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Tikhonov's Solution of Unstable Axisymmetric Initial Value Problem of Wave Propagation: Deteriorated Noisy Measurement Data

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ABSTRACT: *The primary aim of the paper is to solve an unstable axisymmetric initial value problem of wave propagation when given initial data that is deteriorated by noise such as measurement error. To overcome the instability of the problem, Tikhonov's regularization, known as a non-iterative numerical regularization method, is introduced to solve the problem. The L-curvecriterion is introduced to find the optimal regularization parameter for the solution. It is confirmed that fairly stable solutions are realized and that they are accurate when compared to the exact solution.*

1.Introduction

The axisymmetric Cauchy-Poisson problem of dispersive water waves has been known as well-posed one in the sense of stability of Hadamard's third condition (1923). That is, the solution for the initial value problem of Cauchy-Poisson type depends continuously on the initial wave condition: there is the exact solution of time evolution for the initial value problem expressed as Fourier integral representation (Stoker, 1957; Debnath, 1994; Whitham, 1974).

However, Jang et al. (2007) showed that for a certain class of initial conditions, the axisymmetric Cauchy - Poisson problem failed to be well-posed in the sense of stability. If the special type of initial condition were given for the problem, usual or conventional numerical treatments would provide meaningless solutions because they lead to an arbitrarily large error for the solution (Kirsch, 1996; Tikhonov, 1963; Kammerer and Nashed, 1972): this phenomenon is due to the discontinuity of the problem (lack of stability). The class of initial conditions was closely related with wave spectrum with compact support (Jang et al., 2007).

Regularization theory has been known as a powerful mathematical tool to tackle common ill-posed inverse problems in the natural sciences and engineering (Groetsch, 1993; Fridman, 1965; Isakov, 1998; Jang and Kinoshita, 2000; Jang et al., 2000a; 2000b; Jang et al., 2007; Jang and Han 2008). Jang et al. (2007) introduced Landweber-Fridman's regularization to the unstable axisymmetric Cauchy-Poisson problem to find stable regularized solution. However, quite a number of iteration was required to achieve an accurate regularized solution because the regularization method is based on iterative numerical process. In order to overcome this difficulty, in this study, Tikhonov's regularization, known as non-iterative regularizaiton, is introduced. Furthermore, L-curve criterion (Hansen, 1992) is also introduced and applied to determine appropriate regularization parameter. Thereby we succeed in obtaining numerical solutions when measured data of wave profile is deteriorated to an extent by noise.

This paper is organized as follows. The axisymmetric Cauchy - Poisson problem for dispersive water waves and its wave spectrum with compact support are reviewed in section 2. Instability of the problem is investigated in section 3. Tikhonov's regularization theory is examined in section 4. Finally, with a given specific initial condition, numerical experiment with L-curve criterion is studied in section 5.

2. The Axisymmetric Cauchy - Poisson Problem : Spectrum with Compact Support

The axisymmetric Cauchy - Poisson problem is considered for an inviscid incompressible water of infinite depth with a free surface horizontal surface. By the prescribed free surface displacement, the free surface waves are generated in the body of the water, initially at rest for time $t<0$. Laplace's equation governs the surface wave motion with the free surface and bottom boundary conditions (Debnath, 1994; Jang et al., 2007): the linearized resulting equations are expressed, in cylindrical polar coordinates (r,θ,z) as follows

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$$\nabla^2\phi = \phi_{rr} + \frac{1}{r}\phi_r + \phi_{zz} = 0, \quad 0 \le r < \infty,\ -\infty < z \le 0 \tag{1}$$

$$\eta_t = \phi_z \quad \text{on} \quad z = 0,\ t > 0 \tag{2}$$

$$\phi_t + g\eta = 0 \quad \text{on} \quad z = 0,\ t > 0 \tag{3}$$

$$\phi_z = 0 \quad \text{as} \quad z \to \infty \tag{4}$$

where φ, η and g denote the velocity potential, the free surface elevation and the acceleration of gravity, respectively.

If we impose the following initial conditions

$$\phi(r,0,0) = 0 \text{ and } \eta(r,0) = \eta_0(r) \tag{5}$$

we obtain the solution for the free surface elevation η, represented as the integral form (Debnath, 1994):

$$\eta(r,t) = \int_0^\infty kJ_0(kr)F(k)\cos\omega t dk \tag{6}$$

by using the Laplace and the zero-order Hankel thransforms(Stakgold, 1967). Here $J_0(kr)$ denotes a Bessel function of the first kind of order zero and the function $F(k)$ is the Hankel transform of $\eta_0(r)$:

$$F(k) = \int_0^\infty \eta_0(r)rJ_0(kr)dr \tag{7}$$

The dispersion relation for deep water $\omega^2 = kg$ is used where k and ω stand for the wave number and the wave frequency, respectively.

Let us consider a spectrum $F(k)$ which is zero for an interval of wave numbers $k_0 < k < \infty$, for a positive real values k_0. Then we have the free surface displacement given by

$$\eta(r,t) = \int_0^{k_0} kJ_0(kr)F(k)\cos\omega t dk \tag{8}$$

This is equivalent to the physical situation of a propagating group of ring waves with a continuous but finite range of wavelengths: that is, the discussion is restricted to a spectrum $F(k)$ defined on a compact support (Rudin, 1991; Bassanini and Elcrat, 1997).

From Eq. (8), the initial disturbance $\eta_0(r)$ at time $t=0$ is related with the spectrum function $F(k)$ (with the compact support $k_0 < k < \infty$): for a positive real number r_0

$$\eta_0(r) = \int_0^{k_0} kJ_0(kr)F(k)dk, \quad 0 \le r \le r_0 \tag{9}$$

It is convenient to use abstract notation for Eq. (9): that is, operator notation

$$\eta_0 = L(F) \tag{10}$$

where the operator L is defined as

$$L(F) = \int_0^{k_0} K(r,k)F(k)dk \tag{11}$$

with its kernel $K(r,k)$

$$K(r,k) = kJ_0(kr) \tag{12}$$

3. Instability

This section is devoted to the question of stability: that is, it is needed to check whether the solution depends continuously on the input data of given initial wave profile or not.

Because the kernel in Eq. (12) is regular (Hochstadt, 1973), it is classified as a Hilbert-Schmidt kernel. This makes the integral operator L compact (Groetsch, 1993; Roman 1975). Thus, Eq. (9) turns out to be a Fredholm integral equation of the first kind with a compact integral operator (Groetsch, 1993; Roman, 1975).

Due to the compactness, the inverse L^{-1} of the operator L is discontinuous even though L is continuous (Kirsch, 1996). This implies that the integral equation in Eq. (9) is ill-posed in the sense of stability. If direct(usual or conventional) numerical methods were introduced into the ill-posed problems, only meaningless solutions with arbitrary large errors would be given: In fact, a direct numerical discretization of the right side of Eq. (9) gives rise to a matrix whose condition number is very large enough that a numerical inverse of the matrix is not viable because the determinant of the matrix is almost zero. In section 5, numerical illustration of the condition number will be detailed. In general, ill-posed problems with lack of stability are known to cause numerical difficulties such as large errors.

4. Tikhonov's regularization

We have discussed about the instability of the present problem. We introduce a special numerical tool to overcome it, called Tikhonov's regularization method: In this section, a brief introduction of the method will be given.

Tikhonov (1963) introduced a functional M which has a damping term with a positive real number (called regularization parameter) α to regularize the Fredholm integral equation (9);

$$M=\left\{\int_0^{r_0}\left|\int_0^{k_0}K(r,k)F(k)dk-\eta_0(r)\right|^2 dr\right\}^{1/2}+\alpha\Omega \tag{13}$$

The functional M in Eq. (13) is called the Thikonov functional and the additional term Ω is defined as a usual norm, in this study, as follow ;

$$\Omega=\int_0^{k_0}|F(k)|^2 dk \tag{14}$$

It is known that Tikhonov functional M has a unique minimum F. This minimum is the unique solution of the normal equation, which is the Fredholm integral equation of second kind (Kirsch, 1996);

$$\alpha F+L^*LF=L^*\eta_0 \tag{15}$$

where L^* is the adjoint operator, that is, for some function f,

$$L^*(f)=\int_0^{r_0}\overline{K(k,r)}f(r)dr \tag{16}$$

Equation (15) is known as well-posed in the sense of stability because Eq. (15) is not first kind integral equation but second kind (Kirsch, 1996). This indicates that the solution F of Eq. (15) depends continuously on the input data of given initial wave profile η_0 such that conventional numerical treatments such as discretization method work for finding the solution F with a small α.

5. Numerical experiments

For the purpose of numerical experiments, we need to specify the initial condition of wave profile η_0 in Eq. (5) as shown in Fig. 1(a):

$$\eta_0(r)=\frac{1}{r}J_1(r) \tag{17}$$

It is known that the exact solution of the integral equation (9) for the initial wave elevation η_0 turns out to be (Spiegel, 1968)

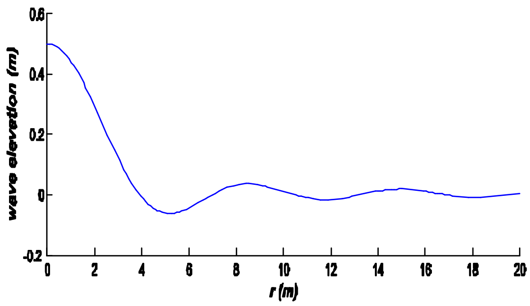
$$F_e=\begin{cases}1, & 0\le k\le 1\\ 0, & 1<k<\infty\end{cases} \tag{18}$$

As previously discussed, the aim of the present study is to identify spectrum function F defined on a compact support from given wave profile. In practice, measured data of the initial wave profile in Eq. (17) are deteriorated to an extent by noise: that is, we never know exactly the left hand-side in Eq. (9) but only up to an error of, say, noise level $\delta>0$. In this study, we assume that we know δ and noisy data η^δ with

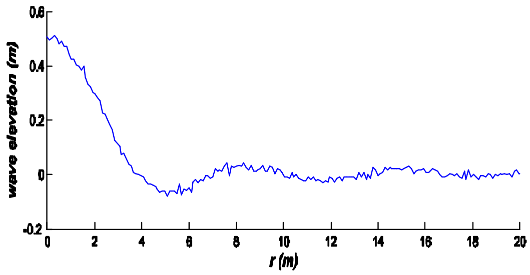
$$\|\eta-\eta^\delta\|_2\le\delta \tag{19}$$

where the symbol $\|\bullet\|_2$ stands for L_2-norm (Kirsch, 1996). Now it is our aim to solve the perturbed equation.

For the numerical experiments, we choose an error intensity $\delta=0.1$ and generate noisy data of wave profile randomly. The randomly generated wave profile is illustrated in Fig. 1(b).



(a) Noise-free η_0



(b) Noisy data : η_0^δ with $\delta=0.1$

Fig. 1 Given initial wave profile $\eta_0=J_1(r)/r$

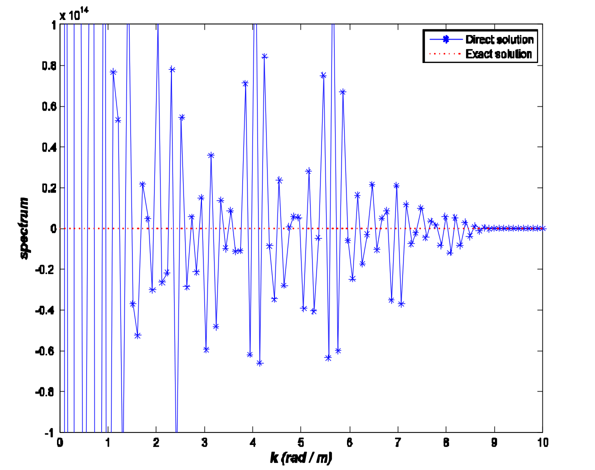


Fig. 2 Numerical solutions of direct discretization (typical example of instability)

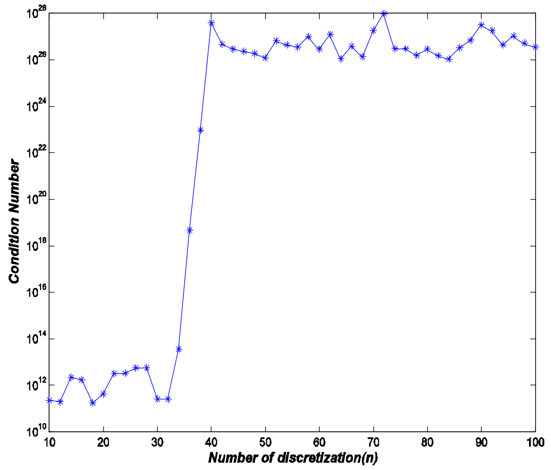


Fig. 3 Distributions of condition numbers for the discretized operator L in Eq. (10)

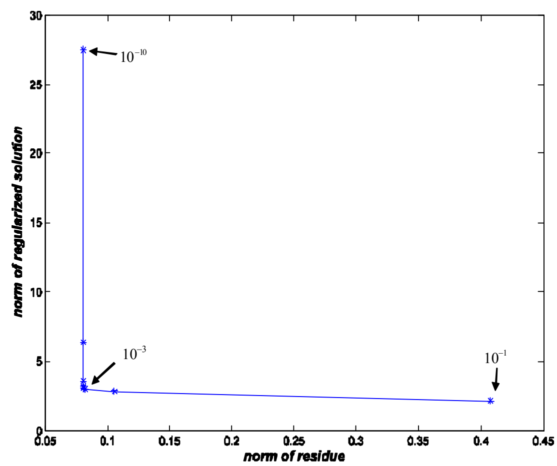


Fig. 4 L-curve criteria when $\delta = 0.1$ (typical example)

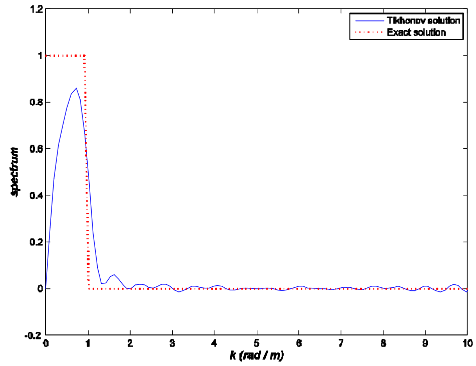
We first discretize the integral equation (9) with the trapezoidal rule, to find a direct numerical solution to Eq. (9) without using the Tikhonov's regularization: the direct numerical solution is depicted in Fig. 2 (the number of discretization is chosen 100). we try $k_0 = 10$ and the measurement length $r_0 = 10 (0 < r < r_0)$ for the present numerical study. All the obtained solutions are found to be meaningless. In fact, the behavior of the solutions has no regularity at all. Furthermore, we note that their magnitudes are unrealistically high.

The source of the trouble can be found by checking the condition number for the discretized form of the operator L in Eq. (10). Extremely large values of condition number occur as shown in Fig. 3: this means that the discretized system is ill-conditioned. This explains why we have the unstable meaningless solutions as shown in Fig. 2.

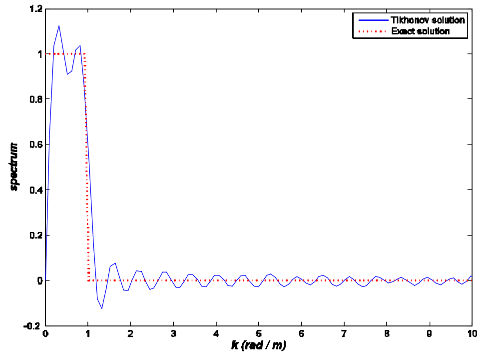
We have shown that direct numerical approach fails to solve the present problem. Therefore, we are to apply the Tikhonov's regularization to the present problem. According to the regularization theory, the regularization parameter α plays an important role in regularization process. In this paper, L-curve criterion (Hansen, 1992) is introduced to determine the appropriate regularization parameter α.

The L-curve is a log-log plot $(\log\|LF - \eta^{\delta}\|_2, \log\|F\|_2)$ of the norm of a regularized solution versus the norm of the corresponding residual norm.

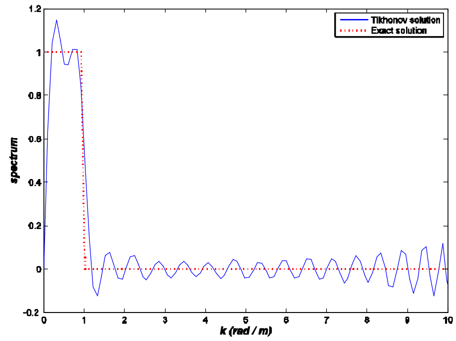
The log-log plot usually gives a typical "L" shape, and the optimal value for the regularization parameter is considered to be the one that corresponds to the corner of the curve (Hansen, 1992) : a heuristic motivation for this choice is that when the regularization parameter is small, then the norm of associated solution is huge and at the same time it is likely to be contaminated by measurement errors. Conversely, when the regularization parameter is large, the solution F is a poor approximation and the norm of associated difference $\|LF - \eta^{\delta}\|_2$ is huge. The corner of the L-curve marks this transition, since it represents a compromise between the minimization of the norm of the residual $\|LF - \eta^{\delta}\|_2$ and norm of the solution $\|F\|_2$. This choice of the regularization parameter α of the Tikhonov's regularization is not guaranteed to be appropriate for all linear systems with very ill-conditioned system. However, considerable computational experience indicates that the L-curve criterion is powerful method for determining a suitable value of the regularization parameter for many problems of interest in science and engineering.



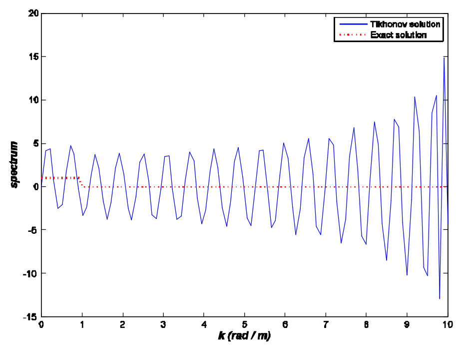
(a) $\alpha = 10^{-1}$



(b) $\alpha = 10^{-3}$



(c) $\alpha = 10^{-4}$

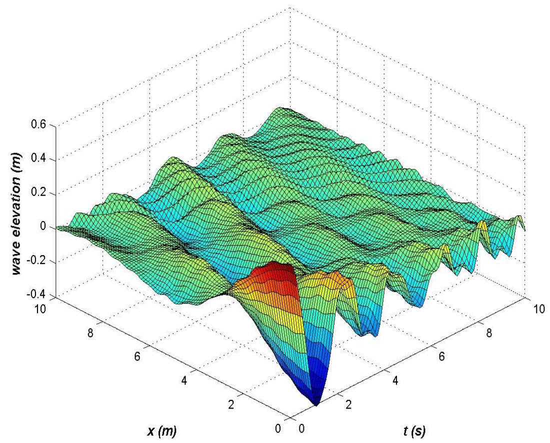


(d) $\alpha = 10^{-8}$

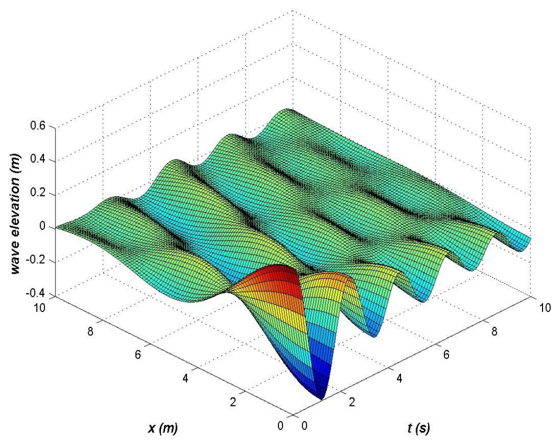
Fig. 5 Regularized solution of Tikhonov's regularization with varing regularization parameter. In the case of (d), note that different vertical scale is used

Figure 4 shows the typical example of the L-curve criteria when the noise level δ is 0.1 . We choose the values of the corner of L-curve as the optimal regularization parameter $\alpha = 10^{-3}$. Figure 5 shows regularized solutions computed through the Tikhonov's regularization for the integral equation (9). The figure clearly shows that the case of the optimal regularization parameter $\alpha = 10^{-3}$ gives most accurate result compared with other cases of regularization parameters. In contrast to the unstable results of direct numerical approach without regularization in Fig. 2, fairly stable solutions are obtained.

The whole time evolution solution then can be determined by using the integral expression in Eq. (6) with the spectrum



(a) Tikhonov's regularization



(b) Exact solution

Fig. 6 Space-time plot of the evolution waves function F. The space-time plots of the wave evolution by using regulized solution obtained from the integral equation (9) are shown in Fig. 6. In comparison with exact one by using exaction solution of Eq. (18), the result is satisfiable.

Finally, we list results corresponding to the regularization parameter α with various noise level δ in the following table, where we show the errors between the exact solution in Eq. (18) and Tikhonov's solution. The errors for the Tikhonov's solution is usually represented as follows, which is normalized by the norm of the exact solution in Eq. (18) :

$$err = \frac{\| F_e - F_{reg} \|_2}{\| F_e \|_2} \tag{20}$$

Here F_{reg} stands for the regularized solution. From the results, it can be easily found that the error decreases when the regularization parameter α decreases in all the cases. However, when the regularization parameter exceeds a certain threshold, which are shaded cells in Table 1, the error increases dramatically.

Table 1 Results of numerical experiments for $\delta > 0$

Regularization parameter α	$\delta = 0.000$	$\delta = 0.001$	$\delta = 0.01$	$\delta = 0.1$
10^{-1}	0.5074	0.5074	0.5070	0.5032
10^{-2}	0.3950	0.3950	0.3948	0.3894
10^{-3}	0.3888	0.3888	0.3888	0.3851
10^{-4}	0.3875	0.3876	0.3873	0.4220
10^{-5}	0.3857	0.3858	0.3850	0.6093
10^{-6}	0.3843	0.3836	0.3816	3.1543
10^{-7}	0.3824	0.3826	0.5101	9.4254
10^{-8}	0.3804	0.4541	2.5866	12.7907
10^{-9}	0.3823	0.6419	7.5324	35.2052
10^{-10}	0.3885	1.2246	29.5605	162.0238

6. Conclusions

Tikhonov's regularization, known as a non-iterative numerical method, is introduced to solve the unstable initial value problem of axisymmetric wave propagation. The optimal regularization parameter corresponding to the noise level δ was found by using the L-curve criterion. The numerical experiments showed that fairly stable solutions are realized and they are accurate when compared with the exact solution so that Tikhonov's regularization can be applicable to the determination of the unstable Cauchy-Poisson problem with the help of the introduced L-curve criterion.

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References

Bassanini, P. and Elcrat, A.R. (1997). A Theory and Applications of Partial Differential Equations, Plenum Press.

Debnath, L. (1994). Nonlinear Water Waves, Academic Press, Inc., pp 35-74.

Fridman, V. (1965). "A Method of Successive Approximations for Fredholm Integral Equations of the First Kind", Uspeki mat. Nauk., Vol 11, pp 233-234.

Groetsch, C.W. (1993). Inverse Problems in the Mathematical Sciences, Vieweg.

Hadamard, J. (1923). Lectures on the Cauchy Problem in Linear Partial Differential Equations, Yale University Press, New Haven.

Hansen, P.C., (1992). "Analysis of Discrete Ill-posed Problems by means of the L-curve", SIAM Rev. Vol 34, pp 561-580.

Hochstadt, H. (1973). Integral Equations, A Wiley Interscience Publication, New York.

Isakov, V. (1998). Inverse Problems for Partial Differential Equations, Springer.

Jang, T.S. and Kinoshita, T. (2000). "An Ill-posed Inverse Problem of a Wing with Locally Given Velocity Data and Its Analysis", Journal of Marine Science and Technology, Vol 5, pp 16-20.

Jang, T.S., Choi, H.S. and Kinoshita, T. (2000a). "Numerical Experiments on an Ill-posed Inverse Problem for a Given Vlocity Around a Hydrofoil by Iterative and Noniterative Regularizations", Journal of Marine Science and Technology, Vol 5, pp 107-111.

Jang, T.S., Choi, H.S. and Kinoshita, T. (2000b). "Solution of an Unstable Inverse Problem: Wave Source Evaluation from Observation of Velocity Distribution", Journal of Marine Science and Technology, Vol 5, pp 181-188.

Jang, T.S., Kwon, S.H. and Kim, B.J. (2007). "Solution of an Unstable Axisymmetric Cauchy-Poisson Problem of Dispersive Water Waves for a Spectrum with Compact Support", Ocean Engineering, Vol 34, pp 676-684

Jang, T.S. and Han, S.L., (2008). "Application of Tikhonov's

Regularization to an Unstable Two Dimensional Water Waves: Spectrum with Compact Support", Ships and Offshore Structures, Vol 3, pp 41-47

Kammerer, W.J. and Nashed, M.Z. (1972). "Iterative Methods for Best Approximate Solutions of Integral Equations of the First and Second Kinds", J. Math. Anal. Appl., Vol 40, pp 547-573.

Kirsch, A. (1996). An Introduction to the Mathematical Theory of Inverse Problems, Springer.

Roman, P. (1975). Some Modern Mathematics for Physicists and Other Outsiders, Pergamon Press Inc. New York, Vol 1, pp 270-290.

Rudin, W. (1991). Functional Analysis, McGraw-Hill, New York.

Spiegel, M. R. (1968). Mathematical Handbook, McGraw-Hill, New York

Stakgold, I. (1967). Boundary Value Problems of Mathematical Physics, Vol 1, Macmillan.

Stoker, J.J. (1957). Water Waves, Interscience Publishers, New York.

Tikhonov, A.N. (1963). Solution of Incorrectly Formulated Problems and the Regularization Method, Sov. Doklady, Vol 4, pp 1035-1038.

Whitham, G.B. (1974). Linear and Nonlinear Waves, John Wiley and Sons, New York, pp 431-476.

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2008년 5월 27일 최종 수정본 채택