdivision so we have 48 triangles (See figure 5). Now we have 8 tetrahedra from original 2 tetrahedra and each tetrahedron has exactly one ideal vertex. We reglue 8 tetrahedra so that ideal vertices are identified to the vertex at infinity (See figure 6).

We may consider that M is homeomorphic to B^3 with face-pairing of 52 2-cells - 48 on bottom and 4 in sides but one vertex at ∞ is deleted from ∂B^3 .

In figure 7, we take a base point in F . A loop is obtained by the sum of lines with arrows the directions of which tell how faces are paired. So it is natural to take the generators of fundamental group as face-pairing maps. The natural correspondence between face-pairing maps and loops can be seen in figure 9.

FIGURE 4. Expand cusp neighborhood until whole M is filled.

But the question is how we get the relations those make. The relations are made from all possible trivial loops. As in figure 8 all possible trivial loops are decomposed into small trivial loops, each of which goes around exactly one edge in M . So if we have face-pairing map expressions for the all such small loops, then we are done.

Now we find expressions for trivial loops, each of which goes around exactly one edge in M .

In figure 9, the loop goes around three edges. Let the face-pairing maps determined by the loop be a, b, c in that order as shown in the figure. Figure 10 shows the previous loop of figure 9 inside the manifold M . It is easy to see

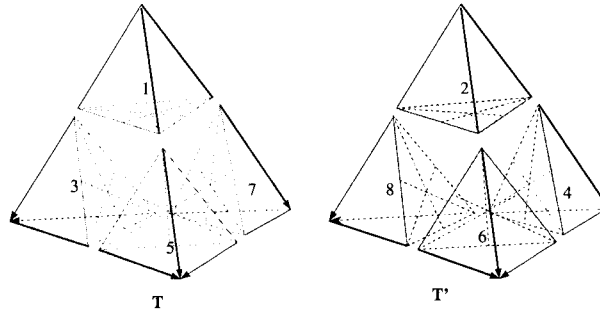


FIGURE 5. Cut along the collision locus.

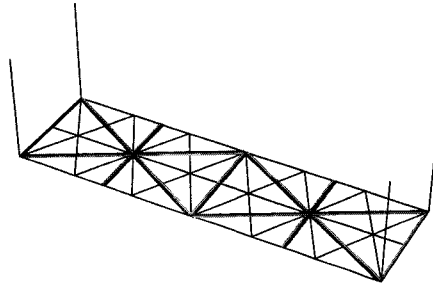


FIGURE 6. Reglue all tetrahedra with flattened bottom faces.

that this loop bounds a 2-cell so it is a trivial loop. This implies we have a relation $abc = \text{id}$.

The fundamental group of M can be described as follows;

$$\pi_1(M) = \langle x_1, x_2, \dots, x_n \mid x_i \text{ is the face-pairing maps,} \\ \text{relations obtained by trivial loops from edge cycles} \rangle$$

But in the case of figure eight knot complement, 26 face-pairing map can exist, so are generators of $\pi_1(M)$ (See figure 11). In figure 11, we use the notation $m - n = l - k$ where $1 \leq m, l \leq 8$ and $1 \leq n, k \leq 6$ to denote that n th triangle in the large triangle with number m is paired with k th triangle in the large triangle with number l . For example, $3 - 1 = 7 - 1$ means that the 1st triangle in the large triangle with number 3 is paired with the 1st triangle in the large triangle with number 7. Those two triangles are paired by a hyperbolic isometry so their shapes (corresponding angles) have to be preserved. Those numbering of triangles are originated from the numbering of triangles in figure 5. Those pairing of triangles are determined by the face pairings by which the figure eight knot complement is obtained from two ideal tetrahedra. If we can reduce the number of 2-cells, then the number of face-pairing maps are reduced and so are the number of generators.

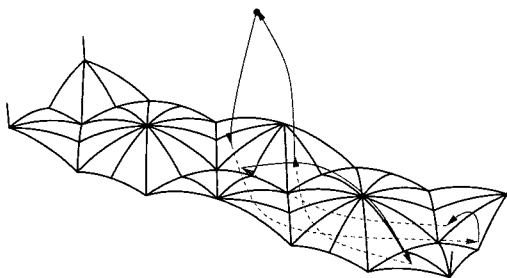
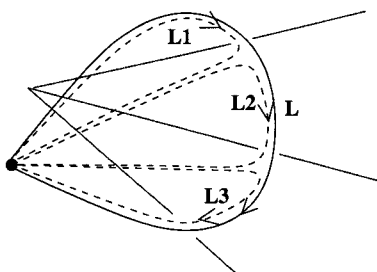
FIGURE 7. A loop in M 

FIGURE 8. A trivial loop can be decomposed into smaller loops each of which goes around only one edge.

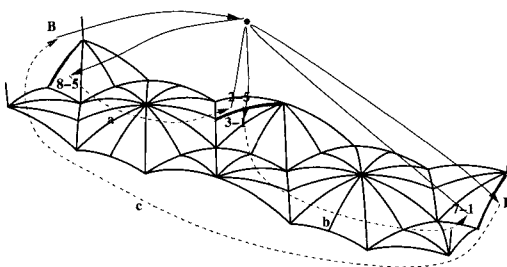
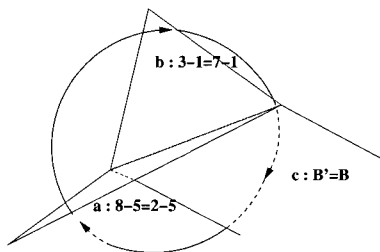


FIGURE 9. A loop which goes around only one edge respectively.

FIGURE 10. The loop of figure 9 in M .

By the cutting and pasting alteration which we will discuss in Section 5, (figure 12) we are able to reduce the number of 2-cells from 52 to 10 (See figure 12). There $F_i = F_j$ means that the face F_i is paired with the face F_j .

Now we obtain face-pairing maps as follows;

$$\begin{aligned}\epsilon_1 : F_1 &\rightarrow F_3 \Rightarrow \begin{pmatrix} Q_1 & Q_{19} & Q_2 & Q_3 & Q_4 \\ Q_5 & Q_{22} & Q_6 & Q_7 & Q_8 \end{pmatrix} \\ \epsilon_2 : F_2 &\rightarrow F_5 \Rightarrow \begin{pmatrix} Q_2 & Q_7 & Q_8 & Q_9 & Q_{21} & Q_{20} & Q_3 \\ Q_{10} & Q_{11} & Q_{12} & Q_{13} & Q_{23} & Q_{14} & Q_{17} \end{pmatrix} \\ \epsilon_3 : F_4 &\rightarrow F_6 \Rightarrow \begin{pmatrix} Q_5 & Q_{10} & Q_{11} & Q_9 & Q_8 \\ Q_{15} & Q_{16} & Q_{17} & Q_{14} & Q_{18} \end{pmatrix} \\ \epsilon_4 : F_7 &\rightarrow F_9 \Rightarrow \\ &\begin{pmatrix} Q_1 & Q_{19} & Q_2 & Q_7 & Q_6 & Q_{22} & Q_5 & Q_{10} & Q_{17} & Q_{16} & Q_{15} & \infty \\ Q_4 & Q_3 & Q_{20} & Q_{21} & Q_9 & Q_{11} & Q_{12} & Q_{13} & Q_{23} & Q_{14} & Q_{18} & \infty \end{pmatrix} \\ \epsilon_5 : F_8 &\rightarrow F_{10} \Rightarrow \begin{pmatrix} Q_{18} & Q_{15} & \infty \\ Q_4 & Q_1 & \infty \end{pmatrix}\end{aligned}$$

3. The figure eight knot complement

To compute the fundamental group of the figure eight knot complement, first we enumerate all possible edge cycles as follows;

$$\begin{aligned}Q_1 Q_{19} &\xrightarrow{\epsilon_1} Q_5 Q_{22} \xrightarrow{\epsilon_4} Q_{12} Q_{11} \xrightarrow{\epsilon_2^{-1}} Q_8 Q_7 \xrightarrow{\epsilon_1^{-1}} Q_4 Q_3 \xrightarrow{\epsilon_4^{-1}} Q_1 Q_{19} \\ Q_{19} Q_2 &\xrightarrow{\epsilon_1} Q_{22} Q_6 \xrightarrow{\epsilon_4} Q_{11} Q_9 \xrightarrow{\epsilon_3} Q_{17} Q_{14} \xrightarrow{\epsilon_2^{-1}} Q_3 Q_{20} \xrightarrow{\epsilon_4^{-1}} Q_{19} Q_2 \\ Q_2 Q_3 &\xrightarrow{\epsilon_1} Q_6 Q_7 \xrightarrow{\epsilon_4} Q_9 Q_{21} \xrightarrow{\epsilon_2} Q_{13} Q_{23} \xrightarrow{\epsilon_4^{-1}} Q_{10} Q_{17} \xrightarrow{\epsilon_2^{-1}} Q_2 Q_3 \\ Q_2 Q_7 &\xrightarrow{\epsilon_2} Q_{10} Q_{11} \xrightarrow{\epsilon_3} Q_{16} Q_{17} \xrightarrow{\epsilon_4} Q_{14} Q_{23} \xrightarrow{\epsilon_2^{-1}} Q_{20} Q_{21} \xrightarrow{\epsilon_4^{-1}} Q_2 Q_7 \\ Q_8 Q_9 &\xrightarrow{\epsilon_2} Q_{12} Q_{13} \xrightarrow{\epsilon_4^{-1}} Q_5 Q_{10} \xrightarrow{\epsilon_3} Q_{15} Q_{16} \xrightarrow{\epsilon_4} Q_{18} Q_{14} \xrightarrow{\epsilon_3^{-1}} Q_8 Q_9 \\ Q_{15} \infty &\xrightarrow{\epsilon_4} Q_{18} \infty \xrightarrow{\epsilon_5} Q_4 \infty \xrightarrow{\epsilon_4^{-1}} Q_1 \infty \xrightarrow{\epsilon_5^{-1}} Q_{15} \infty \\ Q_{18} Q_{15} &\xrightarrow{\epsilon_5} Q_4 Q_1 \xrightarrow{\epsilon_1} Q_8 Q_5 \xrightarrow{\epsilon_3} Q_{18} Q_{15}.\end{aligned}$$

If we denote the generator which is determined by ϵ_i by x_i , then we have the following 7 relations from the above edge cycles;

$$\begin{aligned}x_1 x_4 x_2^{-1} x_1^{-1} x_4^{-1} &= \text{id}, \\ x_1 x_4 x_3 x_2^{-1} x_4^{-1} &= \text{id}, \\ x_1 x_4 x_2 x_4^{-1} x_2^{-1} &= \text{id},\end{aligned}$$

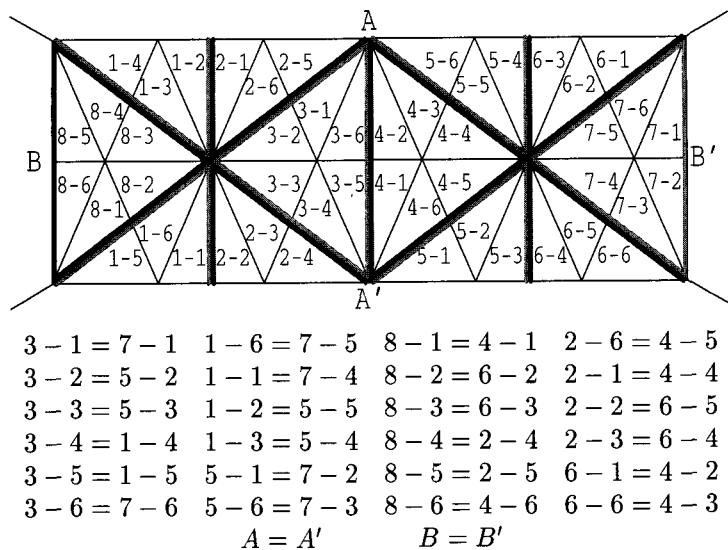


FIGURE 11. The glueing pattern of bottom faces

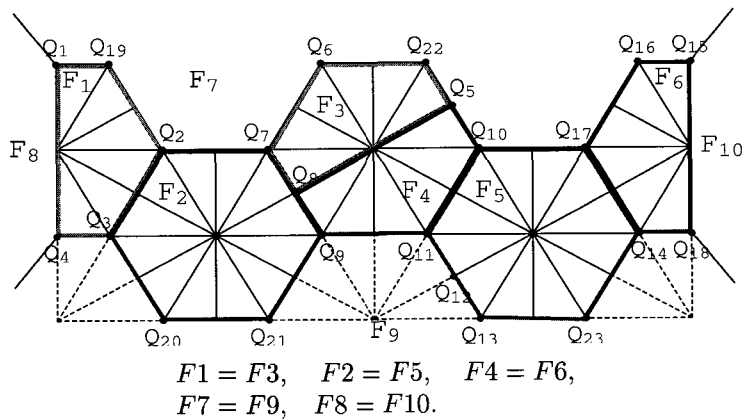


FIGURE 12. Reduce the number of 2-cells by regrouping triangles.

$$x_2 x_3 x_4 x_2^{-1} x_4^{-1} = \text{id},$$

$$x_2 x_4^{-1} x_3 x_4 x_3^{-1} = \text{id},$$

$$x_4 x_5 x_4^{-1} x_5^{-1} = \text{id},$$

$$x_5 x_1 x_3 = \text{id}.$$

Reduce x_5 by using the 7th relation then we have the following 6 relations:

$$x_1 x_4 x_2^{-1} x_1^{-1} x_4^{-1} = \text{id},$$

$$x_1 x_4 x_3 x_2^{-1} x_4^{-1} = \text{id},$$

$$x_1 x_4 x_2 x_4^{-1} x_2^{-1} = \text{id},$$

$$x_2 x_3 x_4 x_2^{-1} x_4^{-1} = \text{id},$$

$$x_2 x_4^{-1} x_3 x_4 x_3^{-1} = \text{id},$$

$$x_4 x_3^{-1} x_1^{-1} x_4^{-1} x_1 x_3 = \text{id}.$$

Reduce x_3 by using the 2nd relation and then we have the following 5 relations:

$$x_1 x_4 x_2^{-1} x_1^{-1} x_4^{-1} = \text{id},$$

$$x_1 x_4 x_2 x_4^{-1} x_2^{-1} = \text{id},$$

$$x_2 x_4^{-1} x_1^{-1} x_4 x_2 x_4 x_2^{-1} x_4^{-1} = \text{id},$$

$$x_2 x_4^{-1} x_4^{-1} x_1^{-1} x_4 x_2 x_4 x_2^{-1} x_4^{-1} x_1 x_4 = \text{id},$$

$$x_4 x_2^{-1} x_4^{-1} x_1 x_4 x_1^{-1} x_4^{-1} x_1 x_4^{-1} x_1^{-1} x_4 x_2 = \text{id}.$$

Reduce x_2 by using the 1st relation then we have 2 relations. Here we point out that the 2nd, 3rd relations are the same and the 4th, 5th relations are the same after reducing x_2 . So we may delete additional 3rd and 5th relations.

$$x_1 x_4 x_1^{-1} x_4^{-1} x_1 x_4^{-1} x_1^{-1} x_4 x_1 = \text{id},$$

$$x_1^{-1} x_4^{-1} x_1 x_4^{-1} x_1^{-1} x_4 x_1^{-1} x_4^{-1} x_1 x_4 x_1^{-1} x_4 x_1 x_4^{-1} x_1 x_4 = \text{id},$$

Reduce $x_1 x_4^{-1} x_1^{-1} x_4$ by using the 1st relation then we have one relation expressed by 9 words.

$$x_1 x_1 x_4 x_1^{-1} x_4^{-1} x_1 x_4^{-1} x_1^{-1} x_4 = x_1 [x_1, x_4] [x_1, x_4^{-1}] = \text{id}.$$

Finally we have the following presentation for the fundamental group of the figure eight knot complement when we replace x_1 by x and x_4 by y . It coincides with what Cannon, Floyd and Parry obtain in section 6 of [3].

Theorem 3.1.

$$\pi_1(M) = \langle x, y \mid x [x, y] [x, y^{-1}] = \text{id} \rangle.$$

Remark. All computations of edge cycles and free-group simplifications are done by the program made by one of the authors (Jungsoo Kim).

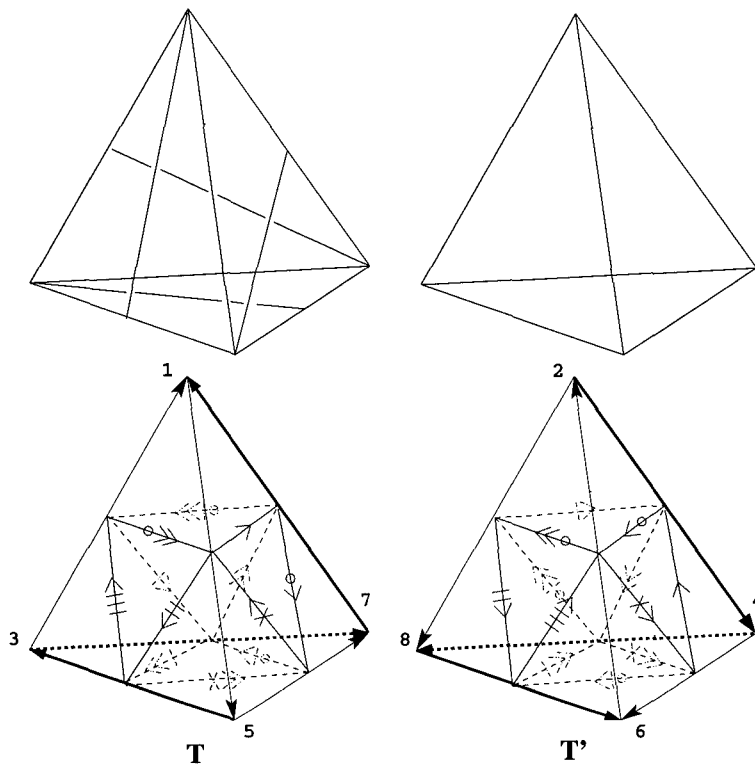


FIGURE 13. The glueing diagram for sibling of figure-8 knot complement.

4. The sibling of the figure eight knot complement

There are only two orientable one cusped hyperbolic 3-manifolds which can be constructed from two ideal tetrahedra (see [2]). They are figure-8 knot complement and its sibling. We will denote the sibling manifold by M' . The sibling manifold is made by glueing each face of the left hand tetrahedron with a corresponding face of the right hand tetrahedron with a flip about the indicated axis as is shown in figure 13. The manifold M' is the punctured torus bundle where the monodromy has trace -3 (see section 12 of [4]). M' can also be obtained by $(5, 1)$ Dehn surgery on one of the Whitehead link component and hence its first homology is $\mathbb{Z} \oplus \mathbb{Z}_5$. We will describe how to construct a Ford domain of sibling manifold on a maximal cusp diagram and compute the fundamental group of M' .

First of all, expand the horoballs and cut along its collision locus and reglue the 8 tetrahedra as in the case of figure-8 knot complement. Γ'_∞ is generated by two parabolic elements of translation length 2 and $\sqrt{3}$ and the angle made by the two translation is $\frac{\pi}{3}$. Figure 14 shows the bottom face of a Ford domain

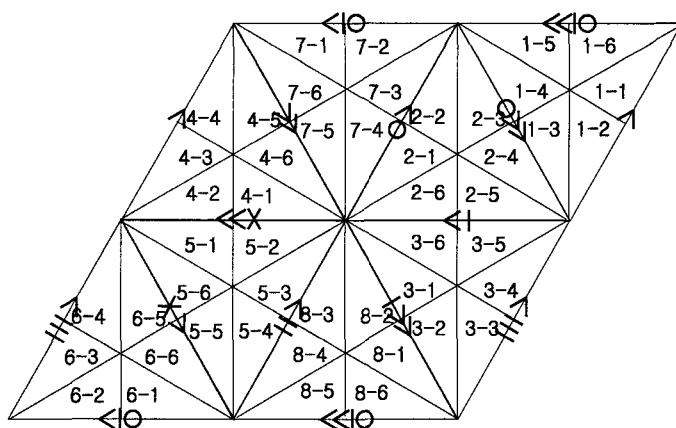
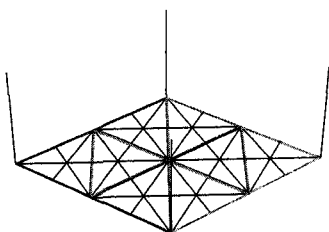
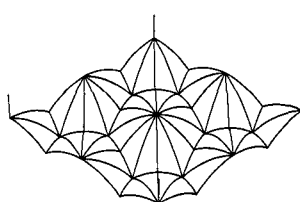


FIGURE 14. Bottom face of Ford domain


 FIGURE 15
Bottom-flattened Ford domain
of M' .

 FIGURE 16
Original shape of Ford domain
of M' .

with labels according to figure 13 and figure 17 describes the glueing pattern of the bottom region.

Figure 15 describes the bottom faces of flatten Ford domain of M' from the original shape of bottom region of the Ford domain (figure 16) of M' . By the cutting and pasting alteration which we will discuss in section 5, we are able to reduce the number of faces from 52 to 12 as in figure 18.

By reshaping the regions appeared in figure 14, we have a region such as figure 18 and by analyzing the gluing pattern in figure 18 originated from figure 17, we obtain six possible face-pairing maps as follows;

$$\epsilon_1 : F_1 \rightarrow F_5 \Rightarrow \begin{pmatrix} Q_{24} & Q_{23} & Q_{19} & Q_{17} & Q_{16} & Q_{15} \\ Q_9 & Q_8 & Q_7 & Q_6 & Q_{12} & Q_{11} \end{pmatrix}$$

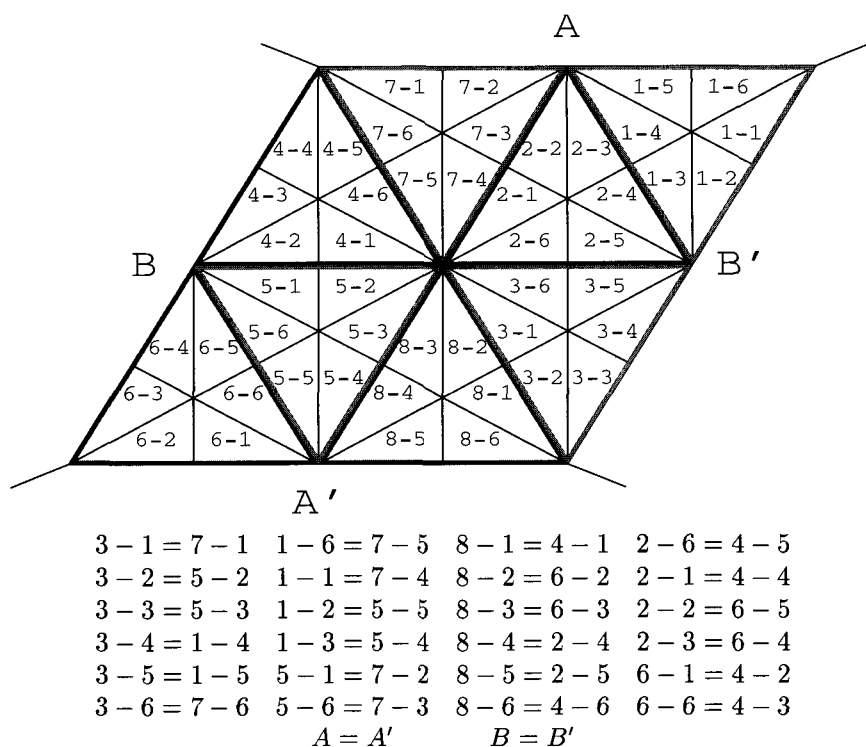


FIGURE 17. The glueing pattern of bottom faces

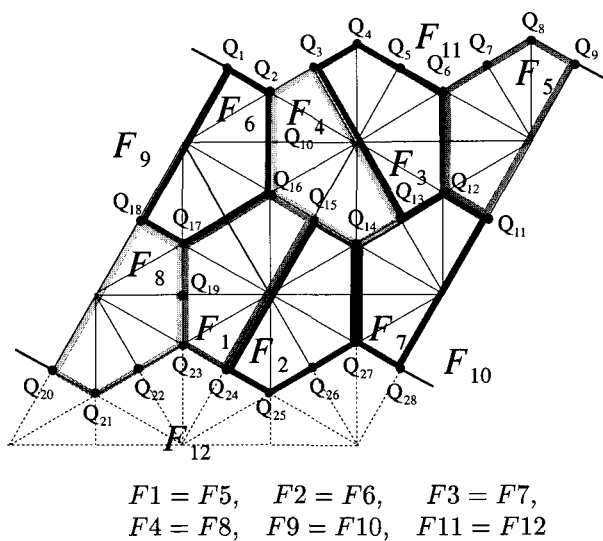


FIGURE 18. The number of faces can be decreased to 12.

$$\begin{aligned}
\epsilon_2 : F_2 \rightarrow F_6 &\Rightarrow \begin{pmatrix} Q_{24} & Q_{25} & Q_{26} & Q_{27} & Q_{14} & Q_{15} \\ Q_1 & Q_2 & Q_{10} & Q_{16} & Q_{17} & Q_{18} \end{pmatrix} \\
\epsilon_3 : F_3 \rightarrow F_7 &\Rightarrow \begin{pmatrix} Q_{13} & Q_{12} & Q_6 & Q_5 & Q_4 & Q_3 \\ Q_{28} & Q_{27} & Q_{14} & Q_{13} & Q_{12} & Q_{11} \end{pmatrix} \\
\epsilon_4 : F_4 \rightarrow F_8 &\Rightarrow \begin{pmatrix} Q_3 & Q_2 & Q_{10} & Q_{16} & Q_{15} & Q_{14} & Q_{13} \\ Q_{18} & Q_{17} & Q_{19} & Q_{23} & Q_{22} & Q_{21} & Q_{20} \end{pmatrix} \\
\epsilon_5 : F_9 \rightarrow F_{10} &\Rightarrow \begin{pmatrix} Q_1 & Q_{18} & Q_{20} & \infty \\ Q_9 & Q_{11} & Q_{28} & \infty \end{pmatrix} \\
\epsilon_6 : F_{11} \rightarrow F_{12} &\Rightarrow \begin{pmatrix} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 & Q_7 & Q_8 & Q_9 & \infty \\ Q_{20} & Q_{21} & Q_{22} & Q_{23} & Q_{24} & Q_{25} & Q_{26} & Q_{27} & Q_{28} & \infty \end{pmatrix}
\end{aligned}$$

Now we calculate all possible edge cycles to compute the relations.

$$\begin{aligned}
Q_{24}Q_{23} &\xrightarrow{\epsilon_1} Q_9Q_8 \xrightarrow{\epsilon_6} Q_{28}Q_{27} \xrightarrow{\epsilon_3^{-1}} Q_{13}Q_{12} \xrightarrow{\epsilon_3^{-1}} Q_5Q_4 \xrightarrow{\epsilon_6} Q_{24}Q_{23} \\
Q_{23}Q_{19} &\xrightarrow{\epsilon_1} Q_8Q_7 \xrightarrow{\epsilon_6} Q_{27}Q_{26} \xrightarrow{\epsilon_2} Q_{16}Q_{10} \xrightarrow{\epsilon_4} Q_{23}Q_{19} \\
Q_{19}Q_{17} &\xrightarrow{\epsilon_1} Q_7Q_6 \xrightarrow{\epsilon_6} Q_{26}Q_{25} \xrightarrow{\epsilon_2} Q_{10}Q_2 \xrightarrow{\epsilon_4} Q_{19}Q_{17} \\
Q_{17}Q_{16} &\xrightarrow{\epsilon_1} Q_6Q_{12} \xrightarrow{\epsilon_3} Q_{14}Q_{27} \xrightarrow{\epsilon_2} Q_{17}Q_{16} \\
Q_{16}Q_{15} &\xrightarrow{\epsilon_1} Q_{12}Q_{11} \xrightarrow{\epsilon_3^{-1}} Q_4Q_3 \xrightarrow{\epsilon_6} Q_{23}Q_{22} \xrightarrow{\epsilon_4^{-1}} Q_{16}Q_{15} \\
Q_{24}Q_{25} &\xrightarrow{\epsilon_2} Q_1Q_2 \xrightarrow{\epsilon_6} Q_{20}Q_{21} \xrightarrow{\epsilon_4^{-1}} Q_{13}Q_{14} \xrightarrow{\epsilon_3^{-1}} Q_5Q_6 \xrightarrow{\epsilon_6} Q_{24}Q_{25} \\
Q_{14}Q_{15} &\xrightarrow{\epsilon_2} Q_{17}Q_{18} \xrightarrow{\epsilon_4^{-1}} Q_2Q_3 \xrightarrow{\epsilon_6} Q_{21}Q_{22} \xrightarrow{\epsilon_4^{-1}} Q_{14}Q_{15} \\
Q_1Q_{18} &\xrightarrow{\epsilon_5} Q_9Q_{11} \xrightarrow{\epsilon_1^{-1}} Q_{24}Q_{15} \xrightarrow{\epsilon_2} Q_1Q_{18} \\
Q_{18}Q_{20} &\xrightarrow{\epsilon_5} Q_{11}Q_{28} \xrightarrow{\epsilon_3^{-1}} Q_3Q_{13} \xrightarrow{\epsilon_4} Q_{18}Q_{20} \\
Q_{20}Q_\infty &\xrightarrow{\epsilon_5} Q_{28}Q_\infty \xrightarrow{\epsilon_6^{-1}} Q_9Q_\infty \xrightarrow{\epsilon_5^{-1}} Q_1Q_\infty \xrightarrow{\epsilon_6} Q_{20}Q_\infty.
\end{aligned}$$

If we denote the generator which is determined by ϵ_i by x_i , then we have the following 10 relations from the above edge cycles;

$$\begin{aligned}
x_1 x_6 x_3^{-1} x_3^{-1} x_6 &= \text{id}, \\
x_1 x_6 x_2 x_4 &= \text{id}, \\
x_1 x_6 x_2 x_4 &= \text{id}, \\
x_1 x_3 x_2 &= \text{id}, \\
x_1 x_3^{-1} x_6 x_4^{-1} &= \text{id},
\end{aligned}$$

$$\begin{aligned}
x_2 x_6 x_4^{-1} x_3^{-1} x_6 &= \text{id}, \\
x_2 x_4^{-1} x_6 x_4^{-1} &= \text{id}, \\
x_5 x_1^{-1} x_2 &= \text{id}, \\
x_5 x_3^{-1} x_4 &= \text{id}, \\
x_5 x_6^{-1} x_5^{-1} x_6 &= \text{id}.
\end{aligned}$$

Reduce x_6 by using the 2nd relation and we have,

$$\begin{aligned}
x_4^{-1} x_2^{-1} x_3^{-1} x_3^{-1} x_1^{-1} x_4^{-1} x_2^{-1} &= \text{id}, \\
x_1 x_3 x_2 &= \text{id}, \\
x_1 x_3^{-1} x_1^{-1} x_4^{-1} x_2^{-1} x_4^{-1} &= \text{id}, \\
x_1^{-1} x_4^{-1} x_2^{-1} x_4^{-1} x_3^{-1} x_1^{-1} x_4^{-1} &= \text{id}, \\
x_2 x_4^{-1} x_1^{-1} x_4^{-1} x_2^{-1} x_4^{-1} &= \text{id}, \\
x_5 x_1^{-1} x_2 &= \text{id}, \\
x_5 x_3^{-1} x_4 &= \text{id}, \\
x_5 x_2 x_4 x_1 x_5^{-1} x_1^{-1} x_4^{-1} x_2^{-1} &= \text{id}.
\end{aligned}$$

Reduce x_5 by using the 6th relation and we have,

$$\begin{aligned}
x_4^{-1} x_2^{-1} x_3^{-1} x_3^{-1} x_1^{-1} x_4^{-1} x_2^{-1} &= \text{id}, \\
x_1 x_3 x_2 &= \text{id}, \\
x_1 x_3^{-1} x_1^{-1} x_4^{-1} x_2^{-1} x_4^{-1} &= \text{id}, \\
x_1 x_4^{-1} x_2^{-1} x_4^{-1} x_3^{-1} x_1^{-1} x_4^{-1} &= \text{id}, \\
x_2 x_4^{-1} x_1^{-1} x_4^{-1} x_2^{-1} x_4 &= \text{id}, \\
x_2^{-1} x_1 x_3^{-1} x_4 &= \text{id}, \\
x_2^{-1} x_1 x_2 x_4 x_2 x_1^{-1} x_4^{-1} x_2^{-1} &= \text{id}.
\end{aligned}$$

Reduce x_4 by using the 6th relation and we have,

$$\begin{aligned}
x_2^{-1} x_1 x_3^{-1} x_2^{-1} x_3^{-1} x_3^{-1} x_1^{-1} x_2^{-1} x_1 x_3^{-1} x_2^{-1} &= \text{id}, \\
x_1 x_3 x_2 &= \text{id}, \\
x_1 x_3^{-1} x_1^{-1} x_2^{-1} x_1 x_3^{-1} x_2^{-1} x_1 x_3^{-1} &= \text{id}, \\
x_1^{-1} x_2^{-1} x_1 x_3^{-1} x_2^{-1} x_2^{-1} x_1 x_3^{-1} x_3^{-1} x_1^{-1} x_2^{-1} x_1 x_3^{-1} &= \text{id}, \\
x_2^{-1} x_1 x_2 x_3 x_1^{-1} x_2 x_2 x_1^{-1} x_2^{-1} x_1 x_3^{-1} x_2^{-1} &= \text{id}.
\end{aligned}$$

Reduce x_3 by using the 2nd relation and we have,

$$\begin{aligned}
x_2^{-1} x_1 x_2 x_1 x_1 x_2 x_1 x_2^{-1} &= \text{id}, \\
x_2^{-1} x_1 x_2 x_1^{-1} x_2^{-1} x_1^{-1} x_2 x_2 x_1^{-1} x_2^{-1} x_1 x_2 x_1 x_2^{-1} &= \text{id}.
\end{aligned}$$

Reduce $x_1 x_2 x_1 x_2^{-1} x_2^{-1} x_1 x_2$ by using the 1st relation and we have,

$$x_1^{-1} x_2^{-1} x_1^{-1} x_2 x_2 x_1^{-1} x_2^{-1} x_1^{-1} x_1^{-1} = \text{id}.$$

Finally we have the following presentation of the fundamental group of M' when we replace x_1^{-1} by x and x_2 by y . It consists of two generators and

one relation of 9 words and it is easy to see from the presentation that the abelianization of $\pi_1(M')$ is the direct sum $\mathbb{Z} \oplus \mathbb{Z}_5$.

Theorem 4.1.

$$\pi_1(M') = \langle x, y | x y^{-1} x y y x y^{-1} x x = \text{id} \rangle.$$

5. Generalization of face reducing procedure

In previous two sections, we reduce the number of the bottom faces of Ford domain of figure-eight knot complement to 6 and that of the sibling to 8. In this section, we will generalize the face reducing procedure.

Theorem 5.1. *If the faces of a Ford domain which contains the ideal vertex are glued by two parabolic elements and the number of edges which are identified by glueing regular ideal tetrahedra is n then there exists a method which reduces the number of bottom faces of the Ford domain less than or equal to $4n$.*

Proof. Since the number of edges which are identified by glueing tetrahedra is n , we may obtain $2n$ vertices of the Ford domain at the height 1. Those are the first collision points of horoballs but some vertices can be identified by parabolic translations. (See figure 19)

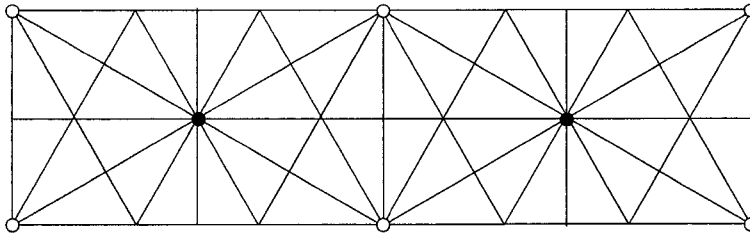


FIGURE 19. Vertices at height 1

After glueing regular ideal tetrahedra, we will have a non compact manifold. Hence the sum of angles around an edge of tetrahedra must be 2π and therefore we observe that 12 small triangles of the bottom face of the Ford domain are glued together around each vertex and they make a hexagon centered at each vertex. For each hexagon, there exists another hexagon which is glued to the corresponding hexagon by a face-pairing map of the Ford domain.

Because n hexagons are glued to another n hexagons and the $2n$ hexagons fill up the whole bottom face of Ford domain by two parabolic translations, we may reduce the number bottom faces of Ford domain to $2n$. (See figure 20)

Note that the vertices on the bottom region which are also adjacent to the vertex at infinity along a face with the vertex at infinity must be identified by parabolic translations. But usually we may have a region which does not hold that restriction as in figure 20. So we need to make some alterations by cutting

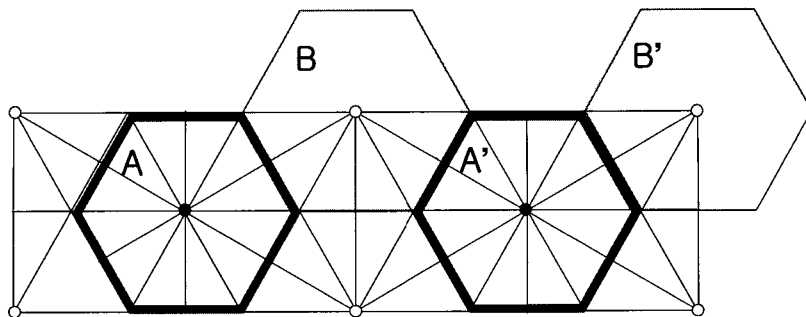


FIGURE 20. A hexagon consists of 12 small triangles and the corresponding hexagon.

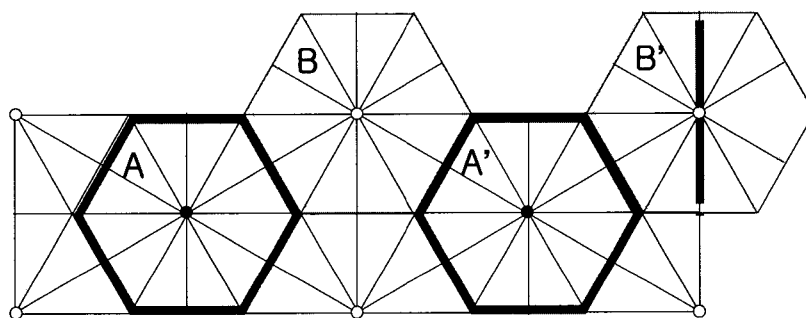


FIGURE 21. Cut a hexagon by the intersection locus.

and pasting so that the bottom region of the Ford domain satisfies the given restriction. Now we will describe a cutting and pasting alteration.

We call the two parabolic translation by α , β and θ is the angle formed by α and β .

We will consider the parabolic translation α as the horizontal translation. From the viewpoint at the point of infinity, the shape of the bottom region of the Ford domain is a parallelogram with angle θ . We will call the lines which extend the edges of parallelogram corresponding to α “horizontal lines” and the extensions of another edges “vertical lines”. In fact “horizontal lines” and “vertical lines” are not genuinely lines, but a set of piecewise geodesic lines, But it looks like Euclidean lines from the viewpoint at the point of infinity.

Now we consider a hexagon which meets a vertical line of the Ford domain. We will call the line segment formed by the intersection of a hexagon and a vertical line the intersection locus. If there exists a hexagon which meets vertical lines Cut the hexagon by the intersection locus as the upper right hexagon in figure (see figure 21) and move the right hand side of the divided hexagon into the region between the two vertical lines by the parabolic translation α .

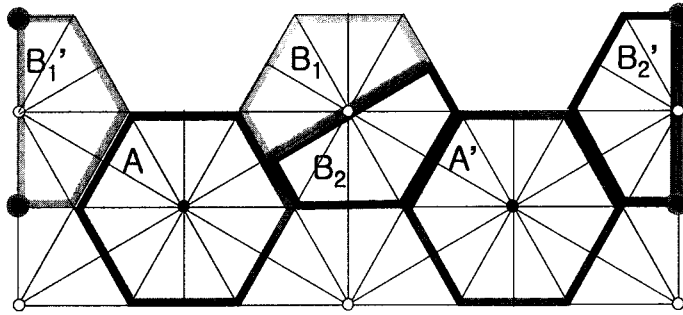


FIGURE 22. Finding the corresponding locus.

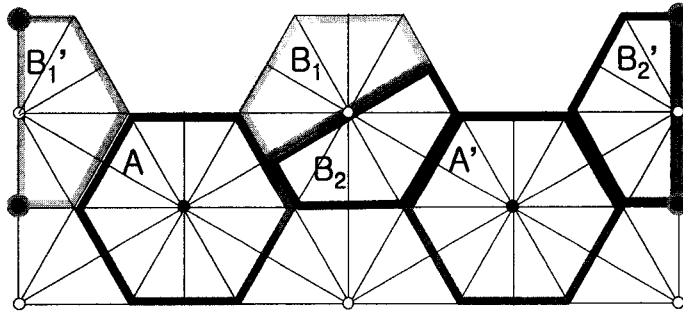


FIGURE 23. The bottom region which satisfies the restriction. Notice the new "big vertices"

We have a corresponding intersection locus in another hexagon which is glued to the given hexagon. (See figure 22)

Now we repeat this cutting and pasting procedure for all the remaining hexagons which are cut by vertical lines. And eventually we may obtain a region which satisfies the above restriction.

The largest possible number of reduced bottom faces can be made when all hexagons are divided by two parts. So the number of reduced bottom faces is less than or equal to $4n$. For example when $n = 2$, the number of bottom face is 8 for the sibling of figure-eight knot complement and the number of bottom face is 6 for the figure-eight knot complement. \square

Finally, we will describe a cutting and pasting alteration procedure to reduce the number of faces on the bottom region of the Ford domain for the sibling of figure-8 knot complement.

At first, we have the bottom region of the Ford domain with the shape of parallelogram and the angle formed by two parabolic elements is $\frac{\pi}{3}$. Since the number of edges after glueing two tetrahedra is 2, there are 4 vertices at height

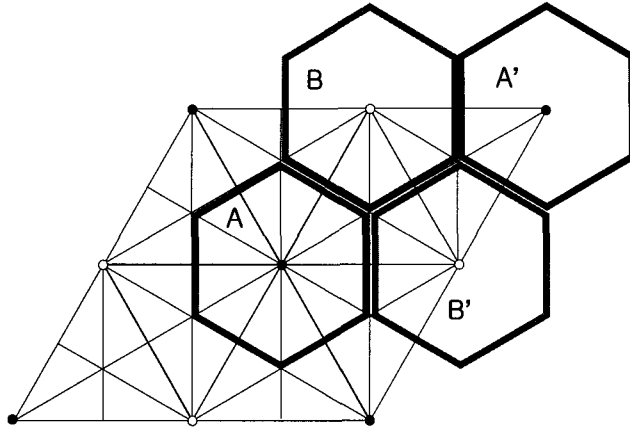


FIGURE 24. Height 1 vertices and its hexagons in the bottom of Ford domain of sibling of figure-8 knot complement

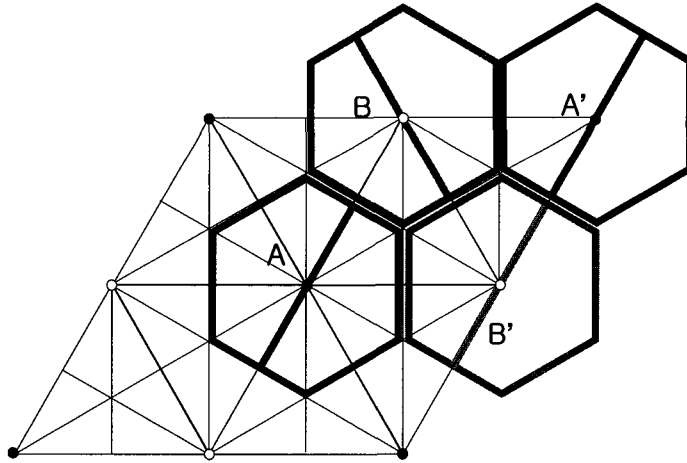


FIGURE 25. Intersection loci of vertical line and hexagons and the corresponding loci.

1. So we have 4 hexagonal faces for the bottom region of the Ford domain. (See figure 24)

Since the region that consists of 4 hexagons does not satisfies the restriction which says that the vertices adjacent to infinity are identified parabolic translations, we need some cutting and pasting alterations.

First note that there are two hexagons which meet vertical lines, so we have two loci and we can find corresponding loci in the other hexagons. (See figure 25)

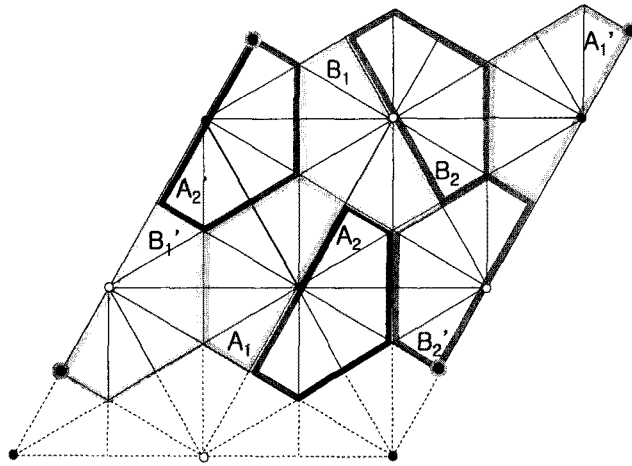


FIGURE 26. The bottom region of the Ford domain which holds the restriction. Notice the new “big vertices”

Move the right hand side of the divided two hexagons which meet vertical line into the region between the vertical lines by parabolic translations. Then we obtain a bottom region of the Ford domain which satisfies the restriction. (See figure 26)

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