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HYPERCYCLICITY ON INVARIANT SUBSPACES

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Abstract. A continuous linear operator $T : \mathcal{X} \to \mathcal{X}$ is called hypercyclic if there exists an $x \in \mathcal{X}$ such that the orbit $\{T^n x\}_{n\geq 0}$ is dense. We consider the problem: given an operator $T : \mathcal{X} \to \mathcal{X}$, hypercyclic or not, is the restriction $T|_{\mathcal{Y}}$ to some closed invariant subspace $\mathcal{Y} \subset \mathcal{X}$ hypercyclic? In particular, it is well-known that any non-constant partial differential operator $p(D)$ on $H(\mathbb{C}^d)$ (entire functions) is hypercyclic. Now, if $q(D)$ is another such operator, $p(D)$ maps $\ker q(D)$ invariantly (by commutativity), and we obtain a necessary and sufficient condition on p and q in order that the restriction $p(D) : \ker q(D) \to \ker q(D)$ is hypercyclic. We also study hypercyclicity for other types of operators on subspaces of $H(\mathbb{C}^d)$.

1. Introduction

In all that follows, $\mathcal{X}$ denotes a real or complex separated locally convex space, and $L(\mathcal{X})$ the algebra of continuous linear operators $T : \mathcal{X} \to \mathcal{X}$ on $\mathcal{X}$. An operator $T \in L(\mathcal{X})$ is said to be *hypercyclic* if for some vector $x \in \mathcal{X}$, called hypercyclic for T, the orbit $\{T^n x\}_{n\geq 0}$ is dense. Thus, the existence of a hypercyclic operator in $L(\mathcal{X})$ requires that $\mathcal{X}$ is separable and, see [15], infinite dimensional (unless $\mathcal{X} = \{0\}$). For Fréchet spaces we have the following well-known Hypercyclicity Criterion to establish hypercyclicity, see [1]:

Proposition 1. *Let $\mathcal{X}$ be a separable Fréchet space and assume $T \in L(\mathcal{X})$ satisfies the* Hypercyclicity Criterion (HC): *There exist dense subspaces $\mathcal{Z}, \mathcal{Y} \subseteq \mathcal{X}$, and sequences (S_k) and (n_k) of linear maps $S_k : \mathcal{Y} \to \mathcal{X}$ and of natural numbers n_k, such that:*

(1) $T^{n_k} z \to 0$ *for all* $z \in \mathcal{Z}$,
(2) $S_k y \to 0$ *for all* $y \in \mathcal{Y}$,
(3) $T^{n_k} S_k y \to y$ *for all* $y \in \mathcal{Y}$.

Then T is hypercyclic.

We say that T satisfies the HC with respect to a given sequence $(n_k) \subseteq \mathbb{N}$, if this sequence can be used in the criterion (i.e., in (1) and (3)). It is convenient

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to note that if T satisfies the HC with respect to (n_k), then T satisfies the HC for any subsequence of (n_k).

In this work we highlight the following two problems:

Problem 1. Given a hypercyclic operator $T \in L(\mathcal{X})$, can we find a closed invariant subspace $\mathcal{Y} \subset \mathcal{X}$ for which the restriction $T|_{\mathcal{Y}}$ of T to $\mathcal{Y}$ is also hypercyclic?

Another problem is to go in the other direction:

Problem 2. Given a hypercyclic operator $T : \mathcal{Y} \to \mathcal{Y}$, where $\mathcal{Y}$ is a proper closed subspace of $\mathcal{X}$, can we extend T to a hypercyclic operator $\hat{T} \in L(\mathcal{X})$?

Note that, in the first problem, every hypercyclic vector x for $T : \mathcal{X} \to \mathcal{X}$ must necessarily be outside of $\mathcal{Y}$ (since $\mathcal{Y}$ is closed). Thus the hypercyclic vectors for T and $T|_{\mathcal{Y}}$ (and thus those for $T : \mathcal{Y} \to \mathcal{Y}$ and $\hat{T}$ in Problem 2) are distinct. In other words, $T|_{\mathcal{Y}}$ does not by no means inherit the hypercyclic property of T. Let us recall that every infinite dimensional separable Fréchet space $\mathcal{X}$ supports a hypercyclic operator [2], and hence so does any infinite dimensional closed subspace $\mathcal{Y} \subset \mathcal{X}$.

The following simple observation was one of the factors that motivated us to pose Problems 1 and 2. Recall that the differentiation operators $D_i \equiv \partial/\partial z_i$ are hypercyclic on the Fréchet space $H(\mathbb{C}^d)$ of d-variable entire functions provided with the compact-open topology (in fact, they satisfy the HC with respect to the full sequence $(n_k = k)$). (See Proposition 3 for a more general result.) Consider now the space $H(\mathbb{C}^2)$ and the operators D_1 and D_2. We know that D_1 is hypercyclic on $H(\mathbb{C}^2)$, and it is clear that $\mathcal{Y} \equiv \ker D_2 = H(\mathbb{C}_1)$ ($\equiv \{f(z_1) : f \in H(\mathbb{C})\}$) and D_1 maps $\mathcal{Y}$ invariantly. It follows now that D_1 is hypercyclic on $\mathcal{Y}$, since D is hypercyclic on $H(\mathbb{C})$. Even more can be said. $\mathcal{Y}$ is complemented in $H(\mathbb{C}^2)$. Indeed, $H(\mathbb{C}^2) = \mathcal{Z} \oplus \mathcal{Y}$ is a topological decomposition where $\mathcal{Z} \equiv z_2 H(\mathbb{C}^2)$. (Recall that a subspace $\mathcal{Y} \subseteq \mathcal{X}$ is said to be *complemented* in $\mathcal{X}$ if there exists a subspace $\mathcal{Z} \subseteq \mathcal{X}$ such that $\mathcal{X} = \mathcal{Z} \oplus \mathcal{Y}$, and where the corresponding projector $\mathcal{X} \to \mathcal{Y}$ (or equivalently $\mathcal{X} \to \mathcal{Z}$) is continuous. In this case we say that $\mathcal{X} = \mathcal{Z} \oplus \mathcal{Y}$ is a *topological decomposition* of $\mathcal{X}$.) We notice that D_1 maps $\mathcal{Z}$ invariantly, and if $f = f(z_1, z_2)$ is a hypercyclic vector for D_1 acting on $H(\mathbb{C}^2)$, then $z_2 f \in \mathcal{Z}$ is evidently hypercyclic for $D_1 : \mathcal{Z} \to \mathcal{Z}$. Thus, we can decompose $H(\mathbb{C}^2)$ into a sum of two closed invariant subspaces $\mathcal{Z}, \mathcal{Y}$ for D_1, and for which the corresponding restrictions of D_1 are hypercyclic. At this point the following question arises; if $z \in \mathcal{Z}$ is hypercyclic for $D_1 : \mathcal{Z} \to \mathcal{Z}$ and $y \in \mathcal{Y}$ for $D_1 : \mathcal{Y} \to \mathcal{Y}$, does $h \equiv z + y$ form a hypercyclic vector for D_1 acting on the full space $H(\mathbb{C}^2)$? What we know is that for any given $f \in H(\mathbb{C}^2)$ and neighborhood U_f of f, there exist $n, m \in \mathbb{N}$ such that $D_1^n z + D_1^m y \in U_f$. Our question remains thus open, since we do not know if, for a *general* choice of z and y, we always can choose $n = m$. However, we shall see (by Proposition 7) that there exist vectors z, y for which this is always possible:

Proposition 2. *$\mathcal{Y} \equiv H(\mathbb{C}_1)$ and $\mathcal{Z} \equiv z_2 H(\mathbb{C}^2)$ are closed invariant subspaces for $D_1 : H(\mathbb{C}^2) \to H(\mathbb{C}^2)$ with $H(\mathbb{C}^2) = \mathcal{Z} \oplus \mathcal{Y}$ (topological decomposition), and the restrictions $D_1 : \mathcal{Y} \to \mathcal{Y}$ and $D_1 : \mathcal{Z} \to \mathcal{Z}$ are hypercyclic. Further, for any hypercyclic vector $y \in \mathcal{Y}$ for $D_1 : \mathcal{Y} \to \mathcal{Y}$, there exists a hypercyclic vector $z \in \mathcal{Z}$ for $D_1 : \mathcal{Z} \to \mathcal{Z}$ such that $z + y$ forms a hypercyclic vector for $D_1 : H(\mathbb{C}^2) \to H(\mathbb{C}^2)$, and the analogue holds if we start with a hypercyclic vector $z \in \mathcal{Z}$ for $D_1 : \mathcal{Z} \to \mathcal{Z}$.*

The idea of this paper is to consider Problems 1 and 2 in the complex analysis setting. In particular we approximate kernels to PDE:s in the following way. If $p(D)$ and $q(D)$ are partial differential operators with constant coefficients acting on $H(\mathbb{C}^d)$, then $p(D)$ and $q(D)$ commute. Consequently, $p(D)$ maps $\ker q(D)$ invariantly, and in Section 2 we establish a necessary and sufficient condition on the polynomials p and q in order that $p(D) : \ker q(D) \to \ker q(D)$ is hypercyclic (Theorem 1). For any such pair p, q, there exists thus a solution $f \in H(\mathbb{C}^d)$ to $q(D)f = 0$ such that any other homogeneous solution lies arbitrarily close to some $p(D)^n f = p^n(D)f$. In Section 3, we construct certain closed subspaces of $H(\mathbb{C}^{2d})$ and prove that for a suitable choice $E \subset H(\mathbb{C}^{2d})$, we can to every operator T on $H(\mathbb{C}^d)$ that satisfies the HC, associate a hypercyclic operator T_E on E (Theorem 2). Further, when $d = 1$, T_E extends to a hypercyclic operator $\hat{T}_E$ on the full space $H(\mathbb{C}^2)$. In the last section, Section 4, we discuss extensions of our obtained results, especially extensions to other spaces than $H(\mathbb{C}^d)$.

2. On Problem 1

Recall that a *convolution operator* T on $H(\mathbb{C}^d)$ is a continuous linear operator that commutes with all translations $\tau_a : f \mapsto f(z + a)$, $a \in \mathbb{C}^d$, and T is called *trivial* if it is a scalar multiple of the identity. The following well-known result of Godefroy and Shapiro [5, Section 5] is some sort of basis for our investigation in this section. Recall first that the algebra $\mathrm{Exp}(\mathbb{C}^d)$ of exponential type functions is formed by all $\varphi \in H(\mathbb{C}^d)$ such that, for some $M, r > 0$, $|\varphi(z)| \leq M e^{r\|z\|}$ where $\|\cdot\|$ denotes the Euclidian norm on $\mathbb{C}^d$. In view of our purposes, it is also convenient to recall that $\mathrm{Exp}(\mathbb{C}^d)$ can be identified with $H'(\mathbb{C}^d)$ via the bilinear form $\langle f, \varphi\rangle \equiv \sum_{\alpha\in\mathbb{N}^d} f^{(\alpha)}(0)\varphi^{(\alpha)}(0)/\alpha!$ ($\alpha! \equiv \prod \alpha_i!$) on $H \times \mathrm{Exp}$.

Proposition 3 (Godefroy, Shapiro [5]). *The map $\varphi \mapsto \varphi(D) \equiv \sum_{\alpha\in\mathbb{N}^d} \varphi_\alpha D^\alpha$, where $\varphi(z) = \sum_{\alpha\in\mathbb{N}^d} \varphi_\alpha z^\alpha$ and $D^\alpha \equiv D_1^{\alpha_1} \cdots D_d^{\alpha_d}$, defines an algebra isomorphism between $\mathrm{Exp}(\mathbb{C}^d)$ and the set $\mathcal{C}$ of convolution operators on $H(\mathbb{C}^d)$. Any nontrivial convolution operator $\varphi(D)$, i.e., φ is not a constant mapping, satisfies the* HC *(with respect to the full sequence) and is thus hypercyclic on $H(\mathbb{C}^d)$.*

Note that $\tau_a = e_a(D)$, where $e_a(z) \equiv \exp \sum z_i a_i$, and $\varphi(D)f(z) = \langle f, \varphi e_z \rangle$. In particular Proposition 3 yields that the operators in $\mathcal{C}$ commute and, accordingly, any $\varphi(D) \in \mathcal{C}$ maps any kernel $\ker \psi(D)$, $\psi \in \mathrm{Exp}(\mathbb{C}^d)$, invariantly. This suggests, in view of Problem 1, the following definition:

Definition 1. Let $\varphi, \psi \in \mathrm{Exp}(\mathbb{C}^d)$. We say that $\varphi(D)$ is ψ-hypercyclic if the restriction $\varphi(D) : \ker \psi(D) \to \ker \psi(D)$ is hypercyclic.

In the Introduction (Proposition 2) we noticed that D_1 is z_2-hypercyclic ($d = 2$) and, in general, a ψ-hypercyclic operator $T = \varphi(D)$ solves Problem 1 with $\mathcal{Y} = \ker \psi(D)$. Let us consider some trivial cases. If $\psi = 0$, then $\ker \psi(D) = H$ and Proposition 3 gives the corresponding ψ-hypercyclic operators. Next, if ψ is a unit in Exp, i.e., $\psi = Ae_a$ where $A \neq 0$, then $\psi(D) = A\tau_a$ and $\ker \psi(D) = \{0\}$. Thus every convolution operator is ψ-hypercyclic in this case, and especially this holds if ψ is a constant $\neq 0$. Any $\varphi(D)$ fails of course to be φ-hypercyclic, unless φ is a unit. Further, when $d = 1$ we know that no nonconstant polynomial ψ supports any ψ-hypercyclic operator, for $\ker \psi(D)$ is then finite dimensional and $\neq \{0\}$. We shall improve this by proving that, when $d = 1$, ψ admits a ψ-hypercyclic operator if and only if $\psi = Ae_a$ for some $A, a \in \mathbb{C}$.

Noteworthy is that for arbitrary d, any kernel $\ker \psi(D)$ is complemented in $H(\mathbb{C}^d)$. (This is a consequence of Taylor and Meise's result from [9], saying that any convolution operator has a continuous linear right inverse.)

After these preliminary observations, the objective is to give a necessary and sufficient condition for $\varphi(D)$ to be ψ-hypercyclic in the case when ψ is a polynomial p. By the discussion above, we may assume p is nonconstant.

Let us recall some terminology from analytic geometry (for an excellent exposition of this theory we refer to [3]). A *regular point* of an analytic set A in $\mathbb{C}^d$, e.g. $A = Z(f) \equiv \{z : f(z) = 0\}$ where $f \in H$, is a point $a \in A$ for which there exists a neighborhood U of a in $\mathbb{C}^d$ such that $A \cap U$ forms a complex manifold. The set of regular points, reg$A$, forms an open and dense subset of A, and its connected components form complex manifolds.

Next, an analytic set A in $\mathbb{C}^d$ is irreducible if it can not be written as a union of two proper analytic subsets. An irreducible analytic subset A' of an analytic set A is called an *irreducible component* of A if every analytic subset $A'' \subseteq A$ with $A' \subset A''$ is reducible. It follows that every irreducible analytic subset of A is contained in an irreducible component. In fact, if $\cup_i R_i$ is the decomposition of reg$A$ into its connected components R_i, then the irreducible components of A are given by the sets $\bar{R}_i$ (= closure in A, which, since any analytic set is closed, equals the closure in $\mathbb{C}^d$). By the density of reg$A$ in A, we have that $A = \cup_i \bar{R}_i$, and this is the decomposition of A into irreducible components. In particular, if $p = p_1^{r_1} \cdots p_m^{r_m}$ is the factorization of a nonconstant polynomial p into irreducible factors p_i with corresponding multiplicities r_i, the irreducible components of $Z(p)$ are formed by $Z(p_i)$, $i = 1, \ldots, m$.

If $f \in H$, we denote by $\mathrm{ord}_f(a)$ the order of the zero $a \in Z(f)$. Thus $\mathrm{ord}_f(a)$ is the largest natural number n for which $D^\alpha f(a) = 0$ whenever $|\alpha| < n$.

Lemma 1. *Let $p = p_1^{r_1} \cdots p_m^{r_m}$ be the factorization of a polynomial p into irreducible factors p_i with corresponding multiplicities $r_i \geq 1$. Let $U \subseteq \mathbb{C}^d$ be any open set such that U meets every irreducible component $Z(p_i)$ of $Z(p)$. Then*

$$(1) \qquad E_U \equiv \{qe_a : a \in U \cap Z(p),\ q \text{ is a polynomial with } \deg p < \mathrm{ord}_p(a)\}$$

forms a total subset of $\ker p(D)$.

Proof. First of all, E_U is indeed a subset of $\ker p(D)$. For let qe_a be any element of E_U. Then $p(D)qe_a = e_a p(a+D)q$. Now $p(a+D)$ is a sum of derivatives D^α of order $|\alpha| \geq \mathrm{ord}_p(a)$, and so $p(a+D)q = 0$.

Next, we intend to prove the statement by induction over the sum $\sum r_i$. For the starting value $\sum r_i = m$, we shall apply the following (cf. [10, Chapter 0.2]):

Sublemma. Let $p = p_1 \cdots p_m$ be a polynomial with distinct irreducible factors p_i and assume $U \cap Z(p_i) \neq \emptyset$ for all i, where $U \subseteq \mathbb{C}^d$ is open. Then $p \cdot \mathrm{Exp} = \{\varphi : U \cap Z(p) \subseteq Z(\varphi)\}$.

Proof of Sublemma. Assume first that $U = \mathbb{C}^d$. We must then prove that $p \cdot \mathrm{Exp} = \{\varphi : Z(p) \subseteq Z(\varphi)\}$. To this end we assume first that p is irreducible. So suppose $Z(p) \subseteq Z(\varphi)$. We must prove that $\varphi = p\psi$ for some $\psi \in \mathrm{Exp}$. But the proof of [18, Lemma 29.2] shows that $\varphi = p\psi$ for some (unique) entire $\psi \in H$. Thus we only have to prove that ψ is of exponential type, which follows by [18, Lemma 28.1]. Next, let $p = p_1 \cdots p_m$, where p_i are distinct irreducible polynomials, and assume $Z(p) \subseteq Z(\varphi)$. Then $Z(p_i) \subseteq Z(\varphi)$ for all i and so, from what we just have proved, $\varphi = p_1\psi_1 = p_2\psi_2 = \cdots = p_m\psi_m$ for some unique $\psi_i \in \mathrm{Exp}$. By virtue of [13, Lemma 4] we conclude that $p|\varphi$ in Exp. Hence the Sublemma holds when $U = \mathbb{C}^d$.

Let now $U \subset \mathbb{C}^d$. From what we just have proved, it suffices to prove that if $\varphi \in \mathrm{Exp}$ vanishes on $U \cap Z(p)$, then φ vanishes on $Z(p)$. But let Z_0 be any connected component of $\mathrm{reg} Z(p)$. Then Z_0 is densely contained in some $Z(p_i)$. Now, U meets $Z(p_i)$, and hence Z_0, so $U_0 \equiv U \cap Z_0$ forms a nonempty open set in the complex manifold Z_0. But φ vanishes on U_0, and hence on all of Z_0. Since Z_0 was arbitrary we conclude that $\varphi|_{\mathrm{reg} Z(p)} = 0$ and finally, by density, $\varphi|_{Z(p)} = 0$ and the Sublemma follows.

The Sublemma shows now that $\{e_a : a \in U \cap Z(p)\}$ is dense in $\ker p(D)$ if $\sum r_i = m$. Indeed, assume $\varphi \in \mathrm{Exp} \simeq H'$ is orthogonal to $\{e_a : a \in U \cap Z(p)\}$. Then $0 = \langle e_a, \varphi\rangle = \varphi(a)$ for all $a \in U \cap Z(p)$. Hence, $\varphi \in p \cdot \mathrm{Exp}$ which equals $\ker p(D)^\perp$, since $p : \varphi \mapsto p\varphi$ is the transpose of (the surjective operator [18, Theorem 28.2]) $p(D)$. Thus the lemma holds when $\sum r_i = m$. Assume that the lemma is proved for $\sum r_i = m, \ldots, n$. Let now $\sum r_i = n$, we must then prove the statement for $p' \equiv p_1^{r_1'} p_2^{r_2} \cdots p_m^{r_m}$ where $r_1' \equiv r_1 + 1$. Assume φ is orthogonal to $\{qe_a : a \in U \cap Z(p'),\ \deg q < \mathrm{ord}_{p'}(a)\}$. But then φ is orthogonal to the smaller set $\{qe_a : a \in U \cap Z(p),\ \deg q < \mathrm{ord}_p(a)\}$ where $p \equiv p_1^{r_1} \cdots p_m^{r_m}$, and so, by the inductive hypothesis, $\varphi = p\psi$ for some $\psi \in \mathrm{Exp}$. We must thus prove that $p_1 | \psi$ which, by the Sublemma, is equivalent to $U \cap Z(p_1) \subseteq Z(\psi)$. So let $a \in U \cap Z(p_1)$ be arbitrary and put $\nu \equiv \mathrm{ord}_{p_1}(a)$. Then $1 \leq \nu < \mathrm{ord}_{p'}(a)$ and there exists a polynomial q with $\deg q = \nu$ such that $q(D)p_1(a) \neq 0$, while $q_0(D)p_1(a) = 0$ when $\deg q_0 < \nu$. Hence, by Leibniz' Formula $q(D)(fg) = \sum_\alpha (q^{(\alpha)}(D)f)D^\alpha g/\alpha!$, we obtain

$$0 = \langle qe_a, \varphi\rangle = q(D)\varphi(a) = q(D)(p_1\psi)(a) = q(D)p_1(a) \cdot \psi(a).$$

Thus $\psi(a) = 0$ and we are done. □

Theorem 1. *Let $p = p_1^{r_1} \cdots p_m^{r_m}$ be the factorization of a nonconstant polynomial p into irreducible factors p_i with corresponding multiplicities $r_i \geq 1$. A necessary and sufficient condition for $\varphi(D)$ to be p-hypercyclic is that the restriction $\varphi|_{Z(p_i)}$ is nonconstant for all i.*

Proof. First we prove the sufficient part, and we shall apply the HC. The proof of the following Sublemma is due to Edgar Lee Stout [17].

Sublemma. Let $\varphi, p \in \mathrm{Exp}$, where p is an irreducible polynomial. Then $\varphi|_{Z(p)}$ is nonconstant if and only if the sets $\Phi_0 \equiv \{z : |\varphi(z)| < 1\}$ and $\Phi_\infty \equiv \{z : |\varphi(z)| > 1\}$ meet $Z(p)$.

Proof of Sublemma. Assume Φ_∞ does not meet $Z(p)$. This means that φ is bounded on $Z(p)$, and we must prove the analogue of Liouville's Theorem, that φ must be constant on $Z(p)$. But $Z(p)$ is an irreducible algebraic variety V and hence, there exists a "projection" $\pi : V \to \mathbb{C}^k$ that is an analytic cover [3, Prop. 7.3.2]. Now the algebra $H(V)$ of analytic functions on V (i.e., analytic in a neighborhood of V), is integral over $\pi^* H(\mathbb{C}^k) = \{f \circ \pi : f \in H(\mathbb{C}^k)\} \subseteq H(V)$. (This is essentially Noether's Normalization Theorem.) In particular $\varphi \in H(V)$ and so there is a monic polynomial $m(x) = x^\nu + g_{\nu-1}x^{\nu-1} + \cdots + g_0$, with coefficients $g_i \in \pi^* H(\mathbb{C}^k)$, such that $m(\varphi) = 0$. The coefficients g_i are symmetric functions in the values of φ. For example, if $p \in V$, $g_0(p) =$ the product of the values of φ on the fiber $\pi^{-1}(\pi(p))$, taken with appropriate multiplicities. Since φ is bounded on V, every g_i must be bounded and hence, by Liouville's Theorem, they must be constant. But then, since V is irreducible and φ is continuous, φ must be constant on $V = Z(p)$.

That $\Phi_0 \cap Z(p) = \emptyset$ implies that φ is constant on $Z(p)$, follows from the arguments above by considering the function $1/\varphi$, which is analytic in a neighborhood of $Z(p)$ if $\Phi_0 \cap Z(p) = \emptyset$.

Since it is evident that $\varphi|_{Z(p)}$ is nonconstant whenever Φ_0 and Φ_∞ meet $Z(p)$, the Sublemma follows.

Hence, our hypothesis is equivalent to that Φ_0 and Φ_∞ meet every irreducible component $Z(p_i)$ of $Z(p)$. But Φ_0 and Φ_∞ are open so, by Lemma 1, the sets $\mathcal{Z} \equiv \mathrm{span}\, E_{\Phi_0}$ and $\mathcal{Y} \equiv \mathrm{span}\, E_{\Phi_\infty}$ are dense in $\mathcal{X} \equiv \ker p(D)$. We prove that

$\varphi(D)^n \to 0$ pointwise on $\mathcal{Z}$. It is enough to prove that $\varphi(D)^n z^\alpha e_a \to 0$ for any α and $a \in \Phi_0$. But $\varphi(D)^n z^\alpha e_a = e_a \varphi^n(a+D) z^\alpha$ and, with usual multi-index notation,

$$\varphi^n(a+D)z^\alpha = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} z^{\alpha-\beta} D^\beta \varphi^n(a).$$

Hence it suffices to prove that $D^\alpha \varphi^n(a) \to 0$ $(n \to \infty)$ for any α and $a \in \Phi_0$, in fact, it is easier to prove the following more general:

Claim. Assume $\varphi \in \mathrm{Exp}$ and $|\varphi(a)| < 1$. Then $n^\nu D^\alpha(\varphi^n \psi)(a) \to 0$ $(n \to \infty)$ for all $\nu > 0$, $\psi \in \mathrm{Exp}$ and multi-indices α.

Proof of Claim. When $|\alpha| = 0$ the statement is evidently true. Assume now the claim holds for $|\alpha| = 0, \ldots, m$. Then for any $i \le d$ and α with $|\alpha| = m$:

$$n^\nu D_i D^\alpha[\varphi^n \psi](a) = n^\nu D^\alpha[n\varphi^{n-1}\psi D_i \varphi + \varphi^n D_i \psi](a) \to 0$$

by the inductive hypothesis. Hence the claim.

Next we construct a right inverse $S : \mathcal{Y} \to \mathcal{Y}$ to $\varphi(D)$ in the following recursive way. (We apply the fact that the exponential polynomials $z^\alpha e_a$, $\alpha \in \mathbb{N}^d$, $a \in \mathbb{C}^d$, form a linearly independent set in H.) Let $a \in \Phi_\infty \cap Z(p)$. Then we define $Se_a \equiv e_a/\varphi(a)$, and extend S linearly to $\mathrm{span}\{e_a\}$. Assume now that S is defined on $\mathcal{Y}_{a:n} \equiv \mathrm{span}\{z^\alpha e_a : |\alpha| \le n\}$ in such a way that S forms a right inverse to $\varphi(D)$ on this set and S maps $\mathcal{Y}_{a:n}$ invariantly. Then if $|\alpha| = n+1 < \mathrm{ord}_p(a)$ we define

$$\begin{aligned} S(z^\alpha e_a) &\equiv z^\alpha e_a/\varphi(a) - S([\varphi(D) - \varphi(a)]z^\alpha e_a/\varphi(a)) \\ &= z^\alpha e_a/\varphi(a) - S(e_a[\varphi(a+D) - \varphi(a)]z^\alpha/\varphi(a)), \end{aligned}$$

and extend S linearly to all of $\mathcal{Y}_{a:n+1}$. In this way we obtain a right inverse to $\varphi(D)$ on $\mathcal{Y}_a \equiv \mathrm{span}\{z^\alpha e_a : |\alpha| < \mathrm{ord}_p(a)\}$. By doing this for all $a \in \Phi_\infty \cap Z(p)$, we obtain a right inverse S to $\varphi(D)$ on $\cup_{\Phi_\infty \cap Z(p)} \mathcal{Y}_a$, which we finally extend linearly to a right inverse S on $\mathcal{Y}$. By our construction S maps $\mathcal{Y}$ invariantly, and trivially we have now $\varphi(D)^n S^n \to \mathrm{Id}_{\mathcal{Y}}$ pointwise. It remains thus only to prove that $S^n \to 0$ on $\mathcal{Y}$, i.e., $S^n(z^\alpha e_a) \to 0$ for any $a \in Z(p) \cap \Phi_\infty$ and α. To prove this, we consider what happens with the lowest degree elements, i.e., when $|\alpha|$ is small. When $|\alpha| = 0$ we have $S^n e_a = e_a/\varphi(a)^n \to 0$. Next we derive that $S(z_1 e_a) = z_1 e_a/\varphi(a) - D_1\varphi(a)e_a/\varphi(a)^2$. The general formula is $S^n(z_1 e_a) = z_1 e_a/\varphi(a)^n - nD_1\varphi(a)e_a/\varphi(a)^{n+1}$, which thus goes to zero. Computation gives

$$\begin{aligned} S^n(z_1^2 e_a) &= \frac{1}{\varphi(a)^n} z_1^2 e_a - 2n\frac{D_1\varphi(a)}{\varphi(a)^{n+1}} z_1 e_a \\ &\quad -n\frac{D_1^2\varphi(a)}{\varphi(a)^{n+1}} e_a + (n^2+n)\frac{(D_1\varphi(a))^2}{\varphi(a)^{n+2}} e_a, \end{aligned}$$

and

$$\begin{aligned} S^n(z_1z_2e_a) &= \frac{1}{\varphi(a)^n}z_1z_2e_a - n\frac{D_1\varphi(a)}{\varphi(a)^{n+1}}z_2e_a - n\frac{D_2\varphi(a)}{\varphi(a)^{n+1}}z_1e_a \\ &\quad -n\frac{D_1D_2\varphi(a)}{\varphi(a)^{n+1}}e_a + (4n-2)\frac{D_1\varphi(a)D_2\varphi(a)}{\varphi(a)^{n+2}}e_a. \end{aligned}$$

The general feature is as follows. For any n, $S^n(z^\alpha e_a)$ is a linear combination of elements $z^\beta e_a$, $\beta \leq \alpha$, and the coefficient for $z^\beta e_a$ is of type $q(n)\mathcal{O}(|\varphi(a)|^{-n})$ where q is a polynomial. From this we conclude that $S^n \to 0$ pointwise on $\mathcal{Y}$. Hence $\varphi(D)$ is p-hypercyclic by the HC.

It remains to prove that it is necessary that $\varphi|_{Z(p_i)}$ is nonconstant for all i. To this end we recall that a necessary condition for an operator $T \in L(\mathcal{X})$ to be hypercyclic is that $T - \lambda$ has dense range for all scalars λ, i.e., the adjoint T' lacks eigenvalues. Indeed, suppose $y \in \mathcal{X}'$ is an eigenvector for T', with corresponding eigenvalue λ, and that T has a hypercyclic vector x. Then $y \neq 0$ and has thus dense range. The image of $\{T^n x\}_{n\geq 0}$ must therefore be dense, but $\{\langle T^n x, y\rangle\}_n = \{\lambda^n\langle x, y\rangle\}_n$ is not dense, hence a contradiction. Now, by way of contradiction, assume $\varphi(D)$ is p-hypercyclic but $\varphi(z) = \lambda$ for all $z \in Z(p_i)$ for some i and λ, we may assume $i = 1$. Then $\varphi - \lambda$ vanishes on $Z(p_1)$ and is thus divisible by p_1 (Sublemma of Lemma 1). We know that the set (1) with $U = \mathbb{C}^d$, which we denote by A, is total in $\ker p(D)$. Since $\varphi(D) - \lambda$ has dense range, $[\varphi(D) - \lambda]A$ must be total in $\ker p(D)$. But

$$\begin{aligned} [\varphi(D) - \lambda]A \subseteq B + C \quad &\equiv \quad \operatorname{span}\{qe_a : a \in Z(p_1),\ \deg q < \operatorname{ord}_p(a) - \operatorname{ord}_{p_1}(a)\} \\ &\quad + \operatorname{span}\{qe_a : a \in Z(p) \setminus Z(p_1),\ \deg q < \operatorname{ord}_p(a)\}, \end{aligned}$$

where, in the definition of B, we tacitly assume $q = 0$ in the case $\operatorname{ord}_p(a) = \operatorname{ord}_{p_1}(a)$. To obtain a contradiction, it suffices to prove that there exists $\varphi_0 \in$ Exp that is orthogonal to $B + C$, but not to A (i.e., not to $\ker p(D)$). We claim that $\varphi_0 \equiv p_1^{r_1-1}p_2^{r_2}\cdots p_m^{r_m}$ will due. Indeed, if $a \in Z(p) \setminus Z(p_1)$, we have that $\operatorname{ord}_{\varphi_0}(a) = \operatorname{ord}_p(a)$ and so if $qe_a \in C$,

$$\langle qe_a, \varphi_0\rangle = q(D)\varphi_0(a) = 0.$$

Next, if $a \in Z(p_1)$, then $\operatorname{ord}_{\varphi_0}(a) = \operatorname{ord}_p(a) - \operatorname{ord}_{p_1}(a)$ which implies $\langle qe_a, \varphi_0\rangle = q(D)\varphi_0(a) = 0$ if $\deg q < \operatorname{ord}_p(a) - \operatorname{ord}_{p_1}(a)$. Thus φ_0 is orthogonal to $B + C$. Now, pick $a \in Z(p_1)$ arbitrarily. Then $\operatorname{ord}_p(a) > \operatorname{ord}_{\varphi_0}(a)$, so there exists a polynomial q with $\deg q < \operatorname{ord}_p(a)$ but $q(D)\varphi_0(a) \neq 0$. But then $qe_a \in A$ and $\langle qe_a, \varphi_0\rangle = q(D)\varphi_0(a) \neq 0$. □

Let $HC(p)$ denote the set of all $\varphi \in$ Exp for which $\varphi(D)$ is p-hypercyclic. From Theorem 1 it is immediate that $HC(pq) = HC(p) \cap HC(q)$ for any pair p, q of nonconstant polynomials, or more generally:

Corollary 1. *Let $p_1, \ldots, p_m$ be nonconstant polynomials. Then*

$$HC(p_1\cdots p_m) = \cap_i HC(p_i) = HC(\operatorname{lcm}\{p_1, \ldots, p_m\}),$$

where $\operatorname{lcm}\{p_1, \dots, p_m\}$ *denotes the least common multiple of* $p_1, \dots, p_m$ *(which, of course, is uniquely determined up to a constant factor).*

Corollary 2. *Let* p, q *be nonconstant polynomials in* $d \geq 2$ *variables. Then* $HC(p) \subseteq HC(q)$ *if and only if* $Z(q) \subseteq Z(p)$*. Thus,* $HC(p) = HC(q)$ *if and only if* $Z(q) = Z(p)$*.*

Proof. Since neither $Z(\cdot)$ nor (by Theorem 1) $HC(\cdot)$ change if we only change the multiplicities of the polynomial under consideration, we can assume p and q have no multiple irreducible factors.

Suppose first that $Z(q) \subseteq Z(p)$. Then every irreducible factor q' of q is an irreducible factor of p, and so $HC(p) \subseteq HC(q)$ by Theorem 1.

Next, assume $HC(p) \subseteq HC(q)$, i.e., by Theorem 1 (Corollary 1), $HC(p) \subseteq HC(q')$ for every irreducible factor q' of q. This means that if $\varphi \in \operatorname{Exp}$ is constant on $Z(q')$, then φ is constant on some component $Z(p')$ of $Z(p)$. We must prove that q' is a factor of p. Assume not. To obtain a contradiction we must construct a φ that is constant on $Z(q')$ but not on any $Z(p')$, where p' is an irreducible factor of p. If q' is nonconstant on every $Z(p')$, we may put $\varphi \equiv q'$. If this is not the case, let $p'_1, \dots, p'_n$ be the irreducible factors of p for which $q'|_{Z(p'_i)}$ is constant. Thus $q'(z) = \alpha_i$ for all $z \in Z(p'_i)$ for some constants α_i. Since q' is distinct from every p'_i, $\alpha_i \neq 0$ for all i.

Claim. Let $d \geq 2$ and q be an irreducible polynomial that is not a factor of a polynomial p. Then there exists a $\psi \in \operatorname{Exp}$ such that $p\psi$ is not constant on $Z(q)$.

Proof of Claim. Assume the claim does not hold, i.e., for every $\psi \in \operatorname{Exp}$ there exist a constant α and $\varphi \in \operatorname{Exp}$ such that $p\psi = \alpha + q\varphi$. This means that image of the operator $T : \mathbb{C} \times \operatorname{Exp} \to \operatorname{Exp}$, defined by $(\alpha, \varphi) \mapsto \alpha + q\varphi$, contains the ideal $p \cdot \operatorname{Exp} = \ker p(D)^{\perp}$. Hence

$$\ker {}^tT = \operatorname{Im} T^{\perp} \subseteq (p \cdot \operatorname{Exp})^{\perp} = \ker p(D),$$

where ${}^tT$ is the transpose of T with respect to the duality (H, Exp) and the natural duality between $\mathbb{C} \times \operatorname{Exp}$ and $\mathbb{C} \times H$. We derive that ${}^tTf = (f(0), q(D)f)$. Hence, to obtain a contradiction, it suffices to prove that there exist a $f \in H$ such that $q(D)f = 0$ and $f(0) = 0$ but $p(D)f \neq 0$. But since q is not a factor of p, there exists $a \in Z(q) \setminus Z(p)$. Since $d \geq 2$, the zeros are not isolated so we may find two different points $a, b \in Z(q) \setminus Z(p)$. Consider now $f \equiv e_a - e_b$. It is evident that $q(D)f = 0$ and $f(0) = 0$ hold, and it follows that $p(D)f = p(a)e_a - p(b)e_b \neq 0$. Indeed, if $p(a)e_a - p(b)e_b = 0$, the linearly independence of e_a, e_b yields that $p(a) = p(b) = 0$, which is a contradiction. Hence the claim.

By the claim there exist for every p'_i, a $\psi_i \in \operatorname{Exp}$ such that $\psi p/p'_i$ is not constant on $Z(p'_i)$. It follows now that

$$\varphi \equiv q'(\psi_1 p/p'_1 + \cdots + \psi_n p/p'_n + 1)$$

is a required function. □

Corollary 3. *Let p be a nonconstant polynomial and assume $\varphi(D)$ is p-hypercyclic. Then $\varphi_\nu(D)$, where $\varphi_\nu \equiv \nu \circ \varphi$, is p-hypercyclic for any nonconstant $\nu \in \mathrm{Exp}(\mathbb{C})$ such that $\varphi_\nu \in \mathrm{Exp}$. (In particular, $\varphi_\nu \in \mathrm{Exp}$ if φ, ν are polynomials or if $\nu = az + b$ and φ is arbitrary.)*

Proof. Let p' be any irreducible factor of p. Since φ is nonconstant and analytic on $\mathrm{reg}Z(p')$, the image $\varphi(Z(p'))$ contains an open set. Thus, since ν is nonconstant, ν cannot be constant on this set, which proves that φ_ν is nonconstant on $Z(p')$. □

When $d = 1$, $\ker p(D)$ is finite dimensional for any polynomial p and thus $HC(p)$ is empty. (We also see that no $\varphi \in \mathrm{Exp}$ satisfies the necessary and sufficient condition of Theorem 1 (since the irreducible components of $Z(p)$ are isolated points in $\mathbb{C}$).) We shall now prove that $HC(\psi) = \emptyset$ in general, except for the trivial cases.

Proposition 4. *If $d = 1$, $\psi \in \mathrm{Exp}$ admits a ψ-hypercyclic operator $\varphi(D)$ if and only if $\psi = Ae^{az}$ for some $A, a \in \mathbb{C}$, i.e., if and only if $\ker \psi(D) = \{0\}, H$.*

Proof. If $\psi = Ae_a$, then $\ker \psi(D) = H$ (if $A = 0$) or $\ker \psi(D) = \{0\}$ (if $A \neq 0$), and so $HC(\psi) \neq \emptyset$ (See the discussion following Definition 1). So we must prove the converse.

Assume ψ is not of the form Ae^{az}. In particular this means that $Z(\psi)$ is countable and not empty. Indeed, any zero free entire function is of the form e^g, where $g \in H$, and $e^g \in \mathrm{Exp}$ if and only if $g(z) = az + b$. Now, the key is to note that Lemma 1 (we assume $U = \mathbb{C}^d$) extends to the non polynomial case. That is, we claim: $\{qe_a : a \in Z(\psi),\ \deg q < \mathrm{ord}_\psi(a)\}$ is total in $\ker \psi(D)$. To see this we show first that the Sublemma of Lemma 1 extends in the sense that if $\phi \in \mathrm{Exp}$ has only simple zeros and $Z(\phi) \subseteq Z(\varphi)$ ($\varphi \in \mathrm{Exp}$), then ϕ divides φ in Exp. Indeed, any element $\phi \in \mathrm{Exp}$ can be written in the form $\phi(z) = Az^m e^{az}\Pi(1 - z/a_n)e^{z/a_n}$, see [8, Chapter 2.1]. Here (a_n) are the zeros $\neq 0$ for ϕ, multiple zeros are repeated in the sequence, ordered so that $|a_n| \leq |a_{n+1}|$. From this it evident that $Z(\phi) \subseteq Z(\varphi)$ implies $\varphi \in \phi \cdot \mathrm{Exp} = \ker \phi(D)^\perp$. From this point the proof of our claim goes parallel to that of Lemma 1.

Now, let $\varphi \in \mathrm{Exp}$ and let a_0 be any zero point of ψ, and put $\lambda \equiv \varphi(a_0)$. From the proof of Theorem 1, we recall that in order that $\varphi(D)$ is ψ-hypercyclic, $[\varphi(D) - \lambda]A$, where $A \equiv \{qe_a : a \in Z(\psi),\ \deg q < \mathrm{ord}_\psi(a)\}$, must be total in $\ker \psi(D)$. That this is not the case follows, as in the proof of Theorem 1, by the absence of terms $z^\alpha e_{a_0}$, $\alpha = |\alpha| = \mathrm{ord}_\psi(a_0) - 1$, in $\mathrm{span}[\varphi(D) - \lambda]A$. □

3. On Problem 2

In the previous section we studied hypercyclicity of restrictions, and in this section we consider the converse problem; the problem of finding hypercyclic extensions of hypercyclic operators on smaller spaces. We continue to work with the space $H(\mathbb{C}^d)$.

By $\|\cdot\|_r$ ($r > 0$) we denote the generating family of seminorms on $H(\mathbb{C}^d)$ defined by $\|f\|_r \equiv \sum_{n\geq 0} \|H_n f\| r^n$, where $H_n f$ denotes the n:th homogeneous part of the Taylor expansion of $f \in H(\mathbb{C}^d)$ about the origin and $\|p\| \equiv \sup_{\|z\|\leq 1} |p(z)|$. Next, E denotes the subspace of $H(\mathbb{C}^d \times \mathbb{C}^d) \simeq H(\mathbb{C}^{2d})$ formed by all $f = f(z,\xi)$ of the form $\sum_{n=0}^{\infty} f_n(\xi)\langle z,\xi\rangle^n/n!$ ($f_n \in H(\mathbb{C}^d)$) with absolute convergence in $H(\mathbb{C}^{2d})$, or equivalently, for which

$$\limsup_{n\to\infty} \left(\frac{\|f_n\|_r}{n!}\right)^{1/n} = 0 \tag{2}$$

for all $r > 0$. Recall that $\langle z,\xi\rangle \equiv \sum z_i\xi_i$. Note that the functions f_i are unique for $f \in E$, and we shall see that E forms an infinite dimensional closed (and when $d = 1$, complemented) subspace of $H(\mathbb{C}^{2d})$.

The main result in this section reads:

Theorem 2. *Any operator T on $H(\mathbb{C}^d)$ that satisfies the HC (e.g. any non-trivial $\varphi(D)$), defines a hypercyclic operator $T_E : E \to E$ by*

$$\sum_{n\geq 0} f_n(\xi)\frac{\langle z,\xi\rangle^n}{n!} \mapsto \sum_{n\geq 0} (Tf_n)(\xi)\frac{\langle z,\xi\rangle^n}{n!}.$$

Further, if $d = 1$, T_E extends to a hypercyclic operator $\hat{T}_E : H(\mathbb{C}^2) \to H(\mathbb{C}^2)$.

We shall also establish an explicit hypercyclic extension $\hat{T}_E$ of T_E when $d = 1$ (see Corollary 5). However, before we prove Theorem 2, we shall give another characterization of the space E, which is of independent interest. Further, our proof of Theorem 2 goes via general results applicable for Problem 2.

We equip $\mathrm{Exp} = \mathrm{Exp}(\mathbb{C}^d)$ with its standard inductive limit topology. That is, by $\mathrm{Exp}_r = \mathrm{Exp}_r(\mathbb{C}^d)$ we denote the Banach space of all functions $\varphi \in H = H(\mathbb{C}^d)$ such that $\|\varphi\|_r \equiv \sup_z |\varphi(z)|e^{-r\|z\|} < \infty$, provided with the norm $\|\cdot\|_r$. Then $\mathrm{Exp} = \cup_{r>0}\mathrm{Exp}_r$ and we endow Exp with the corresponding inductive locally convex topology. It follows then that the identification $\mathrm{Exp} \simeq H'$ (see the previous section) is a topological isomorphism, where H' carries the strong topology, and so Exp is a nuclear (because H is) DF-space. By $L(\mathrm{Exp}, H)$ we denote the space of all continuous linear mappings $\mathrm{Exp} \to H$ provided with the topology of uniform convergence on bounded (or equivalently, since Exp is nuclear, compact) sets. Since Exp is a DF-space and H a Fréchet space, $L(\mathrm{Exp}, H)$ is a Fréchet space. In fact:

Proposition 5. *Let, as before, $e_\xi \equiv e^{\langle\cdot,\xi\rangle} \in \mathrm{Exp} = \mathrm{Exp}(\mathbb{C}^d)$, $\xi \in \mathbb{C}^d$. Then the map $T \mapsto f(z,\xi) \equiv Te_\xi(z)$ is a topological isomorphism between $L(\mathrm{Exp}, H)$ and $H(\mathbb{C}^{2d})$.*

Proof. Let $T \in L(\mathrm{Exp}, H)$ and consider $f(z,\xi) \equiv Te_\xi(z)$. We prove that $f \in H(\mathbb{C}^{2d})$. It is clear that $f(\cdot,\xi) \in H(\mathbb{C}^d)$ for any fixed $\xi \in \mathbb{C}^d$. But we note that $Te_\xi(z) = {}^tTe_z(\xi)$, and so $f(z,\cdot) \in H(\mathbb{C}^d)$ for fixed z, and by Hartog's Theorem we conclude that $f \in H(\mathbb{C}^{2d})$. That $i : T \to f$ is one-to-one follows from the fact that the elements e_ξ, $\xi \in \mathbb{C}^d$, form a total set in Exp. Indeed, since H is reflexive, $\mathrm{Exp}' \simeq H$ in the sense of the duality between H and Exp. So if $f \in H$ is orthogonal to $\{e_\xi : \xi \in \mathbb{C}^d\}$, then $0 = \langle f, e_\xi\rangle = f(\xi)$ for all ξ and hence $f = 0$. This shows that $\{e_\xi : \xi \in \mathbb{C}^d\}$ is total and thus i is injective. Next, let $f \in H(\mathbb{C}^{2d})$ and define T by $T\varphi(z) \equiv \langle f(z,\cdot), \varphi\rangle$. It is easily checked that $T \in L(\mathrm{Exp}, H)$, and we note that $iT(z,\xi) = \langle f(z,\cdot), e_\xi\rangle = f(z,\xi)$. Hence i is a bijection and therefore an isomorphism by the Open-Mapping Theorem. □

It follows that the set of convolution operators on $\mathrm{Exp}(\mathbb{C}^d)$ is given by all $f(D) \equiv \sum_{\alpha\in\mathbb{N}^d} f_\alpha D^\alpha$ (pointwise convergence in Exp) where $f = \sum_\alpha f_\alpha z^\alpha \in H(\mathbb{C}^d)$. (Given a convolution operator T, we have that $T = f(D)$ where $f(z) \equiv Te_z(0) = {}^tT1(z)$.) In particular we have that ${}^tf(D)$ is "multiplication by f" and $f(D)g(D) = (fg)(D)$, so convolution operators on Exp commute.

Definition 2. An operator $T \in L(\mathrm{Exp}, H)$ is said to be PDE-preserving for a set $\mathbb{A} \subseteq \mathrm{Exp} \times H$ if $T \ker f(D) \subseteq \ker \varphi(D)$ for all $(\varphi, f) \in \mathbb{A}$. The set of PDE-preserving operators for $\mathbb{A}$ is denoted by $\mathcal{O}(\mathbb{A})$.

Note that $\mathcal{O}(\mathbb{A})$ forms a closed subspace of $L(\mathrm{Exp}, H)$ ($\simeq H(\mathbb{C}^{2d})$) for any set $\mathbb{A}$. PDE-preserving operators, in other settings, have been studied in e.g. [11, 12, 13].

Proposition 6. *Let $\mathbb{H}$ denote the set of homogeneous polynomials ($d \geq 1$ variables) and $\mathbb{E} \equiv \{(p,p) : p \in \mathbb{H}\} \subset \mathrm{Exp} \times H$. Then $\mathcal{O}(\mathbb{E})$ is formed by all operators T of the form $T = \sum_{n\geq 0} H_n \circ f_n(D)$, where the sequence (f_n) satisfies* (2) *and is unique for T. In particular, $E = \mathcal{O}(\mathbb{E})$ in the sense of the isomorphism in Proposition 5, and so E is closed.*

Proof. By the fact that any $f_n(D)$ commutes with any $p(D)$, $p \in \mathbb{H}$, and, if $p \in \mathbb{H}$ is m-homogeneous, $p(D)H_n$ equals $H_{n-m}p(D)$ if $n \geq m$ and 0 otherwise, it is easily checked that any operator T of the form $\sum_{n\geq 0} H_n \circ f_n(D)$ (where (f_n) satisfies (2)), belongs to $\mathcal{O}(\mathbb{E})$. In fact, the growth condition (2) implies that $\sum_{n\geq 0} H_n \circ f_n(D)$ converges pointwise and, by Banach-Steinhaus Theorem, defines thus a continuous operator. Further, $f_n(D)e_\xi = f_n(\xi)e_\xi$, and hence $Te_\xi(z) = \sum f_n(\xi)\langle z,\xi\rangle^n/n!$. It remains thus only to prove that every $T \in \mathcal{O}(\mathbb{E})$ is of this form.

For any $n \in \mathbb{N}$ and $z \in \mathbb{C}^d$, $\varphi \mapsto H_nT\varphi(z)$ is a continuous linear functional on $\mathrm{Exp} = \mathrm{Exp}(\mathbb{C}^d)$. Hence, there exists a unique $g_{n:z} \in H \simeq \mathrm{Exp}'$ such that $\langle g_{n:z}, \varphi\rangle = H_nT\varphi(z)$ for all $\varphi \in \mathrm{Exp}$. We prove that $g_{n:z} \in \langle z,\cdot\rangle^n \cdot H$ if $n \geq 1$ and $z \neq 0$. Multiplication by $\langle z,\cdot\rangle^n$, $f \mapsto \langle z,\cdot\rangle^n f$, is the transpose of $p_{n,z}(D) : \mathrm{Exp} \to \mathrm{Exp}$ where $p_{n:z}$ is the homogeneous polynomial $\langle z,\cdot\rangle^n$ (i.e., $p_{n:z}(D)\varphi$ is the n:th directional derivative of $\varphi \in \mathrm{Exp}$ along z). Now, since $T \in \mathcal{O}(\mathbb{E})$, it is evident that $g_{n:z} \in \ker p_{n:z}(D)^\perp$, which equals $\mathrm{Im}\langle z,\cdot\rangle^n$, because $p_{n:z}(D)$ is surjective [18, Theorem 28.2]. So, for every $n \geq 1$ and $z \neq 0$, there exists a unique $f_{n:z} \in H$ with $g_{n:z} = \langle z, \cdot\rangle^n f_{n:z}/n!$. Hence

$$H_n f_{n:z}(D)\varphi(z) = \langle\langle z,\cdot\rangle^n/n!, f_{n:z}(D)\varphi\rangle = \langle f_{n:z}\langle z,\cdot\rangle^n/n!, \varphi\rangle = H_n T\varphi(z)$$

if $z \neq 0$, and we note that the identity $H_n f_{n:z}(D)\varphi(z) = H_n T\varphi(z)$ holds even if $z = 0$ (both sides vanish). We must prove that, for fixed $n \geq 1$, the $f_{n:z}$ are independent of $z \neq 0$, i.e., equal to some $f_n \in H$. To this end we only have to refer to the proof of Theorem 2 in [11], where an analogous step is solved. When $n = 0$, $g_{n:z}$ is independent of z and we put $f_0 \equiv g_{0:z}$ and notice that $H_0 f_0(D)\varphi = H_0 T\varphi$. Thus, for any $\varphi \in \mathrm{Exp}$ we have

$$T\varphi = \sum H_n T\varphi = \sum H_n f_n(D)\varphi$$

pointwise. It remain thus only to prove that (f_n) satisfies the growth condition (2). From the identity $H_n f_n(D)e_\xi = H_n T e_\xi$ (consider the $(n+m)$-homogeneous part in ξ of both sides) we derive that

$$\langle z,\xi\rangle^n f_{n,m}(\xi) = \langle T\frac{\langle\cdot,\xi\rangle^{n+m}}{(n+m)!}, \langle z,\cdot\rangle^n\rangle, \tag{3}$$

where $f_{n,m} \equiv H_m f_n$. Now, for any $\varepsilon > 0$ we have that

$$B \equiv \{\varepsilon^{-n-m}\langle\cdot,\xi\rangle^{n+m}/(n+m)! : n, m \in \mathbb{N},\ \|\xi\| = 1\}$$

forms a bounded set in Exp (it is contained and bounded in some Exp_r). Hence, TB is bounded in H and so, for every $r > 0$ there exists $M_r = M_r(\varepsilon) > 0$ such that

$$\|T\frac{\langle\cdot,\xi\rangle^{n+m}}{(n+m)!}\|_r \leq \varepsilon^{n+m} M_r \tag{4}$$

for all ξ on the unit sphere and all $n, m \geq 0$. Next we need the following simple consequence of Cauchy's estimates, whose proof we omit:

Sublemma. If $\varphi \in \mathrm{Exp}_r(\mathbb{C}^d)$, then $|\langle f,\varphi\rangle| \leq \|f\|_{d^2er}\|\!|\varphi|\!\|_r$ for all $f \in H(\mathbb{C}^d)$.

We see now that $\langle z,\cdot\rangle^n \in \mathrm{Exp}_1$ with $\|\!|\langle z,\cdot\rangle^n|\!\|_1 \leq n!\|z\|^n$. Hence, from (3), (4) and the Sublemma, we conclude that

$$|\langle z,\xi\rangle^n f_{n,m}(\xi)| \leq \varepsilon^{n+m} M_{d^2e} n!\|z\|^n, \qquad \|\xi\| = 1.$$

If p and q are an n and m-homogeneous polynomial respectively, we have that $\|\!|p|\!\|\,\|\!|q|\!\| \leq (2e)^{n+m}\|\!|pq|\!\|$ (see [4, p. 72]). This yields with $p \equiv \langle z,\cdot\rangle^n$ and $q \equiv f_{n,m}$:

$$\|z\|^n\|\!|f_{n,m}|\!\| \leq (2e)^{n+m}\|\!|\langle z,\cdot\rangle^n f_{n,m}|\!\| \leq (2e\varepsilon)^{n+m} M_{d^2e} n!\|z\|^n. \tag{5}$$

Accordingly, for any given $r > 0$ (take z with $\|z\| = r$ in (5) and $\varepsilon = \varepsilon_r$ small enough), we have

$$r^n\|f_n\|_r = \sum_{m\geq 0} r^n\|\!|f_{n,m}|\!\| r^m \leq M_{d^2e} n!(2e\varepsilon r)^n \sum_{m\geq 0}(2e\varepsilon r)^m \leq Nn!$$

for some constant $N = N(r,\varepsilon)$. Hence (f_n) satisfies (2). □

Note that when $d = 1$, $\mathcal{O}(\mathbb{E})$ consists precisely of the continuous operators $T : \mathrm{Exp}(\mathbb{C}) \to H(\mathbb{C})$ that, for every m, maps polynomials of degree at most m onto polynomials of degree at most m.

Next we prove some general results related to our problem, Problem 2.

We recall that a sequence $(T_n)_{n\geq 0} \subset L(\mathcal{X})$ is called hypercyclic if, for some (hypercyclic) $x \in \mathcal{X}$, $\{T_n x\}_{n\geq 0}$ is dense, and the HC (Proposition 1) extends to sequences of operators on a separable Fréchet space in the sense that we may replace T^n by T_n [1]. (Thus an operator T is hypercyclic if and only if the sequence $(T^n)_n$ is.)

Proposition 7. *Let $\mathcal{X}$ be a separable Fréchet space and assume we have a topological decomposition $\mathcal{X} = \mathcal{Z} \oplus \mathcal{Y}$. If $S \in L(\mathcal{Z})$ and $T \in L(\mathcal{Y})$ satisfies the* HC *with respect to some common sequence $(n_k) \subseteq \mathbb{N}$, then $S \oplus T : z + y \mapsto Sz + Ty$ forms a hypercyclic operator on $\mathcal{X}$. In fact, for any hypercyclic vector $y \in \mathcal{Y}$ for (T^{n_k}), there exists a hypercyclic vector $z \in \mathcal{Z}$ for S such that $z + y$ is hypercyclic for $S \oplus T$.*

Proof. It suffices to prove the last part, i.e., given a hypercyclic vector $y \in \mathcal{Y}$ for (T^{n_k}), there exists a hypercyclic vector $z \in \mathcal{Z}$ for S such that $z+y$ is hypercyclic for $S\oplus T$. Let $(y_i)_{i\geq 0}$ be a countable dense set in $\mathcal{Y}$. There exists a subsequence $(n_{i,j})_j$ of (n_k) such that $T^{n_{i,j}} y \to y_i$ $(j \to \infty)$. Consider now the denumerable family of sequences $(S^{n_{i,j}})_j \subset L(\mathcal{Z})$, $i = 0, 1, \ldots$. We know that $(S^{n_k})_k$, and hence every $(S^{n_{i,j}})_j$, satisfies the HC. Accordingly, by [6, Prop. 3], $(S^{n_{i,j}})_j$, $i = 0, 1, \ldots$ have a common hypercyclic vector $z \in \mathcal{Z}$. We prove that $z + y$ is a required hypercyclic vector for $S \oplus T$. So let $x = x_Z + x_Y \in \mathcal{Z} \oplus \mathcal{Y} = \mathcal{X}$ be arbitrary, and choose a continuous seminorm p and $\varepsilon > 0$ arbitrarily. Pick i_0 so that $p(y_{i_0} - x_Y) \leq \varepsilon/3$. Next we may find a j_ε such that $p(T^{n_{i_0,j}} y - y_{i_0}) \leq \varepsilon/3$ for all $j \geq j_\varepsilon$. Now z is hypercyclic for $(S^{n_{i_0,j}})_j$, so there exists a $j_0 \geq j_\varepsilon$ with $p(S^{n_{i_0,j_0}} z - x_Z) \leq \varepsilon/3$. The triangle inequality gives

$$p((S\oplus T)^{n_{i_0,j_0}}(z+y)-x) \leq p(S^{n_{i_0,j_0}} z - x_Z) + p(T^{n_{i_0,j_0}} y - y_{i_0}) + p(y_{i_0} - x_Y) \leq \varepsilon,$$

thus $z + y$ is indeed a hypercyclic vector. □

Thus, in order to extend an operator $T \in L(\mathcal{Y})$ that satisfies the HC to a hypercyclic operator on $\mathcal{X}$, where $\mathcal{Y} \subset \mathcal{X}$ is complemented, we only have to construct an operator $S \in L(\mathcal{Z})$, where $\mathcal{X} = \mathcal{Z} \oplus \mathcal{Y}$, that satisfies the HC with respect to a sequence for which T satisfies the HC.

Corollary 4. *Let $\mathcal{X}$ be a separable Fréchet space and assume $\mathcal{Y} \subset \mathcal{X}$ is complemented in $\mathcal{X}$ and* $\operatorname{codim} \mathcal{Y} = \infty$. *Then every operator $T \in L(\mathcal{Y})$ that satisfies the* HC *with respect to the full sequence $(n_k = k)$, has a hypercyclic extension $\hat{T} \in L(\mathcal{X})$.*

Proof. Let $\mathcal{X} = \mathcal{Z} \oplus \mathcal{Y}$ be a topological decomposition. Since $\operatorname{codim} \mathcal{Y} = \infty$, $\mathcal{Z}$ forms an infinite dimensional separable Fréchet space and hence supports a hypercyclic operator $S \in L(\mathcal{Z})$ [2]. In fact, from the proof in [2], we may choose S so that S satisfies the HC with respect to some sequence (n_k). Now

T satisfies the HC with respect to the full sequence and hence with respect to (n_k). Thus $S \oplus T$ is a required extension by Proposition 7. $\square$

Remark 1. In some settings, Corollary 4 can strengthened. Indeed, let $\mathcal{X}$ be a separable Hilbert space and assume $\mathcal{Y} \subseteq \mathcal{X}$ is closed and has infinite codimension. Then $\mathcal{Y}$ is complemented and if $\mathcal{X} = \mathcal{Z} \oplus \mathcal{Y}$ is a topological decomposition, $\dim \mathcal{Z} = \infty$. Thus $\mathcal{Z} \simeq \ell_2$ and so $\mathcal{Z}$ supports (because ℓ_2 does) an operator $S \in L(\mathcal{Z})$ that satisfies the HC with respect to the full sequence. From this we conclude: *Every hypercyclic operator $T \in L(\mathcal{Y})$, where $\mathcal{Y}$ is an infinite codimensional closed subspace of a separable Hilbert space $\mathcal{X}$, has a hypercyclic extension $\hat{T} \in L(\mathcal{X})$.*

Example 1. We recall that any kernel $\ker \psi(D)$, $\psi \in \mathrm{Exp}$, is complemented in $H = H(\mathbb{C}^d)$. We claim that $\ker \psi(D)$ has infinite codimension if $\psi \neq 0$. Indeed, the complement of $Z(\psi)$ is infinite for any such ψ, and we have that $\{\hat{e}_a \equiv e_a + \ker \psi(D) : a \notin Z(\psi)\}$ forms a linearly independent set in $H/\ker\psi(D)$. For if $\sum_1^n A_i \hat{e}_{a_i} = 0$, $a_i \notin Z(\psi)$, then $\sum A_i \psi(a_i) e_{a_i} = 0$. But $\{e_a : a \in \mathbb{C}^d\}$ is a linearly independent set in H and so, since $\psi(a_i) \neq 0$, $A_i = 0$ for all i. Hence any operator $T : \ker\psi(D) \to \ker\psi(D)$ that satisfies the HC with respect to the full sequence admits a hypercyclic extension $\hat{T} \in L(H)$. In particular, let p be a nonconstant homogeneous polynomial. Then we have that $H = \bar{p} \cdot H \oplus \ker p(D)$ is a topological decomposition of H [16]. Here $\bar{p}$ denotes the homogeneous polynomial obtained from p by conjugating the coefficients. Now, assume $T : \ker p(D) \to \ker p(D)$ satisfies the HC with respect to the full sequence, and let $S \in L(H)$ be any operator that satisfies the HC. (In particular we may have $T = \varphi(D)|_{\ker p(D)}$ where $\varphi \in \mathrm{Exp}$ satisfies the hypothesis of Theorem 1 with respect to p.) Then S induces an operator $\tilde{S}$ on $\bar{p}H$, that satisfies the HC, by $\bar{p}f \mapsto \bar{p}Sf$. Thus, by Proposition 7, $\tilde{S} \oplus T \in L(H)$ forms a hypercyclic extension of T. (Note that all this latter is a generalization of Proposition 2, which corresponds to the case $p = \bar{p} = z_2$, $T = D_1|_{\ker D_2}$ and $S = D_1$, because in this case $\tilde{S} = S|_{\bar{p}H}$.)

Proposition 8. *If $d = 1$, E is complemented in $H(\mathbb{C}^2)$. In fact, $H = E \oplus F$ forms a topological decomposition of $H = H(\mathbb{C}^2)$, where F is formed by all $p = \sum_{n \geq 1} p_n(\xi) z^n/n!$ where each p_n is a polynomial with $\deg p_n < n$ and the sequence (p_n) satisfies* (2).

Proof. Since $H = H(\mathbb{C}^2)$ is a Fréchet space and E is closed, it suffices, by the Closed-Graph Theorem, to prove that F is closed and $H = E \oplus F$. Let $f \in H$. Then f can uniquely be written in the form $\sum_{n \geq 0} f_n(\xi) z^n/n!$ where the sequence $(f_n) \subset H(\mathbb{C})$ satisfies (2). But each f_n has a unique decomposition $f_n(\xi) = p_n + g_n \xi^n$ where $g_n \in H(\mathbb{C})$ and $p_n = \sum_{m<n} f_n^{(m)}(0)\xi^m/m!$ is a polynomial of degree $< n$ if $n \geq 1$ and $p_0 = 0$. We prove that (g_n) and (p_n) satisfy (2). Put $h_n \equiv g_n \xi^n = \sum_{m \geq n} f_n^{(m)}(0)\xi^m/m!$. We note that $||h_n||_r, ||p_n||_r \leq ||p_n||_r + ||h_n||_r = ||f_n||_r$ for any $r > 0$. Hence (p_n) and (h_n), and therefore (g_n), satisfy (2). This implies that $g \equiv \sum g_n(\xi) z^n \xi^n/n!$ and $p \equiv \sum p_n(\xi) z^n/n!$ converge in H and belong to E and F respectively. Thus $f = g + p$ and so $H = E + F$. It is evident that $E \cap F = \{0\}$ and hence $H = E \oplus F$ holds. Finally, the fact that the space of polynomials of degree $\leq m$ is closed in H for all m, implies F is closed in H, and we are done. □

Proof of Theorem 2. It is clear that T_E indeed defines a continuous operator on E, and we prove that T_E is hypercyclic. We shall apply the HC. Let Z, Y be dense subspaces of $H = H(\mathbb{C}^d)$ and $S_k : Y \to H$ linear maps for which for some sequence (n_k); $T^{n_k} \to 0$ on Z and $S_k \to 0$, $T^{n_k} S_k \to \mathrm{Id}_Y$ on Y. By $\mathcal{Z}$ ($\mathcal{Y}$ resp.) we denote the set in E of all finite sums $\sum_n f_n(\xi)\langle z, \xi\rangle^n/n!$ where $f_i \in Z$ (Y resp.). It is easily checked that $\mathcal{Z}$ and $\mathcal{Y}$ are dense in E, and $T_E^{n_k}$ goes to zero on $\mathcal{Z}$. Next we define mappings $S_E^k : \mathcal{Y} \to E$ by $\sum f_n(\xi)\langle z, \xi\rangle^n/n!) \mapsto \sum (S_k f_n)(\xi)\langle z, \xi\rangle^n/n!$. Again it is easily seen that $T_E^{n_k} S_E^k \to \mathrm{Id}_{\mathcal{Y}}$ and $S_E^k \to 0$ pointwise on $\mathcal{Y}$, and hence T_E satisfies the HC with respect to (n_k).

It remains to prove that, when $d = 1$, T_E can be extended to a hypercyclic operator on $H(\mathbb{C}^2)$, and we shall apply Proposition 7. In view of Proposition 8, we only have to construct an operator $S \in L(F)$ that satisfies the HC with respect to the full sequence. Indeed, then $T_E \oplus S$ forms a required extension of T_E as T_E satisfies the HC with respect to some sequence. We shall prove that S defined by

$$\sum_{n\geq 1} p_n(\xi) z^n/n! \mapsto \sum_{n\geq 1} p'_{n+1}(\xi) z^n/n! \tag{6}$$

is a required hypercyclic operator. Note that $\deg p'_{n+1} \leq n - 1$, and it is an easy exercise to show that $S \in L(F)$. Let $\mathcal{Z} = \mathcal{Y}$ be the dense subspace of F formed by all $\sum_{n\geq 1} p_n(\xi) z^n/n! \in F$ where $p_n = 0$ for all but a finite number of n. It is then obvious that $S^n \to 0$ pointwise on $\mathcal{Z}$. (In fact, $S^n f = 0$ for all n sufficiently large if $f \in \mathcal{Z}$). Next we define a map $\rho : \mathcal{P} \to \mathcal{P}$ on the space $\mathcal{P}$ of one-variable polynomials in the following way. Let $\rho(z^n) \equiv z^{n+1}/(n+1)$, $n \geq 0$, and then extend ρ linearly. Note that ρ forms a right inverse to the differentiation operator D on $\mathcal{P}$, and ρ maps polynomials of degree $\leq n$ to polynomials of degree at most $n+1$. From this we conclude that $\sum_{n\geq 1} p_n(\xi) z^n/n! \mapsto \sum_{n\geq 2} \rho(p_{n-1})(\xi) z^n/n!$ defines a right inverse $R : \mathcal{Y} \to \mathcal{Y}$ to S. Hence, we only have to prove that $R^n \to 0$ pointwise on $\mathcal{Y}$. It suffices to prove that $R^n y_{i,j} \to 0$ for all $0 \leq i < j$ where $y_{i,j} \equiv \xi^i z^j \in \mathcal{Y}$. We derive that $R^n y_{i,j} = i!\xi^{i+n} z^{j+n}/[(i+n)!(j+n)!] \to 0$ $(n \to \infty)$. □

Corollary 5. *The operator S defined by* (6) *is hypercyclic on $F \subseteq H(\mathbb{C}^2)$, and $\hat{T}_E \equiv T_E \oplus S$ is a hypercyclic extension of T_E. If T satisfies the* HC *with respect to (n_k), then so does T_E (thus $(T_E^{n_k})_k$ is hypercyclic). For any hypercyclic vector f for S ($(T_E^{n_k})_k$ resp.), there exists a hypercyclic vector g for T_E (S resp.) such that $f + g$ is hypercyclic for $\hat{T}_E$.*

Remark 2. From the proof of Theorem 2 we see that the theorem extends in the following way: *If* $(T_n)_{n\geq 0}$ *is an equicontinuous sequence of operators* $T_n \in L(H(\mathbb{C}^d))$ *that satisfy the* HC *with respect to some common sequence* (n_k), *then* $\sum f_n(\xi)\langle z,\xi\rangle^n/n! \mapsto \sum (T_n f_n)(\xi)\langle z,\xi\rangle^n/n!$ *defines a hypercyclic operator on* E. (Note that equicontinuous means that there for every $r > 0$ exist $M, R > 0$ such that $\|T_n f\|_r \leq M\|f\|_R$ for all n and $f \in H$.) In particular, any nontrivial $\varphi(D)$ satisfies the HC with respect to the full sequence, and we conclude that *any sequence* $(\varphi_n)_{n\geq 0} \subset \mathrm{Exp} \setminus \mathbb{C}$ *such that* $\sup_{n,z} |\varphi_n(z)|e^{-r\|z\|} < \infty$ *for some* $r > 0$, *defines a hypercyclic operator on* E *by* $\sum f_n(\xi)\langle z,\xi\rangle^n/n! \mapsto \sum(\varphi_n(D)f_n)(\xi)\langle z,\xi\rangle^n/n!$ *on* E.

4. Some extensions

The results of Section 2 extend to other spaces, i.e., we may replace H by some other suitable function or power series space. Indeed, consider for example the space $C^\infty = C^\infty(\mathbb{R}^d)$ of all complex-valued smooth functions on $\mathbb{R}^d$ equipped with its usual Fréchet space topology [7, p. 44]. From Paley-Wiener-Schwartz' Theorem [7, p. 181], we know that the dual of C^∞ can be identified with the space $\mathrm{Exp} = \mathrm{Exp}(\mathbb{C}^d)$ of all $\varphi \in H$ such that (for some $C, r, n > 0$) $|\varphi(z)| \leq C(1+\|z\|)^n e^{r\|\mathrm{im} z\|}$, via the Fourier-Laplace transform $\lambda \mapsto \varphi(z) \equiv \lambda(e_z) \in \mathrm{Exp}$. Here $\mathrm{im} z \equiv (\mathrm{im} z_i) \in \mathbb{R}^d$ and $e_z(x) = e_x(z) \equiv e^{-i\langle x,z\rangle}$, $(x,z) \in \mathbb{R}^d \times \mathbb{C}^d$. Thus C^∞ and Exp form a dual pair and, given $\varphi \in \mathrm{Exp}$, we define the convolution operator $\varphi(\partial)f(x) \equiv \langle f, \varphi e_x\rangle$ on C^∞. It follows that we obtain all the convolution operators on C^∞ in this way [14, Prop. 2]. In particular, if p is a polynomial, $p(\partial)$ is the differential operator obtained by replacing each variable in z_j in p by $i\partial_j = i\partial/\partial x_j$. Moreover, Godefroy and Shapiro's result extends in the sense that any nontrivial convolution operator $\varphi(\partial)$ is hypercyclic on C^∞ [14, Theorem 3]. Now, the key-lemma, Lemma 1, remains true for any kernel $\ker p(\partial)$, and we have the following analogue of Theorem 1:

Theorem 3. *Let* $p = p_1^{r_1} \cdots p_m^{r_m}$ *be the factorization of a nonconstant polynomial* p *into irreducible factors* p_i *with corresponding multiplicities* $r_i \geq 1$. *A necessary and sufficient condition for the restriction* $\varphi(\partial) : \ker p(\partial) \to \ker p(\partial)$ *to be hypercyclic is that* $\varphi|_{Z(p_i)}$ *is nonconstant for all* i.

Theorem 1 (and Theorem 3) extends also naturally to spaces such as the ring, and Fréchet space, F of formal power series (d variables, complex coefficients) equipped with the topology of convergence at each coefficient. (Replace φ in Theorem 1 by a nonconstant polynomial, so $\varphi(D)$ is a well-defined continuous operator on F.)

We conclude by considering another type of extension of our study. In [5] the authors showed that an operator T acting on a separable Fréchet space is chaotic, in the sense of Devaney, if and only if T is hypercyclic and has a dense set of periodic points. (Recall that a periodic point is a point for which $T^n x = x$

for some n.) In the same paper they extended Proposition 3 by proving that any nontrivial convolution operator on $H(\mathbb{C}^d)$ is in fact chaotic. Thus, we are led to the question; is the restriction $\varphi(D)|_{\ker p(D)}$ of any convolution operator $\varphi(D)$, satisfying the hypothesis of Theorem 1 with respect to p, in fact chaotic? We shall not here give a full treatment of this problem, this chaotic problem seems to be more delicate in the sense that multiple factors in the polynomial p affects. However, we prove the following:

Theorem 4. *Let $p = p_0 \cdots p_m$ be a nonconstant polynomial whose irreducible factors p_i are distinct. Then $\varphi(D)|_{\ker p(D)}$ is chaotic if and only if $\varphi|_{Z(p_i)}$ is nonconstant for all i.*

Proof. We must prove that $\varphi(D)|_{\ker p(D)}$ has a dense set of periodic points provided $\varphi|_{Z(p_i)}$ is nonconstant for all i. In particular we have that $d \geq 2$. We know, from the proof of Lemma 1, that $\ker p(D)^{\perp} = \{\psi : Z(p) \subseteq Z(\psi)\}$. Thus we must show that there exists a set of periodic points for $\varphi(D)$ in $\ker p(D)$ such that if $\psi \in \mathrm{Exp}$ is orthogonal to this set, then ψ vanishes on $Z(p)$. From this point the proof goes very close to that of [5, Theorem 6.2], however, we give some details. Assume first that $d = 2$. Our hypothesis on φ implies that for every i there is a point $a_i \in \mathrm{reg}Z(p_i)$ such that $|\varphi(a_i)| = 1$. Since every $\mathrm{reg}Z(p_i)$ forms a one dimensional complex manifold, there exist for every i a neighborhood U_i of a_i in $\mathrm{reg}Z(p_i)$, a domain $\Omega_i \subseteq \mathbb{C}$ and a biholomorphic map $u_i : \Omega_i \to U_i$ such that $\varphi_i \equiv \varphi \circ u_i \in H(\Omega_i)$. Now, $\varphi_i(\Omega_i)$ meets the unit circle, and we may find an open relatively compact set $G_i \subseteq \Omega_i$ such that $\varphi_i(G_i)$ meets the unit circle. Since a holomorphic map is open, $\varphi_i(G_i)$ contains a nontrivial arc of the unit circle, and hence contains infinitely many roots of unity. Thus, there exists an infinite set $E_i \subseteq G_i$ for which $\varphi_i(E_i) = \varphi(u_i(E))$ is formed by roots of unity. We now claim that $P \equiv \cup_i\{e_a : a \in u_i(E_i)\}$ is a total set of periodic points in $\ker p(D)$ and hence, since linear combinations of periodic points are again periodic, its span is the required dense set of periodic points. Indeed, if ψ is orthogonal to P then ψ vanishes on every $u_i(E_i)$. But, since G_i is relatively compact, E_i has a limit point in Ω_i. Hence $\psi \circ u_i = 0$ and so ψ vanishes on every $\mathrm{reg}Z(p_i)$. Thus $\psi|_{Z(p)} = 0$ and our claim follows.

When $d > 2$ we only have to refine the proof in the same way as in the proof in [5]. □

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