

**BLENDING INSTANTANEOUS AND CONTINUOUS  
PHENOMENA IN FEYNMAN'S OPERATIONAL CALCULI:  
THE CASE OF TIME DEPENDENT NONCOMMUTING  
OPERATORS**

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**ABSTRACT.** Feynman's operational calculus for noncommuting operators was studied via measures on the time interval. We investigate some properties of Feynman's operational calculi which include a variety of blends of discrete and continuous measures in the time dependent setting.

**1. Introduction**

Feynman's 1951 paper on the operational calculus for noncommuting operators arose out of his ingenious work on quantum electrodynamics and was inspired in part by his earlier work on the Feynman path integral. Much surprisingly varied work on the subject has been done since by mathematicians and physicists. References can be found in the recent books of Johnson and Lapidus [8] and Nazaikinskii, Shatalov and Sternin [12].

A new approach to the mathematically rigorous theory of Feynman's operational calculus was begun recently by Jefferies and Johnson [3]-[7]. Each of the  $n$  operators involved has associated with it a measure on an appropriate time interval, and the resulting  $n$ -vector of measures determines a particular operational calculus. Here we begin the study of a broader theory which includes a variety of blends of discrete and continuous measure in the time dependent setting.

We now introduce some notation and begin to our discussion more precise. Let  $X$  be a separable Banach space over the complex numbers and let  $\mathcal{L}(X)$  denote the space of bounded linear operators on  $X$ . Fix  $T > 0$ . For  $i = 1, \dots, n$  let  $A_i : [0, T] \rightarrow \mathcal{L}(X)$  be maps that are measurable in the sense that  $A_i^{-1}(E)$  is a Borel set in  $[0, T]$  for any strong operator open set  $E \subset \mathcal{L}(X)$ . To each  $A_i(\cdot)$  we associate a finite Borel measure  $\lambda_i$  on  $[0, T]$  and we require that, for

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each  $i$ ,

$$r_i = \int_{[0,T]} \|A_i(s)\|_{\mathcal{L}(X)} |\lambda_i|(ds) < \infty.$$

Given a positive integer  $n$  and  $n$  positive numbers  $r_1, \dots, r_n$ , let  $\mathbb{A}(r_1, \dots, r_n)$  be the space of complex-valued functions of  $n$  complex variables  $f(z_1, \dots, z_n)$ , which are analytic at  $(0, \dots, 0)$ , and are such that their power series expansion

$$(1) \quad f(z_1, \dots, z_n) = \sum_{m_1, \dots, m_n=0}^{\infty} c_{m_1, \dots, m_n} z_1^{m_1} \cdots z_n^{m_n}$$

converges absolutely, at least on the closed polydisk  $|z_1| \leq r_1, \dots, |z_n| \leq r_n$ . Such functions are analytic at least in the open polydisk  $|z_1| < r_1, \dots, |z_n| < r_n$ .

For  $f \in \mathbb{A}(r_1, \dots, r_n)$  given by (1), we let

$$(2) \quad \|f\| = \|f\|_{\mathbb{A}(r_1, \dots, r_n)} := \sum_{m_1, \dots, m_n=0}^{\infty} |c_{m_1, \dots, m_n}| r_1^{m_1} \cdots r_n^{m_n}.$$

The function on  $\mathbb{A}(r_1, \dots, r_n)$  defined by (2) makes  $\mathbb{A}(r_1, \dots, r_n)$  into a commutative Banach algebra [3].

To the algebra  $\mathbb{A}(r_1, \dots, r_n)$  we associate a disentangling algebra by replacing the  $z_i$ 's with formal commuting objects  $(A_i(\cdot), \lambda_i)$ ,  $i = 1, \dots, n$ . Consider the collection  $\mathbb{D}((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n))$  of all expressions of the form

$$\begin{aligned} & f((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n)) \\ &= \sum_{m_1, \dots, m_n=0}^{\infty} c_{m_1, \dots, m_n} ((A_1(\cdot), \lambda_1))^{m_1} \cdots ((A_n(\cdot), \lambda_n))^{m_n}, \end{aligned}$$

where  $c_{m_1, \dots, m_n} \in \mathbb{C}$  for all  $m_1, \dots, m_n = 0, 1, \dots$ , and

$$\begin{aligned} & \|f((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n))\| \\ &= \|f((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n))\|_{\mathbb{D}((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n))} \\ (3) \quad &:= \sum_{m_1, \dots, m_n=0}^{\infty} |c_{m_1, \dots, m_n}| r_1^{m_1} \cdots r_n^{m_n} < \infty, \end{aligned}$$

where  $r_i = \int_{[0,T]} \|A_i(s)\|_{\mathcal{L}(X)} |\lambda_i|(ds)$  for  $i = 1, 2, \dots, n$ .

Rather than using the notation  $(A_i(\cdot), \lambda_i)$  below, we will often abbreviate to  $A_i(\cdot)$ , especially when carrying out calculations. We will often write  $\mathbb{D}$  in place of  $\mathbb{D}((A_1(\cdot)), \dots, (A_n(\cdot)))$  or  $\mathbb{D}((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n))$ .

Adding and scalar multiplying such expressions coordinatewise, we can easily see that  $\mathbb{D}((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n))$  is a vector space and that  $\|\cdot\|_{\mathbb{D}}$  defined by (3) is a norm. The normed linear space  $(\mathbb{D}((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n))), \|\cdot\|_{\mathbb{D}}$  can be identified with the weighted  $l_1$ -space, where the weight at the index  $(m_1, \dots, m_n)$  is  $r_1^{m_1} \cdots r_n^{m_n}$ . It follows that  $\mathbb{D}((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n))$  is a commutative Banach algebra with identity [7].

We refer to  $\mathbb{D}((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n))$  as the disentangling algebra associated with the  $n$ -tuple  $((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n))$ .

For  $m = 0, 1, \dots$ , let  $S_m$  denote the set of all permutations of the integers  $\{1, \dots, m\}$ , and given  $\pi \in S_m$ , we let

$$\Delta_m(\pi) = \{(s_1, \dots, s_m) \in [0, T]^m : 0 < s_{\pi(1)} < \dots < s_{\pi(m)} < T\}.$$

Now for nonnegative integers  $m_1, \dots, m_n$  and  $m = m_1 + \dots + m_n$ , we define

$$C_i(s) = \begin{cases} A_1(s), & \text{if } i \in \{1, \dots, m_1\} \\ A_2(s), & \text{if } i \in \{m_1 + 1, \dots, m_1 + m_2\} \\ \vdots \\ A_n(s), & \text{if } i \in \{m_1 + \dots + m_{n-1} + 1, \dots, m\} \end{cases}$$

for  $i = 1, \dots, m$  and for all  $0 \leq s \leq T$ . Next, in order to accommodate the use of discrete measures, we will need a refined version of the time-ordered sets  $\Delta_m(\pi)$ . Let  $\tau_1, \dots, \tau_h \in [0, T]$  be such that  $0 < \tau_1 < \dots < \tau_h < T$ . Given  $m \in \mathbb{N}$  and  $\pi \in S_m$ , and nonnegative integers  $r_1, \dots, r_{h+1}$  such that  $r_1 + \dots + r_{h+1} = m$ , we define

$$\begin{aligned} \Delta_{m;r_1, \dots, r_{h+1}}(\pi) = \{ (s_1, \dots, s_m) \in [0, T]^m : & 0 < s_{\pi(1)} < \dots < s_{\pi(r_1)} \\ & < \tau_1 < s_{\pi(r_1+1)} < \dots < s_{\pi(r_1+r_2)} \\ & < \tau_2 < s_{\pi(r_1+r_2+1)} < \dots < s_{\pi(r_1+\dots+r_h)} \\ & < \tau_h < s_{\pi(r_1+\dots+r_h+1)} < \dots < s_{\pi(m)} < T \}. \end{aligned}$$

Now let  $\lambda_1, \dots, \lambda_n$  be finite Borel measures on  $[0, T]$  such that

$$\lambda_l = \mu_l + \eta_l$$

for  $l = 1, \dots, n$  where  $\mu_l$  is a continuous measure and  $\eta_l$  is a finitely supported discrete measure for each  $l$ . Let  $\{\tau_1, \dots, \tau_h\}$  be the set obtained by taking the union of the supports of the discrete measures  $\eta_1, \dots, \eta_n$  and write

$$\eta_l = \sum_{i=1}^h p_{li} \delta_{\tau_i}$$

for each  $l = 1, \dots, n$ . With this notation it may be that many of the  $p_{li}$ 's are equal to zero.

Now we define the map  $T_{\lambda_1, \dots, \lambda_n}$  which will take us from the commutative framework of the disentangling algebra  $\mathbb{D}(A_1(\cdot), \dots, A_n(\cdot))$  to the noncommutative setting of  $\mathcal{L}(X)$ .

**Definition 1.** Let  $P^{m_1, \dots, m_n}(z_1, \dots, z_n) = z_1^{m_1} \cdots z_n^{m_n}$ . We define the action of the disentangling map on this monomial by

$$\begin{aligned} & T_{\lambda_1, \dots, \lambda_n} P^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)) \\ &= T_{\lambda_1, \dots, \lambda_n}((A_1(\cdot))^{m_1} \cdots (A_n(\cdot))^{m_n}) \end{aligned}$$

$$\begin{aligned}
&:= \sum_{q_{11}+q_{12}=m_1} \sum_{q_{21}+q_{22}=m_2} \cdots \sum_{q_{n1}+q_{n2}=m_n} \left( \frac{m_1! \cdots m_n!}{q_{11}! q_{12}! q_{21}! q_{22}! \cdots q_{n1}! q_{n2}!} \right) \\
&\quad \sum_{\pi \in S_{q_{11}+q_{21}+\cdots+q_{n1}}} \sum_{r_1+\cdots+r_{h+1}=q_{11}+q_{21}+\cdots+q_{n1}} \cdots \\
&\quad \sum_{j_{11}+\cdots+j_{1h}=q_{12}} \sum_{j_{21}+\cdots+j_{2h}=q_{22}} \cdots \sum_{j_{n1}+\cdots+j_{nh}=q_{n2}} \\
&\quad \left( \frac{q_{12}! q_{22}! \cdots q_{n2}!}{j_{11}! \cdots j_{1h}! j_{21}! \cdots j_{2h}! \cdots j_{n1}! \cdots j_{nh}!} \right) \int_{\Delta_{q_{11}+q_{21}+\cdots+q_{n1}; r_1, \dots, r_{h+1}(\pi)}} \\
&\quad C_{\pi(q_{11}+q_{21}+\cdots+q_{n1})}(s_{\pi(q_{11}+q_{21}+\cdots+q_{n1})}) \cdots \\
&\quad C_{\pi(r_1+\cdots+r_{h+1})}(s_{\pi(r_1+\cdots+r_{h+1})}) [p_{nh} A_n(\tau_h)]^{j_{nh}} \cdots [p_{2h} A_2(\tau_h)]^{j_{2h}} \\
&\quad [p_{1h} A_1(\tau_h)]^{j_{1h}} C_{\pi(r_1+\cdots+r_h)}(s_{\pi(r_1+\cdots+r_h)}) \cdots C_{\pi(r_1+1)}(s_{\pi(r_1+1)}) \\
&\quad [p_{n1} A_n(\tau_1)]^{j_{n1}} \cdots [p_{21} A_2(\tau_1)]^{j_{21}} [p_{11} A_1(\tau_1)]^{j_{11}} C_{\pi(r_1)}(s_{\pi(r_1)}) \cdots \\
&\quad C_{\pi(1)}(s_{\pi(1)}) (\mu_1^{q_{11}} \times \cdots \times \mu_n^{q_{n1}})(ds_1, \dots, ds_{q_{11}+q_{21}+\cdots+q_{n1}}).
\end{aligned}$$

Finally for  $f \in \mathbb{D}((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n))$  given by

$$f(A_1(\cdot), \dots, A_n(\cdot)) = \sum_{m_1, \dots, m_n=0}^{\infty} c_{m_1, \dots, m_n} (A_1(\cdot))^{m_1} \cdots (A_n(\cdot))^{m_n}$$

we set

$$\begin{aligned}
&\mathcal{T}_{\lambda_1, \dots, \lambda_n} f(A_1(\cdot), \dots, A_n(\cdot)) \\
&:= \sum_{m_1, \dots, m_n=0}^{\infty} c_{m_1, \dots, m_n} \mathcal{T}_{\lambda_1, \dots, \lambda_n} P^{m_1, \dots, m_n} (A_1(\cdot), \dots, A_n(\cdot)).
\end{aligned}$$

We will often use the alternative notation :

$$P_{\lambda_1, \dots, \lambda_n}^{m_1, \dots, m_n} (A_1(\cdot), \dots, A_n(\cdot)) = \mathcal{T}_{\lambda_1, \dots, \lambda_n} P^{m_1, \dots, m_n} (A_1(\cdot), \dots, A_n(\cdot))$$

and

$$f_{\lambda_1, \dots, \lambda_n} (A_1(\cdot), \dots, A_n(\cdot)) = \mathcal{T}_{\lambda_1, \dots, \lambda_n} f(A_1(\cdot), \dots, A_n(\cdot)).$$

## 2. The time dependent disentangling map

In this section, we obtain basic properties of the time dependent disentangling map which can be reduced to the time independent disentangling map defined in [7]

**Proposition 2.1.** *Suppose that each of the measures  $\lambda_1, \dots, \lambda_n$  is continuous on  $[0, T]$ . Then the expression of the disentangling map defined in Definition 1 is identical to that defined in Definition 2.3 of [7].*

*Proof.* If each of the measures  $\lambda_1, \dots, \lambda_n$  is continuous then all of the  $\lambda$ 's have 0 discrete part. So  $q_{12} = \cdots = q_{n2} = 0$  and  $q_{i1} = m_i$  for  $i = 1, \dots, n$ . Thus  $q_{11} + \cdots + q_{n1} = m_1 + \cdots + m_n = m$  and so  $S_{q_{11}+\cdots+q_{n1}} = S_{m_1+\cdots+m_n} = S_m$ .

Also,  $r_1 = \dots = r_{h+1} = 0$  and all of the  $j'$ s are 0. Further, both of the quotients of products of factorials are equal to 1. Hence we have

$$\begin{aligned}
& \mathcal{T}_{\lambda_1, \dots, \lambda_n} P^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)) \\
&= \sum_{q_{11}+q_{12}=m_1} \sum_{q_{21}+q_{22}=m_2} \dots \sum_{q_{n1}+q_{n2}=m_n} \left( \frac{m_1! \cdots m_n!}{q_{11}! q_{12}! q_{21}! q_{22}! \cdots q_{n1}! q_{n2}!} \right) \\
&\quad \sum_{\pi \in S_{q_{11}+q_{21}+\dots+q_{n1}}} \sum_{r_1+\dots+r_{h+1}=q_{11}+q_{21}+\dots+q_{n1}} \dots \\
&\quad \sum_{j_{11}+\dots+j_{1h}=q_{12}} \sum_{j_{21}+\dots+j_{2h}=q_{22}} \dots \sum_{j_{n1}+\dots+j_{nh}=q_{n2}} \\
&\quad \left( \frac{q_{12}! q_{22}! \cdots q_{n2}!}{j_{11}! \cdots j_{1h}! j_{21}! \cdots j_{2h}! \cdots j_{n1}! \cdots j_{nh}!} \right) \\
&\quad \int_{\Delta_{q_{11}+q_{21}+\dots+q_{n1}; r_1, \dots, r_{h+1}}(\pi)} C_{\pi(q_{11}+q_{21}+\dots+q_{n1})}(s_{\pi(q_{11}+q_{21}+\dots+q_{n1})}) \cdots \\
&\quad C_{\pi(r_1+\dots+r_{h+1})}(s_{\pi(r_1+\dots+r_{h+1})}) [p_{nh} A_n(\tau_h)]^{j_{nh}} \cdots [p_{2h} A_2(\tau_h)]^{j_{2h}} \\
&\quad [p_{1h} A_1(\tau_h)]^{j_{1h}} C_{\pi(r_1+\dots+r_h)}(s_{\pi(r_1+\dots+r_h)}) \cdots C_{\pi(r_1+1)}(s_{\pi(r_1+1)}) \\
&\quad [p_{n1} A_n(\tau_1)]^{j_{n1}} \cdots [p_{21} A_2(\tau_1)]^{j_{21}} [p_{11} A_1(\tau_1)]^{j_{11}} C_{\pi(r_1)}(s_{\pi(r_1)}) \cdots \\
&\quad C_{\pi(1)}(s_{\pi(1)})(\mu_1^{q_{11}} \times \cdots \times \mu_n^{q_{n1}})(ds_1, \dots, ds_{q_{11}+q_{21}+\dots+q_{n1}}) \\
&= \sum_{\pi \in S_{q_{11}+q_{21}+\dots+q_{n1}}} \int_{\Delta_{q_{11}+q_{21}+\dots+q_{n1}}(\pi)} C_{\pi(q_{11}+q_{21}+\dots+q_{n1})}(s_{\pi(q_{11}+q_{21}+\dots+q_{n1})}) \cdots \\
&\quad C_{\pi(1)}(s_{\pi(1)})(\mu_1^{q_{11}} \times \cdots \times \mu_n^{q_{n1}})(ds_1, \dots, ds_{q_{11}+q_{21}+\dots+q_{n1}}) \\
&= \sum_{\pi \in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \\
&\quad (\lambda_1^{m_1} \times \cdots \times \lambda_n^{m_n})(ds_1, \dots, ds_m).
\end{aligned}$$

The last equation is the identical expression for the disentangling of

$$P^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot))$$

in Definition 2.3 of [7].  $\square$

**Theorem 2.2.** *The disentangling map  $\mathcal{T}_{\lambda_1, \dots, \lambda_n}$  is a bounded linear operator from  $\mathbb{D}((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n))$  to  $\mathcal{L}(X)$ . In fact,  $\|\mathcal{T}_{\lambda_1, \dots, \lambda_n}\| \leq 1$ .*

*Proof.* The linearity of  $\mathcal{T}_{\lambda_1, \dots, \lambda_n}$  is clear. We have

$$\|\mathcal{T}_{\lambda_1, \dots, \lambda_n} P^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot))\|$$

$$\begin{aligned}
&\leq \sum_{q_{11}+q_{12}=m_1} \sum_{q_{21}+q_{22}=m_2} \cdots \sum_{q_{n1}+q_{n2}=m_n} \left( \frac{m_1! \cdots m_n!}{q_{11}! q_{12}! q_{21}! q_{22}! \cdots q_{n1}! q_{n2}!} \right) \\
&\quad \sum_{\pi \in S_{q_{11}+q_{21}+\cdots+q_{n1}}} \sum_{r_1+\cdots+r_{h+1}=q_{11}+q_{21}+\cdots+q_{n1}} \cdots \\
&\quad \sum_{j_{11}+\cdots+j_{1h}=q_{12}} \sum_{j_{21}+\cdots+j_{2h}=q_{22}} \cdots \sum_{j_{n1}+\cdots+j_{nh}=q_{n2}} \\
&\quad \left( \frac{q_{12}! q_{22}! \cdots q_{n2}!}{j_{11}! \cdots j_{1h}! j_{21}! \cdots j_{2h}! \cdots j_{n1}! \cdots j_{nh}!} \right) \int_{\Delta_{q_{11}+q_{21}+\cdots+q_{n1}; r_1, \dots, r_{h+1}}(\pi)} \\
&\quad \|C_{\pi(q_{11}+q_{21}+\cdots+q_{n1})}(s_{\pi(q_{11}+q_{21}+\cdots+q_{n1})})\| \cdots \\
&\quad \|C_{\pi(r_1+\cdots+r_{h+1})}(s_{\pi(r_1+\cdots+r_{h+1})})\| [p_{nh} \|A_n(\tau_h)\|]^{j_{nh}} \cdots \\
&\quad [p_{2h} \|A_2(\tau_h)\|]^{j_{2h}} [p_{1h} \|A_1(\tau_h)\|]^{j_{1h}} \|C_{\pi(r_1+\cdots+r_h)}(s_{\pi(r_1+\cdots+r_h)})\| \cdots \\
&\quad \|C_{\pi(r_1+1)}(s_{\pi(r_1+1)})\| [p_{n1} \|A_n(\tau_1)\|]^{j_{n1}} \cdots [p_{21} \|A_2(\tau_1)\|]^{j_{21}} \\
&\quad [p_{11} \|A_1(\tau_1)\|]^{j_{11}} \|C_{\pi(r_1)}(s_{\pi(r_1)})\| \cdots \|C_{\pi(1)}(s_{\pi(1)})\| \\
&\quad |\mu_1^{q_{11}} \times \cdots \times \mu_n^{q_{n1}}|(ds_1, \dots, ds_{q_{11}+\cdots+q_{n1}}) \\
&= \sum_{q_{11}+q_{12}=m_1} \sum_{q_{21}+q_{22}=m_2} \cdots \sum_{q_{n1}+q_{n2}=m_n} \left( \frac{m_1! \cdots m_n!}{q_{11}! q_{12}! q_{21}! q_{22}! \cdots q_{n1}! q_{n2}!} \right) \\
&\quad \sum_{j_{11}+\cdots+j_{1h}=q_{12}} \sum_{j_{21}+\cdots+j_{2h}=q_{22}} \cdots \sum_{j_{n1}+\cdots+j_{nh}=q_{n2}} \\
&\quad \left( \frac{q_{12}! q_{22}! \cdots q_{n2}!}{j_{11}! \cdots j_{1h}! j_{21}! \cdots j_{2h}! j_{n1}! \cdots j_{nh}!} \right) \\
&\quad \sum_{\pi \in S_{q_{11}+q_{21}+\cdots+q_{n1}}} \int_{\Delta_{q_{11}+q_{21}+\cdots+q_{n1}}(\pi)} \\
&\quad \|A_1(s_1)\| \cdots \|A_1(s_{q_{11}})\| \|A_2(s_{q_{11}+1})\| \cdots \|A_2(s_{q_{11}+q_{21}})\| \cdots \\
&\quad \|A_n(s_{q_{11}+\cdots+q_{(n-1)1}+1})\| \cdots \|A_n(s_{q_{11}+\cdots+q_{n1}})\| [p_{n1} \|A_n(\tau_1)\|]^{j_{n1}} \cdots \\
&\quad [p_{21} \|A_2(\tau_1)\|]^{j_{21}} [p_{11} \|A_1(\tau_1)\|]^{j_{11}} \cdots [p_{nh} \|A_n(\tau_h)\|]^{j_{nh}} \cdots \\
&\quad [p_{2h} \|A_2(\tau_h)\|]^{j_{2h}} [p_{1h} \|A_1(\tau_h)\|]^{j_{1h}} \\
&\quad |\mu_1^{q_{11}} \times \cdots \times \mu_n^{q_{n1}}|(ds_1, \dots, ds_{q_{11}+\cdots+q_{n1}}) \\
&= \sum_{q_{11}+q_{12}=m_1} \sum_{q_{21}+q_{22}=m_2} \cdots \sum_{q_{n1}+q_{n2}=m_n} \left( \frac{m_1! \cdots m_n!}{q_{11}! q_{12}! q_{21}! q_{22}! \cdots q_{n1}! q_{n2}!} \right) \\
&\quad \sum_{j_{11}+\cdots+j_{1h}=q_{12}} \sum_{j_{21}+\cdots+j_{2h}=q_{22}} \cdots \sum_{j_{n1}+\cdots+j_{nh}=q_{n2}} \\
&\quad \left( \frac{q_{12}! q_{22}! \cdots q_{n2}!}{j_{11}! \cdots j_{1h}! j_{21}! \cdots j_{2h}! \cdots j_{n1}! \cdots j_{nh}!} \right) \\
&\quad \int_{[0, T]^{q_{11}}} \|A_1(s_{11})\| \|A_1(s_{12})\| \cdots \|A_1(s_{1q_{11}})\| |\mu_1|^{q_{11}}(ds_{11}, \dots, ds_{1q_{11}})
\end{aligned}$$

$$\begin{aligned}
& [p_{11}\|A_1(\tau_1)\|]^{j_{11}} \cdots [p_{1h}\|A_1(\tau_h)\|]^{j_{1h}} \int_{[0,T]^{q_{21}}} \|A_2(s_{21})\| \cdots \\
& \|A_2(s_{2q_{21}})\| |\mu_2|^{q_{21}} (ds_{21}, \dots, ds_{2q_{21}}) [p_{21}\|A_2(\tau_1)\|]^{j_{21}} \cdots \\
& [p_{2h}\|A_2(\tau_h)\|]^{j_{2h}} \cdots \int_{[0,T]^{q_{n1}}} \|A_n(s_{n1})\| \|A_n(s_{n2})\| \cdots \|A_n(s_{nq_{n1}})\| \\
& |\mu_n|^{q_{n1}} (ds_{n1}, \dots, ds_{nq_{n1}}) [p_{n1}\|A_n(\tau_1)\|]^{j_{n1}} \cdots [p_{nh}\|A_n(\tau_h)\|]^{j_{nh}} \\
& = \sum_{q_{11}+q_{12}=m_1} \frac{m_1!}{q_{11}!q_{12}!} \left[ \int_{[0,T]} \|A_1(s)\| |\mu_1|(ds) \right]^{q_{11}} \left[ \sum_{i=1}^h p_{1i} \|A_1(\tau_i)\| \right]^{q_{12}} \\
& \quad \cdots \sum_{q_{n1}+q_{n2}=m_n} \frac{m_n!}{q_{n1}!q_{n2}!} \left[ \int_{[0,T]} \|A_n(s)\| |\mu_n|(ds) \right]^{q_{n1}} \left[ \sum_{i=1}^h p_{ni} \|A_n(\tau_i)\| \right]^{q_{n2}} \\
& = \left[ \int_{[0,T]} \|A_1(s)\| |\mu_1|(ds) + \sum_{i=1}^h p_{1i} \|A_1(\tau_i)\| \right]^{m_1} \cdots \\
& \quad \left[ \int_{[0,T]} \|A_n(s)\| |\mu_n|(ds) + \sum_{i=1}^h p_{ni} \|A_n(\tau_i)\| \right]^{m_n} \\
& = \left[ \int_{[0,T]} \|A_1(s)\| |\lambda_1|(ds) \right]^{m_1} \cdots \left[ \int_{[0,T]} \|A_n(s)\| |\lambda_n|(ds) \right]^{m_n}.
\end{aligned}$$

Hence, for  $f(A_1(\cdot), \dots, A_n(\cdot)) \in \mathbb{D}((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n))$ ,

$$\begin{aligned}
& \|\mathcal{T}_{\lambda_1, \dots, \lambda_n} f(A_1(\cdot), \dots, A_n(\cdot))\| \\
& \leq \sum_{m_1, \dots, m_n=0}^{\infty} |c_{m_1, \dots, m_n}| \|\mathcal{T}_{\lambda_1, \dots, \lambda_n} P^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot))\| \\
& \leq \sum_{m_1, \dots, m_n=0}^{\infty} |c_{m_1, \dots, m_n}| \left[ \int_{[0,T]} \|A_1(s)\| |\lambda_1|(ds) \right]^{m_1} \\
& \quad \cdots \left[ \int_{[0,T]} \|A_n(s)\| |\lambda_n|(ds) \right]^{m_n} \\
& = \|f((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n))\|_{\mathbb{D}((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n))}.
\end{aligned}$$

This finishes the proof.  $\square$

**Theorem 2.3.** Suppose that  $A_i(s)A_j(t) = A_j(t)A_i(s)$  for  $i$  and  $j = 1, \dots, n$  whenever the products are defined, then we have

$$\begin{aligned}
& \mathcal{T}_{\lambda_1, \dots, \lambda_n} P^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)) \\
(4) \quad & = \left[ \int_{[0,T]} A_1(s) \lambda_1(ds) \right]^{m_1} \cdots \left[ \int_{[0,T]} A_n(s) \lambda_n(ds) \right]^{m_n}.
\end{aligned}$$

Further, for all  $f(A_1(\cdot), \dots, A_n(\cdot)) \in \mathbb{D}((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n))$ ,

$$\begin{aligned} & \mathcal{T}_{\lambda_1, \dots, \lambda_n} f(A_1(\cdot), \dots, A_n(\cdot)) \\ &= f \left( \int_{[0, T]} A_1(s) \lambda_1(ds), \dots, \int_{[0, T]} A_n(s) \lambda_n(ds) \right), \end{aligned}$$

where  $f$  given by

$$f(z_1, \dots, z_n) = \sum_{m_1, \dots, m_n=0}^{\infty} c_{m_1, \dots, m_n} z_1^{m_1} \cdots z_n^{m_n}$$

is an element of  $\mathbb{A}(r_1, \dots, r_n)$  where

$$r_i = \int_{[0, T]} \|A_i(s)\|_{\mathcal{L}(X)} |\lambda_i|(ds).$$

*Proof.* The operator

$$\mathcal{T}_{\lambda_1, \dots, \lambda_n} f(A_1(\cdot), \dots, A_n(\cdot))$$

is given in terms of

$$\mathcal{T}_{\lambda_1, \dots, \lambda_n} P^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot))$$

and so it suffices to show equation (4). We have

$$\begin{aligned} & \mathcal{T}_{\lambda_1, \dots, \lambda_n} P^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)) \\ &= \sum_{q_{11}+q_{12}=m_1} \sum_{q_{21}+q_{22}=m_2} \cdots \sum_{q_{n1}+q_{n2}=m_n} \left( \frac{m_1! \cdots m_n!}{q_{11}! q_{12}! q_{21}! q_{22}! \cdots q_{n1}! q_{n2}!} \right) \\ & \quad \sum_{\pi \in S_{q_{11}+q_{21}+\cdots+q_{n1}}} \sum_{r_1+\cdots+r_{n+1}=q_{11}+q_{21}+\cdots+q_{n1}} \cdots \\ & \quad \sum_{j_{11}+\cdots+j_{1h}=q_{12}} \sum_{j_{21}+\cdots+j_{2h}=q_{22}} \cdots \sum_{j_{n1}+\cdots+j_{nh}=q_{n2}} \\ & \quad \left( \frac{q_{12}! q_{22}! \cdots q_{n2}!}{j_{11}! \cdots j_{1h}! j_{21}! \cdots j_{2h}! \cdots j_{n1}! \cdots j_{nh}!} \right) \int_{\Delta_{q_{11}+q_{21}+\cdots+q_{n1}; r_1, \dots, r_{n+1}}(\pi)} \\ & \quad C_{\pi(q_{11}+q_{21}+\cdots+q_{n1})}(s_{\pi(q_{11}+q_{21}+\cdots+q_{n1})}) \cdots \\ & \quad C_{\pi(r_1+\cdots+r_{n+1})}(s_{\pi(r_1+\cdots+r_{n+1})}) [p_{nh} A_n(\tau_h)]^{j_{nh}} \cdots \\ & \quad [p_{2h} A_2(\tau_h)]^{j_{2h}} [p_{1h} A_1(\tau_h)]^{j_{1h}} C_{\pi(r_1+\cdots+r_h)}(s_{\pi(r_1+\cdots+r_h)}) \cdots \\ & \quad C_{\pi(r_1+1)}(s_{\pi(r_1+1)}) [p_{n1} A_n(\tau_1)]^{j_{n1}} \cdots [p_{21} A_2(\tau_1)]^{j_{21}} \\ & \quad [p_{11} A_1(\tau_1)]^{j_{11}} C_{\pi(r_1)}(s_{\pi(r_1)}) \cdots C_{\pi(1)}(s_{\pi(1)}) \\ & \quad (\mu_1^{q_{11}} \times \cdots \times \mu_n^{q_{n1}})(ds_1, \dots, ds_{q_{11}+\cdots+q_{n1}}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{q_{11}+q_{12}=m_1} \sum_{q_{21}+q_{22}=m_2} \cdots \sum_{q_{n1}+q_{n2}=m_n} \left( \frac{m_1! \cdots m_n!}{q_{11}! q_{12}! q_{21}! q_{22}! \cdots q_{n1}! q_{n2}!} \right) \\
&\quad \sum_{\pi \in S_{q_{11}+q_{21}+\cdots+q_{n1}}} \sum_{r_1+\cdots+r_{h+1}=q_{11}+q_{21}+\cdots+q_{n1}} \cdots \\
&\quad \sum_{j_{11}+\cdots+j_{1h}=q_{12}} \sum_{j_{21}+\cdots+j_{2h}=q_{22}} \cdots \sum_{j_{n1}+\cdots+j_{nh}=q_{n2}} \\
&\quad \left( \frac{q_{12}! q_{22}! \cdots q_{n2}!}{j_{11}! \cdots j_{1h}! j_{21}! \cdots j_{2h}! \cdots j_{n1}! \cdots j_{nh}!} \right) \int_{\Delta_{q_{11}+q_{21}+\cdots+q_{n1}; r_1, \dots, r_{h+1}}(\pi)} \\
&\quad A_1(s_1) \cdots A_1(s_{q_{11}}) A_2(s_{q_{11}+1}) \cdots A_2(s_{q_{11}+q_{21}}) \cdots \\
&\quad A_n(s_{q_{11}+\cdots+q_{(n-1)1}+1}) \cdots A_n(s_{q_{11}+\cdots+q_{n1}}) [p_{11} A_n(\tau_1)]^{j_{n1}} \cdots \\
&\quad [p_{21} A_2(\tau_1)]^{j_{21}} [p_{11} A_1(\tau_1)]^{j_{11}} \cdots [p_{nh} A_n(\tau_h)]^{j_{nh}} \cdots \\
&\quad [p_{2h} A_2(\tau_h)]^{j_{2h}} [p_{1h} A_1(\tau_h)]^{j_{1h}} (\mu_1^{q_{11}} \times \cdots \times \mu_n^{q_{n1}})(ds_1, \dots, ds_{q_{11}+\cdots+q_{n1}}) \\
&= \sum_{q_{11}+q_{12}=m_1} \sum_{q_{21}+q_{22}=m_2} \cdots \sum_{q_{n1}+q_{n2}=m_n} \left( \frac{m_1! \cdots m_n!}{q_{11}! q_{12}! q_{21}! q_{22}! \cdots q_{n1}! q_{n2}!} \right) \\
&\quad \sum_{j_{11}+\cdots+j_{1h}=q_{12}} \sum_{j_{21}+\cdots+j_{2h}=q_{22}} \cdots \sum_{j_{n1}+\cdots+j_{nh}=q_{n2}} \\
&\quad \left( \frac{q_{12}! q_{22}! \cdots q_{n2}!}{j_{11}! \cdots j_{1h}! j_{21}! \cdots j_{2h}! \cdots j_{n1}! \cdots j_{nh}!} \right) \\
&\quad \int_{[0,T]^{q_{11}+q_{21}+\cdots+q_{n1}}} A_1(s_1) \cdots A_1(s_{q_{11}}) A_2(s_{q_{11}+1}) \cdots \\
&\quad A_2(s_{q_{11}+q_{21}}) \cdots A_n(s_{q_{11}+\cdots+q_{(n-1)1}+1}) \cdots A_n(s_{q_{11}+\cdots+q_{n1}}) \\
&\quad [p_{11} A_1(\tau_1)]^{j_{11}} \cdots [p_{1h} A_1(\tau_h)]^{j_{1h}} [p_{21} A_2(\tau_1)]^{j_{21}} \cdots \\
&\quad [p_{2h} A_2(\tau_h)]^{j_{2h}} \cdots [p_{n1} A_n(\tau_1)]^{j_{n1}} \cdots [p_{nh} A_n(\tau_h)]^{j_{nh}} \\
&\quad (\mu_1^{q_{11}} \times \cdots \times \mu_n^{q_{n1}})(ds_1, \dots, ds_{q_{11}+\cdots+q_{n1}}) \\
&= \sum_{q_{11}+q_{12}=m_1} \frac{m_1!}{q_{11}! q_{12}!} \left[ \int_{[0,T]} A_1(s) \mu_1(ds) \right]^{q_{11}} \left[ \sum_{i=1}^h p_{1i} A_1(\tau_i) \right]^{q_{12}} \\
&\quad \cdots \sum_{q_{n1}+q_{n2}=m_n} \frac{m_n!}{q_{n1}! q_{n2}!} \left[ \int_{[0,T]} A_n(s) \mu_n(ds) \right]^{q_{n1}} \left[ \sum_{i=1}^h p_{ni} A_n(\tau_i) \right]^{q_{n2}} \\
&= \left[ \int_{[0,T]} A_1(s) \mu_1(ds) + \sum_{i=1}^h p_{1i} A_1(\tau_i) \right]^{m_1} \cdots \left[ \int_{[0,T]} A_n(s) \mu_n(ds) + \sum_{i=1}^h p_{ni} A_n(\tau_i) \right]^{m_n} \\
&= \left[ \int_{[0,T]} A_1(s) \lambda_1(ds) \right]^{m_1} \cdots \left[ \int_{[0,T]} A_n(s) \lambda_n(ds) \right]^{m_n}.
\end{aligned}$$

We obtain the result.  $\square$

### 3. Stability properties

In this section, we obtain stability properties for the disentangling map  $T_{\lambda_1, \dots, \lambda_n}$  which was introduced in the previous section. Let  $S$  be a metric space and let  $\{\lambda_k\}_{k=1}^\infty$  be a sequence of finite Borel measures on  $S$ . We say that  $\lambda_k$  converges weakly to a finite Borel measure  $\lambda$  on  $S$  and write  $\lambda_k \rightharpoonup \lambda$  if for every bounded continuous real-valued function  $f$  on  $S$  we have  $\int_S f(s) \lambda_k(ds) \rightarrow \int_S f(s) \lambda(ds)$  as  $k \rightarrow \infty$ . The following result is Lemma 3.1 of [13].

**Lemma 3.1.** *Let  $\eta = \sum_{i=1}^h p_i \delta_{\tau_i}$  be a purely discrete probability measure on  $[0, T]$  with finite support. Assume that  $0 < \tau_1 < \dots < \tau_h < T$ . Let*

$$\alpha_i = \min\{\tau_i - \tau_{i-1}, \tau_{i+1} - \tau_i\}$$

for  $i = 1, \dots, h$  where we take  $\tau_0 = 0$  and  $\tau_{h+1} = T$ . In each interval  $(\tau_i - \alpha_i, \tau_i + \alpha_i)$ ,  $i = 1, \dots, h$  choose sequences  $\{\tau_{ik}\}_{k=1}^\infty$ . For each  $i = 1, \dots, h$  choose a sequence  $\{p_{ik}\}_{k=1}^\infty$  such that  $\eta_k = \sum_{i=1}^h p_{ik} \delta_{\tau_{ik}}$  is a probability measure for each  $k$ . Then  $\eta_k \rightharpoonup \eta$  if and only if

$$\begin{cases} p_{ik} \rightarrow p_i & \text{and } \tau_{ik} \rightarrow \tau_i & \text{if } p_i \neq 0, \\ p_{ik} \rightarrow p_i & \text{and } \{\tau_{ik}\}_{k=1}^\infty \text{ bounded} & \text{if } p_i = 0 \end{cases}$$

for  $i = 1, \dots, h$ .

First we consider the disentangling map for  $P^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot))$ .

**Theorem 3.2.** *Let  $A_l : [0, T] \rightarrow \mathcal{L}(X)$  be continuous with respect to the norm topology on  $\mathcal{L}(X)$  for each  $l = 1, 2, \dots, n$ . And let  $\lambda_1, \dots, \lambda_n$  be finite Borel measures on  $[0, T]$  such that*

$$\lambda_l = \mu_l + \eta_l$$

for  $l = 1, \dots, n$  where  $\mu_l$  is a continuous probability measure and  $\eta_l$  is a finitely supported discrete probability measure for each  $l$ . Let  $\{\tau_1, \dots, \tau_h\}$  be the set obtained by taking the union of the supports of the discrete measures  $\eta_1, \dots, \eta_n$  and write

$$\eta_l = \sum_{i=1}^h p_{li} \delta_{\tau_i}$$

for each  $l = 1, \dots, n$ . Choose sequences  $\{\mu_{lk}\}_{k=1}^\infty, l = 1, \dots, n$  of continuous Borel probability measures on  $[0, T]$  such that  $\mu_{lk} \rightharpoonup \mu_l$ . Also choose sequences  $\{\eta_{lk}\}_{k=1}^\infty, l = 1, \dots, n$  of discrete probability measures on  $[0, T]$  as in Lemma 3.1 such that  $\mu_{lk} \rightharpoonup \mu_l$ ; i.e., write

$$\eta_{lk} = \sum_{i=1}^h p_{li}^k \delta_{\tau_{ik}},$$

where, as in the Lemma 3.1,  $p_{li}^k \rightarrow p_{li}$  and  $\tau_{ik} \rightarrow \tau_i$  as  $k \rightarrow \infty$  for all  $i, l$  assuming that for  $p_{li} \neq 0$  for all  $i, l$ . Finally let  $\lambda_{lk} = \mu_{lk} + \eta_{lk}$  for  $l = 1, \dots, n$ .

Then for any nonnegative integers  $m_1, \dots, m_n$  and for any  $\Lambda \in \mathcal{L}(X)^*$

$$\begin{aligned} & \lim_{k \rightarrow \infty} \Lambda(P_{\lambda_{1k}, \dots, \lambda_{nk}}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot))) \\ &= \Lambda(P_{\lambda_1, \dots, \lambda_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot))). \end{aligned}$$

*Proof.* We see that for any  $\Lambda \in \mathcal{L}(X)^*$

$$\begin{aligned} & |\Lambda(P_{\lambda_{1k}, \dots, \lambda_{nk}}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot))) - \Lambda(P_{\lambda_1, \dots, \lambda_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)))| \\ & \leq \sum_{q_{11}+q_{12}=m_1} \sum_{q_{21}+q_{22}=m_2} \cdots \sum_{q_{n1}+q_{n2}=m_n} \left( \frac{m_1! \cdots m_n!}{q_{11}! q_{12}! q_{21}! q_{22}! \cdots q_{n1}! q_{n2}!} \right) \\ & \quad \sum_{\pi \in S_{q_{11}+q_{21}+\cdots+q_{n1}}} \sum_{r_1+\cdots+r_{h+1}=q_{11}+q_{21}+\cdots+q_{n1}} \cdots \\ & \quad \sum_{j_{11}+\cdots+j_{1h}=q_{12}} \sum_{j_{21}+\cdots+j_{2h}=q_{22}} \cdots \sum_{j_{n1}+\cdots+j_{nh}=q_{n2}} \\ & \quad \left( \frac{q_{12}! q_{22}! \cdots q_{n2}!}{j_{11}! \cdots j_{1h}! j_{21}! \cdots j_{2h}! \cdots j_{n1}! \cdots j_{nh}!} \right) \\ & \quad | \int_{\Delta_{q_{11}+q_{21}+\cdots+q_{n1}; r_1, \dots, r_{h+1}}(\pi)} \Lambda(C_{\pi(q_{11}+q_{21}+\cdots+q_{n1})}(s_{\pi(q_{11}+q_{21}+\cdots+q_{n1})}) \cdots \\ & \quad C_{\pi(r_1+\cdots+r_{h+1})}(s_{\pi(r_1+\cdots+r_{h+1})}) [p_{nh}^k A_n(\tau_{hk})]^{j_{nh}} \cdots [p_{2h}^k A_2(\tau_{hk})]^{j_{2h}} \\ & \quad [p_{1h}^k A_1(\tau_{hk})]^{j_{1h}} C_{\pi(r_1+\cdots+r_h)}(s_{\pi(r_1+\cdots+r_h)}) \cdots C_{\pi(r_1+1)}(s_{\pi(r_1+1)}) \\ & \quad [p_{n1}^k A_n(\tau_{1k})]^{j_{n1}} \cdots [p_{21}^k A_2(\tau_{1k})]^{j_{21}} [p_{11}^k A_1(\tau_{1k})]^{j_{11}} C_{\pi(r_1)}(s_{\pi(r_1)}) \cdots \\ & \quad C_{\pi(1)}(s_{\pi(1)})) (\mu_{1k}^{q_{11}} \times \cdots \times \mu_{nk}^{q_{n1}})(ds_1, \dots, ds_{q_{11}+q_{21}+\cdots+q_{n1}}) \\ & \quad - \int_{\Delta_{q_{11}+q_{21}+\cdots+q_{n1}; r_1, \dots, r_{h+1}}(\pi)} \Lambda(C_{\pi(q_{11}+q_{21}+\cdots+q_{n1})}(s_{\pi(q_{11}+q_{21}+\cdots+q_{n1})}) \cdots \\ & \quad C_{\pi(r_1+\cdots+r_{h+1})}(s_{\pi(r_1+\cdots+r_{h+1})}) [p_{nh}^k A_n(\tau_h)]^{j_{nh}} \cdots [p_{2h}^k A_2(\tau_h)]^{j_{2h}} \\ & \quad [p_{1h}^k A_1(\tau_h)]^{j_{1h}} C_{\pi(r_1+\cdots+r_h)}(s_{\pi(r_1+\cdots+r_h)}) \cdots C_{\pi(r_1+1)}(s_{\pi(r_1+1)}) \\ & \quad [p_{n1}^k A_n(\tau_1)]^{j_{n1}} \cdots [p_{21}^k A_2(\tau_1)]^{j_{21}} [p_{11}^k A_1(\tau_1)]^{j_{11}} C_{\pi(r_1)}(s_{\pi(r_1)}) \cdots \\ & \quad C_{\pi(1)}(s_{\pi(1)})) (\mu_1^{q_{11}} \times \cdots \times \mu_n^{q_{n1}})(ds_1, \dots, ds_{q_{11}+q_{21}+\cdots+q_{n1}})|. \end{aligned}$$

For each  $l = 1, \dots, n, i = 1, \dots, h$ ,  $p_{li}^k \rightarrow p_{li}$  and  $\tau_{ik} \rightarrow \tau_i$  as  $k \rightarrow \infty$ . Hence since  $A_l$  is continuous we have

$$p_{li}^k A_l(\tau_{ik}) \rightarrow p_{li} A_l(\tau_i)$$

as  $k \rightarrow \infty$ . Therefore, we have, for any  $\Lambda \in \mathcal{L}(X)^*$

$$\begin{aligned} & \chi_{\Delta_{q_{11}+q_{21}+\dots+q_{n1};r_1,\dots,r_{h+1}}}(\pi) \Lambda(C_{\pi(q_{11}+q_{21}+\dots+q_{n1})}(s_{\pi(q_{11}+q_{21}+\dots+q_{n1})}) \dots \\ & C_{\pi(r_1+\dots+r_h+1)}(s_{\pi(r_1+\dots+r_h+1)}) [p_{nh}^k A_n(\tau_{hk})]^{j_{nh}} \dots [p_{2h}^k A_2(\tau_{hk})]^{j_{2h}} \\ & [p_{1h}^k A_1(\tau_{hk})]^{j_{1h}} C_{\pi(r_1+\dots+r_h)}(s_{\pi(r_1+\dots+r_h)}) \dots C_{\pi(r_1+1)}(s_{\pi(r_1+1)}) \\ & [p_{n1}^k A_n(\tau_{1k})]^{j_{n1}} \dots [p_{21}^k A_2(\tau_{1k})]^{j_{21}} [p_{11}^k A_1(\tau_{1k})]^{j_{11}} C_{\pi(r_1)}(s_{\pi(r_1)}) \dots \\ & C_{\pi(1)}(s_{\pi(1)})) \rightarrow \\ & \chi_{\Delta_{q_{11}+q_{21}+\dots+q_{n1};r_1,\dots,r_{h+1}}}(\pi) \Lambda(C_{\pi(q_{11}+q_{21}+\dots+q_{n1})}(s_{\pi(q_{11}+q_{21}+\dots+q_{n1})}) \dots \\ & C_{\pi(r_1+\dots+r_h+1)}(s_{\pi(r_1+\dots+r_h+1)}) [p_{nh}^k A_n(\tau_h)]^{j_{nh}} \dots [p_{2h}^k A_2(\tau_h)]^{j_{2h}} \\ & [p_{1h}^k A_1(\tau_h)]^{j_{1h}} C_{\pi(r_1+\dots+r_h)}(s_{\pi(r_1+\dots+r_h)}) \dots C_{\pi(r_1+1)}(s_{\pi(r_1+1)}) \\ & [p_{n1}^k A_n(\tau_1)]^{j_{n1}} \dots [p_{21}^k A_2(\tau_1)]^{j_{21}} [p_{11}^k A_1(\tau_1)]^{j_{11}} C_{\pi(r_1)}(s_{\pi(r_1)}) \dots \\ & C_{\pi(1)}(s_{\pi(1)})) \end{aligned}$$

uniformly on  $[0, T]^{q_{11}+\dots+q_{n1}}$ .  $\{\mu_{1k}^{q_{11}} \times \dots \times \mu_{nk}^{q_{n1}}\}$  is a sequence of continuous probability measures on  $[0, T]^{q_{11}+\dots+q_{n1}}$  since each term in the product is a continuous probability measure. And  $[0, T]^{q_{11}+\dots+q_{n1}}$  is separable. By Theorem 3.2 of [1]  $\mu_{1k}^{q_{11}} \times \dots \times \mu_{nk}^{q_{n1}} \rightharpoonup \mu_1^{q_{11}} \times \dots \times \mu_n^{q_{n1}}$  since  $\mu_{ik} \rightharpoonup \mu_i$  for each  $i$ . Hence we have, using Lemma 3.2 of [13],

$$\begin{aligned} & \lim_{k \rightarrow \infty} \int_{\Delta_{q_{11}+q_{21}+\dots+q_{n1};r_1,\dots,r_{h+1}}}(\pi) \\ & \Lambda(C_{\pi(q_{11}+q_{21}+\dots+q_{n1})}(s_{\pi(q_{11}+q_{21}+\dots+q_{n1})}) \dots \\ & C_{\pi(r_1+\dots+r_h+1)}(s_{\pi(r_1+\dots+r_h+1)}) [p_{nh}^k A_n(\tau_{hk})]^{j_{nh}} \dots [p_{2h}^k A_2(\tau_{hk})]^{j_{2h}} \\ & [p_{1h}^k A_1(\tau_{hk})]^{j_{1h}} C_{\pi(r_1+\dots+r_h)}(s_{\pi(r_1+\dots+r_h)}) \dots C_{\pi(r_1+1)}(s_{\pi(r_1+1)}) \\ & [p_{n1}^k A_n(\tau_{1k})]^{j_{n1}} \dots [p_{21}^k A_2(\tau_{1k})]^{j_{21}} [p_{11}^k A_1(\tau_{1k})]^{j_{11}} C_{\pi(r_1)}(s_{\pi(r_1)}) \dots \\ & C_{\pi(1)}(s_{\pi(1)})) (\mu_{1k}^{q_{11}} \times \dots \times \mu_{nk}^{q_{n1}})(ds_1, \dots, ds_{q_{11}+q_{21}+\dots+q_{n1}}) \\ = & \int_{\Delta_{q_{11}+q_{21}+\dots+q_{n1};r_1,\dots,r_{h+1}}}(\pi) \\ & \Lambda(C_{\pi(q_{11}+q_{21}+\dots+q_{n1})}(s_{\pi(q_{11}+q_{21}+\dots+q_{n1})}) \dots \\ & C_{\pi(r_1+\dots+r_h+1)}(s_{\pi(r_1+\dots+r_h+1)}) [p_{nh}^k A_n(\tau_h)]^{j_{nh}} \dots [p_{2h}^k A_2(\tau_h)]^{j_{2h}} \\ & [p_{1h}^k A_1(\tau_h)]^{j_{1h}} C_{\pi(r_1+\dots+r_h)}(s_{\pi(r_1+\dots+r_h)}) \dots C_{\pi(r_1+1)}(s_{\pi(r_1+1)}) \\ & [p_{n1}^k A_n(\tau_1)]^{j_{n1}} \dots [p_{21}^k A_2(\tau_1)]^{j_{21}} [p_{11}^k A_1(\tau_1)]^{j_{11}} C_{\pi(r_1)}(s_{\pi(r_1)}) \dots \\ & C_{\pi(1)}(s_{\pi(1)})) (\mu_1^{q_{11}} \times \dots \times \mu_n^{q_{n1}})(ds_1, \dots, ds_{q_{11}+q_{21}+\dots+q_{n1}}). \end{aligned}$$

Hence the conclusion follows.  $\square$

The following results can be obtained easily.

**Lemma 3.3.** Let  $\lambda_1, \dots, \lambda_n, \lambda_{1k}, \dots, \lambda_{nk}, k = 1, 2, \dots$  be finite Borel measures. Suppose for  $l = 1, 2, \dots, n$

$$\bar{r}_l = \sup\{r_l, r_{l1}, \dots, r_{lk}, \dots\} < \infty,$$

where  $r_l = \int_{[0,T]} |A_l(s)| |\lambda_l|(ds)$  and  $r_{lk} = \int_{[0,T]} |A_l(s)| |\lambda_{lk}|(ds)$ . Then for any  $f \in \mathbb{A}(\bar{r}_1, \dots, \bar{r}_n)$ ,  $f((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n)) \in \mathbb{D}((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n))$  and  $f((A_1(\cdot), \lambda_{1k}), \dots, (A_n(\cdot), \lambda_{nk})) \in \mathbb{D}((A_1(\cdot), \lambda_{1k}), \dots, (A_n(\cdot), \lambda_{nk}))$  for any  $k = 1, 2, \dots$ .

**Theorem 3.4.** Let the hypotheses of Theorem 3.1 be satisfied. Further suppose that for each  $l = 1, 2, \dots, n$  and  $k = 1, 2, \dots, \bar{r}_l$ ,  $r_l, r_{lk}$  are given as in Lemma 3.3. Let  $\mathcal{T}_{\lambda_{1k}, \dots, \lambda_{nk}}$  denote the disentangling map corresponding to the  $k^{\text{th}}$  term of sequences of measures. Then for any  $f \in \mathbb{A}(\bar{r}_1, \dots, \bar{r}_n)$ , and for any  $\Lambda \in \mathcal{L}(X)^*$

$$\begin{aligned} & \lim_{k \rightarrow \infty} \Lambda(\mathcal{T}_{\lambda_{1k}, \dots, \lambda_{nk}} f((A_1(\cdot), \lambda_{1k}), \dots, (A_n(\cdot), \lambda_{nk}))) \\ &= \Lambda(\mathcal{T}_{\lambda_1, \dots, \lambda_n} f((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n))). \end{aligned}$$

*Proof.* We have

$$\begin{aligned} & |\Lambda(\mathcal{T}_{\lambda_{1k}, \dots, \lambda_{nk}} f((A_1(\cdot), \lambda_{1k}), \dots, (A_n(\cdot), \lambda_{nk}))) \\ & - \Lambda(\mathcal{T}_{\lambda_1, \dots, \lambda_n} f((A_1(\cdot), \lambda_1), \dots, (A_n(\cdot), \lambda_n)))| \\ & \leq \sum_{m_1, \dots, m_n=0}^{\infty} |c_{m_1, \dots, m_n}| |\Lambda(P_{\lambda_{1k}, \dots, \lambda_{nk}}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot))) \\ & - \Lambda(P_{\lambda_1, \dots, \lambda_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)))|. \end{aligned}$$

Note that

$$\begin{aligned} & \sum_{m_1, \dots, m_n=0}^{\infty} |c_{m_1, \dots, m_n}| |\Lambda(P_{\lambda_{1k}, \dots, \lambda_{nk}}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot))) \\ & - \Lambda(P_{\lambda_1, \dots, \lambda_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot)))| \\ & \leq \sum_{m_1, \dots, m_n=0}^{\infty} |c_{m_1, \dots, m_n}| \|\Lambda\| \left[ \|P_{\lambda_{1k}, \dots, \lambda_{nk}}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot))\| \right. \\ & \quad \left. + \|P_{\lambda_1, \dots, \lambda_n}^{m_1, \dots, m_n}(A_1(\cdot), \dots, A_n(\cdot))\| \right] \\ & \leq \|\Lambda\| \sum_{m_1, \dots, m_n=0}^{\infty} |c_{m_1, \dots, m_n}| \left[ \left[ \int_{[0,T]} |A_1(s)| |\lambda_{1k}|(ds) \right]^{m_1} \dots \right. \\ & \quad \left. \left[ \int_{[0,T]} |A_n(s)| |\lambda_{nk}|(ds) \right]^{m_n} + \left[ \int_{[0,T]} |A_1(s)| |\lambda_1|(ds) \right]^{m_1} \dots \right. \\ & \quad \left. \left[ \int_{[0,T]} |A_n(s)| |\lambda_n|(ds) \right]^{m_n} \right] \end{aligned}$$

$$\begin{aligned}
&= \|\Lambda\| \sum_{m_1, \dots, m_n=0}^{\infty} |c_{m_1, \dots, m_n}| [r_{1k}^{m_1} \cdots r_{nk}^{m_n} + r_1^{m_1} \cdots r_n^{m_n}] \\
&\leq 2\|\Lambda\| \sum_{m_1, \dots, m_n=0}^{\infty} |c_{m_1, \dots, m_n}| \bar{r}_1^{m_1} \cdots \bar{r}_n^{m_n}.
\end{aligned}$$

Since  $\sum_{m_1, \dots, m_n=0}^{\infty} |c_{m_1, \dots, m_n}| \bar{r}_1^{m_1} \cdots \bar{r}_n^{m_n} < \infty$ , by Theorem 3.2 and the Lebesgue Dominated Convergence Theorem, we obtain a result.  $\square$

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