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ON A FUNCTIONAL CENTRAL LIMIT THEOREM FOR THE LINEAR PROCESS GENERATED BY ASSOCIATED RANDOM VARIABLES IN A HILBERT SPACE

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Abstract. Let $\{\xi_k, k \in \mathbb{Z}\}$ be a strictly stationary associated sequence of H-valued random variables with $E\xi_k = 0$ and $E\|\xi_k\|^2 < \infty$ and $\{a_k, k \in \mathbb{Z}\}$ a sequence of linear operators such that $\sum_{j=-\infty}^{\infty} \|a_j\|_{L(H)} < \infty$. For a linear process $X_k = \sum_{j=-\infty}^{\infty} a_j \xi_{k-j}$ we derive that $\{X_k\}$ fulfills the functional central limit theorem.

1. Introduction

Let H be a separable real Hilbert space with the norm $\|\cdot\|_H$ generated by an inner product, $\langle \cdot, \cdot \rangle_H$ and let $\{e_k, k \geq 1\}$ be an orthonormal basis in H. Let $L(H)$ be the class of bounded linear operators from H to H and denote by $\|\cdot\|_{L(H)}$ its usual norm. Let $\{\xi_k, k \in \mathbb{Z}\}$ be a strictly stationary sequence of H-valued random variables, and $\{a_k, k \in \mathbb{Z}\}$ be a sequence of operators, $a_k \in L(H)$. We define the stationary Hilbert space process by:

$$X_k = \sum_{j=-\infty}^{\infty} a_j \xi_{k-j}, k \in \mathbb{Z}. \tag{1.1}$$

The sequence $\{X_k, k \in \mathbb{Z}\}$ is a natural extension of the multivariate linear processes (Brockwell and Davis [5], Chap. 11). These types of processes with values in functional spaces also facilitate the study of estimation and forecasting problems for several classes of continuous time processes. For more details see Bosq [3].

We define

$$W_n(t) = n^{-\frac{1}{2}} \sum_{k=1}^{[nt]} X_k,\ t \in [0, 1]. \tag{1.2}$$

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When $\{\xi_k, k \in \mathbb{Z}\}$ is a sequence of H-valued i.i.d. random variables such that $E\|\xi_k\|^2 < \infty$ and $E\xi_k = 0$ if $\sum_{j=-\infty}^{\infty} \|a_j\|_{L(H)} < \infty$, then the series in (1.1) converges almost surely and in $L_1(H)$ (Denisevskii and Dorogovtsev [8]). Moreover, X_k satisfies a functional central limit theorem (Bosq [3]) and the Berry-Esseen inequality (Bosq [4]).

A sequence $\{\xi_i, 1 \le i \le n\}$ of real-valued random variables is said to be associated if for any coordinatewise increasing functions $f, g : \mathbb{R}^n \to \mathbb{R}$

$$Cov(f(\xi_1, \dots, \xi_n), g(\xi_1, \dots, \xi_n)) \ge 0$$

whenever this exists. Associated sequences are widely encountered in applications; e.g. in reliability, in mathematical physics and percolation theory (c.f. Barlow and Proschan [1], Newman [11], Cox and Grimmett [7]). Newman [11] proved the central limit theorem, Newman and Wright [12] extended this to a functional central limit theorem.

Recently Kim and Ko [10] derived a functional central limit theorem for the linear process generated by associated random variables as follows.

Theorem 1.1 (Kim and Ko [10]). *Let $\{\xi_k\}$ be a strictly stationary sequence of centered and associated random variables having finite second moment and let $\{a_k\}$ be a sequence of numbers such that*

$$\sum_{j=-\infty}^{\infty} |a_j| < \infty.$$

Define X_k by (1.1)*, W_n by* (1.2) *and assume*

$$\sigma^2 = E\xi_1^2 + 2\sum_{j=2}^{\infty} E(\xi_1 \xi_j) < \infty.$$

Then, as $n \to \infty$

$$W_n(t) \Rightarrow W^1,$$

where $\Rightarrow$ indicates weak convergence and W^1 is a Wiener process with variance $(\sum_{j=-\infty}^{\infty} a_j)^2 \sigma^2$.

In the studying the infinite-dimensional case, our question is to what extent Theorem 1.1 remains valid in the new context when we replace $\{\xi_k\}$ by an infinite-dimensional space valued random variables, the constants by linear bounded operators and absolute values by the corresponding norms. To see new possible quality effects, we consider a simplest case of infinite dimensional Hilbert space H in this paper.

2. Preliminaries

Theorem 2.1 (Newman, Wright [12]). *Let $\{\xi_1, \dots, \xi_m\}$ be a sequence of associated random variables with $E|\xi_i|^2 < \infty$ and $E\xi_i = 0$ $i \ge 1$, and let $M_m = \max(S_1, \dots, S_m)$, where $S_m = \xi_1 + \cdots + \xi_m$. Then*

$$E(M_m^2) \le E(S_m). \tag{2.1}$$

As the notion of weakly associated random vectors in Burton et al. [6], we introduce the concept of associated random vectors.

Definition 2.2. A finite sequence $\{\xi_i,\ 1 \le i \le n\}$ of $\mathbb{R}^d$-valued random vectors is said to be associated if for all coordinatewise increasing functions $f, g : \mathbb{R}^{nd} \to \mathbb{R}$, $Cov(f(\xi_1, \dots, \xi_n), g(\xi_1, \dots, \xi_n)) \ge 0$ whenever this is defined. An infinite family of $\mathbb{R}^d$-valued random vectors is associated if every finite subfamily is associated.

From the functional central limit theorem of weakly associated random vectors in Burton et al. [6], we can obtain the following functional central limit theorem for stationary associated random vectors.

Theorem 2.3. *Let $\{\xi_i,\ i \ge 1\}$ be a strictly stationary associated sequence of $\mathbb{R}^d$-valued random vectors with $E\xi_1 = \mathbb{O}$ and $E\|\xi_1\|^2 < \infty$. If*

$$\sigma^2 = E\|\xi_1\|^2 + 2\sum_{i=2}^{\infty}\sum_{j=1}^{d} E(\xi_{1j}\xi_{ij}) < \infty \tag{2.2}$$

then, as $n \to \infty$

$$n^{-\frac{1}{2}} \sum_{i=1}^{[nt]} \xi_i \Rightarrow W^d, \tag{2.3}$$

where W^d is a d-dimensional Wiener process with covariance matrix $\Gamma = [\sigma_{kj}]$,

$$\sigma_{kj} = E(\xi_{1k}\xi_{1j}) + \sum_{i=2}^{\infty} [E(\xi_{1k}\xi_{ij}) + E(\xi_{1j}\xi_{ik})]. \tag{2.4}$$

From Definition 2.2 we consider the following notion:

Definition 2.4 (Burton et al. [6])**.** Let $\{\xi_i,\ i \ge 1\}$ be a sequence of random variables taking values in a separable Hilbert space H. $\{\xi_i,\ i \ge 1\}$ is called associated if for some orthonormal basis $\{e_k,\ k \ge 1\}$ in H and for any $d \ge 1$ the d-dimensional sequence $(\langle \xi_i, e_1\rangle, \dots, \langle \xi_i, e_d\rangle), i \ge 1$, is associated.

Definition 2.5 (Burton et al. [6])**.** Let $\{\xi_i,\ i \ge 1\}$ be a strictly stationary associated sequence H-valued random variables with $E\xi_1 = 0$ and $E\|\xi_1\|^2 < \infty$. If

$$\sigma^2 = E\|\xi_1\|^2 + 2\sum_{i=2}^{\infty} E(\langle \xi_1, \xi_i\rangle) < \infty, \tag{2.5}$$

then

$$n^{-\frac{1}{2}} \sum_{i=1}^{[nt]} \xi_i \Rightarrow W,$$

where W is a Wiener process on H with covariance operator $\Gamma = (\sigma_{kl}), k, l = 1, 2, \ldots$,

$$\sigma_{kl} = E(\langle e_k, \xi_1\rangle\langle e_l, \xi_1\rangle) + \sum_{i=2}^{\infty} [E(\langle e_k, \xi_1\rangle\langle e_l, \xi_i\rangle) + E(\langle e_l, \xi_1\rangle\langle e_k, \xi_i\rangle)]. \tag{2.6}$$

3. Main results

To prove the main theorem we need the following lemmas:

Lemma 3.1. *Let $\{\xi_k,\ k \in \mathbb{Z}\}$ be a strictly stationary associated sequence of H-valued random variables with $E\xi_1 = 0$ and $E\|\xi_1\|^2 < \infty$ and $\{c_k\}$ be a sequence of bounded linear operators satisfying*

$$\sum_{j=-\infty}^{\infty} \|c_j\|_{L(H)} < \infty. \tag{3.1}$$

If (2.5) *holds, then there is a constant K such that, for every $-\infty < p < q < \infty$,*

$$E\|\sum_{j=p}^{q} c_j\xi_j\|_H^2 \le K(\sum_{j=p}^{q} \|c_j\|_{L(H)}^2). \tag{3.2}$$

Proof. By stationarity, (2.5), and the facts that $\|c_j\xi_j\|_H \le \|c_j\|_{L(H)}\|\xi_j\|_H$ and $E(\langle\xi_i, \xi_j\rangle) \ge 0$ we have

$$\begin{aligned} E\|\sum_{j=p}^{q} c_j\xi_j\|_H^2 &\le \sum_{j=p}^{q} \|c_j\|_{L(H)}^2 E\|\xi_j\|_H^2 \\ &\quad +2\sum_{i=p}^{q-1}\sum_{j=i+1}^{q} \|c_i\|_{L(H)}\|c_j\|_{L(H)} E(\langle\xi_i, \xi_j\rangle) \\ &\le \sum_{j=p}^{q} \|c_j\|_{L(H)}^2 E\|\xi_j\|_H^2 + 2\sum_{j=2}^{\infty} E\langle\xi_1, \xi_j\rangle(\sum_{j=p}^{q} \|c_j\|_{L(H)}^2) \\ &\le K(\sum_{j=p}^{q} \|c_j\|_{L(H)}^2). \end{aligned}$$

□

Lemma 3.2. *Let $\{b_k,\ k \in \mathbb{Z}\}$ be a sequence of bounded linear operators in a Hilbert space $(H, \|\cdot\|_H)$ such that*

$$\sum_{k=-\infty}^{\infty} \|b_k\|_{L(H)} < \infty \tag{3.3}$$

and

$$\sum_{k=-\infty}^{\infty} b_k = 0. \tag{3.4}$$

Then we have

$$\frac{1}{n}\sum_{j=-\infty}^{\infty} \| \sum_{i=1-j}^{n-j} b_i\|^2_{L(H)} \to 0 \text{ as } n \to \infty. \tag{3.5}$$

Proof. Denote by $D_n = \sum_{|j|\geq n} \|b_j\|_{L(H)}$. By taking into account (3.3) we observe that

$$\begin{aligned} \frac{1}{n}\sum_{|j|\geq 2n} \| \sum_{i=1-j}^{n-j} b_i\|^2_{L(H)} \leq & \quad (\sum_{|j|\geq n} \|b_j\|_{L(H)})\frac{1}{n}\sum_{j=-\infty}^{\infty}(\sum_{i=1-j}^{n-j}\|b_i\|_{L(H)}) \\ = & \quad D_n \sum_{j=-\infty}^{\infty} \|b_j\|_{L(H)} \to 0 \text{ as } n\to\infty. \end{aligned} \tag{3.6}$$

Now for a fixed x in the interval $[-2,2]$, we define

$$h_n(x) = \| \sum_{i=1-[nx]}^{n-[nx]} b_i\|^2_{L(H)} \ .$$

One can easily see that, under the conditions (3.3) and (3.4), for every $x \neq 1$ we have $h_n(x) \to 0$, as $n \to \infty$ and $0 \leq h_n(x) \leq (\sum_{j=-\infty}^{\infty} \|b_i\|_{L(H)})^2$. Hence by Lebesgue's dominated convergence theorem, we obtain

$$\frac{1}{n}\sum_{j=-2n}^{2n-1} \| \sum_{i=1-j}^{n-j} b_i\|^2_H = \int_0^2 h_n(x)dx \to 0 \text{ as } n \to \infty. \tag{3.7}$$

Therefore the conclusion (3.5) is a consequence of (3.6) and (3.7). □

Theorem 3.3. *Let $\{\xi_k,\ k \in \mathbb{Z}\}$ be a strictly stationary associated sequence of H-valued random variables with $E\xi_1 = 0$ and $E\|\xi_1\|^2 < \infty$. Let $\{a_k, k \in \mathbb{Z}\}$ be a sequence of linear bounded operators such that*

$$\sum_{j=-\infty}^{\infty} \|a_j\|_{L(H)} < \infty. \tag{3.8}$$

If (2.5) *holds, then*

$$\frac{\sum_{k=1}^{[nt]} X_k}{\sqrt{n}} \Rightarrow W, \tag{3.9}$$

where X_k is defined by (1.1)*, W is a Wiener process on H with covariance operator $A\Gamma A^*$, Γ is defined in Theorem 2.5, $A = \sum_{j=-\infty}^{\infty} a_j$ and A^* denotes the adjoint operator of A.*

Proof. First note that by Theorem 2.5 we have

$$\frac{A\sum_{k=1}^{[nt]}\xi_k}{\sqrt{n}} \to^{\mathcal{D}} W, \tag{3.10}$$

where W is a Wiener process on H with covariance operator $A\Gamma A^*$ and that from (1.1) we have

$$\sum_{k=1}^{[nt]} X_k = \sum_{k=1}^{[nt]}\sum_{m=-\infty}^{\infty} a_m \xi_{k-m} = \sum_{j=-\infty}^{\infty}\Big(\sum_{k=1}^{[nt]} a_{k-j}\Big)\xi_j. \tag{3.11}$$

It remains to show that

$$n^{-\frac{1}{2}}\Big\|\sum_{k=1}^{[nt]} X_k - A\sum_{j=1}^{[nt]}\xi_j\Big\| \to^{p} 0 \tag{3.12}$$

by Billingsley [2], Theorem 4.1.

By partitioning the last sum in (3.11) into two sums, one with j between 1 and n, and another containing all the other terms, we get the representation

$$\sum_{k=1}^{[nt]} X_k - A\sum_{j=1}^{[nt]}\xi_j = \sum_{j=-\infty}^{\infty}\Big(\sum_{k=1}^{[nt]} b_{k-j}\Big)\xi_j, \tag{3.13}$$

where

$$b_0 = a_0 - A \quad\text{and}\quad b_i = a_i \quad\text{for}\quad |i| \geq 1. \tag{3.14}$$

Now by Lemma 3.1 and Fatou Lemma, we deduce from (3.13)

$$\begin{aligned} & \frac{1}{n}E\Big\|\sum_{k=1}^{[nt]} X_k - A\sum_{j=1}^{[nt]}\xi_j\Big\|_H^2 \\ \leq\ & \frac{1}{[nt]}E\Big\|\sum_{k=1}^{[nt]} X_k - A\sum_{j=1}^{[nt]}\xi_j\Big\|_H^2 \\ =\ & \frac{1}{[nt]}E\Big\|\sum_{j=-\infty}^{\infty}\Big(\sum_{k=1}^{[nt]} b_{k-j}\Big)\xi_j\Big\|_H^2 \\ \leq\ & K\frac{1}{[nt]}\sum_{j=-\infty}^{\infty}\Big\|\sum_{k=1}^{[nt]} b_{k-j}\Big\|_H^2 \\ =\ & K\frac{1}{[nt]}\sum_{j=-\infty}^{\infty}\Big\|\sum_{i=1-j}^{[nt]-j} b_i\Big\|_{L(H)}^2. \end{aligned} \tag{3.15}$$

Notice that the operators $\{b_i, i \in \mathbb{Z}\}$ being defined by (3.14) satisfy the conditions of Lemma 3.2. Therefore from (3.15), (3.12) follows by applying Lemma 3.2. $\square$

Remark. Obviously, Theorem 3.3 is an extension of Theorem 1.1 to a Hilbert space.

From Theorem 3.3 we obtain the following result.

Corollary 3.4 (Kim et al. [9]). *Let $\{\xi_k,\ k \in \mathbb{Z}\}$ be a strictly stationary associated sequence of $\mathbb{R}^d$-valued random vectors with $E\xi_1 = \mathbb{O}$ and $E\|\xi_1\|^2 < \infty$ and let $\{B_j\}$ be a sequence of matrix such that*

$$\sum_{j=-\infty}^{\infty} \|B_j\| < \infty \quad \sum_{j=-\infty}^{\infty} B_j \neq \mathbb{O}_{d\times d},$$

where for any $d \times d$ matrix $B = (a_{ij})$, $\|B\| = \sum_{i=1}^{d} \sum_{i=1}^{d} |a_{ij}|$ and $\mathbb{O}_{d\times d}$ denotes the $d \times d$ zero matrix. Define X_k an $\mathbb{R}^d$-valued linear process of the form $X_k = \sum_{j=-\infty}^{\infty} B_j \xi_{k-j}$. If (2.2) *holds, then*

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} X_k \Rightarrow W^d,$$

where W^d is a d-dimensional Wiener process with covariance matrix $T = (\sum_{j=-\infty}^{\infty} A_j)\Gamma(\sum_{j=-\infty}^{\infty} A_j)'$ and Γ is defined in (2.4).

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