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A GLOBAL BEHAVIOR OF THE POSITIVE SOLUTIONS OF $x_{n+1} = \frac{\beta x_n + x_{n-2}}{A + Bx_n + x_{n-2}}$

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Abstract. In this paper we prove that every positive solution of the third order rational difference equation

$$x_{n+1} = \frac{\beta x_n + x_{n-2}}{A + Bx_n + x_{n-2}}$$

converges to the positive equilibrium point

$$\overline{x} = \frac{\beta + 1 - A}{B + 1},$$

where $0 < \beta \leq B, 1 < A < \beta + 1$.

1. Introduction

Many authors have studied the periodic behavior or global stability of the positive solution of the rational difference equations [2, 3]. In [1] E. Camouzis obtained the global stability of the positive solutions of the following third order rational difference equation

$$x_{n+1} = \frac{\beta x_n + x_{n-2}}{A + Bx_n + x_{n-1}},$$

where $1 \leq B, 1 \leq A < \beta + 1, \beta > 0$. In this paper we apply the ideas of E. Camouzis to the global stability of the positive solutions of another third order rational difference equation

$$x_{n+1} = \frac{\beta x_n + x_{n-2}}{A + Bx_n + x_{n-2}}, \tag{1}$$

where $0 < \beta \leq B, 1 < A < \beta + 1$. We recall that $\{x_n\}$ is a positive solution of (1) if given initial $x_{-2}, x_{-1}, x_0 (> 0)$ $\{x_n\}$ satisfies (1). And we know that the equilibrium points of both difference equation are $0, \dfrac{\beta + 1 - A}{B + 1}$. Global stability means that every positive solutions of the difference equation converge to a equilibrium point. Specifically the equilibrium point is positive in [1].

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2. Main result

By using the technique 'backward repeated calculation' of [1] we obtained the global stability of the positive solution of the rational difference equation (1).

Lemma 2.1. *Assume* $A > 1, \beta > 0, B > 0$. *Let* $\{x_n\}$ *be a positive solution of the difference equation* (1). *Then it holds that*

$$x_n < \frac{\beta}{B}$$

for all sufficiently large n.

Proof. To obtain a contradiction we assume that there exist infinitely many N such that

$$x_N > \frac{\beta}{B}.$$

For such N we have

$$x_N = \frac{\beta x_{N-1} + x_{N-3}}{A + B x_{N-1} + x_{N-3}} > \frac{\beta}{B}.$$

From this we have

$$\beta x_{N-1} + x_{N-3} > \frac{\beta}{B}(A + B x_{N-1} + x_{N-3}),$$

$$x_{N-3} > \frac{\beta}{B}A + \frac{\beta}{B}x_{N-3} > \frac{\beta}{B}A.$$

Similarly, from $x_{N-3} = \dfrac{\beta x_{N-4} + x_{N-6}}{A + B x_{N-4} + x_{N-6}} > \dfrac{\beta}{B}A$, we obtains

$$x_{N-6} > \frac{\beta}{B}A^2.$$

Inductively

$$x_{N-3k} > \frac{\beta}{B}A^k, (k = 1, 2, \ldots).$$

Since $A > 1$, we have the contradiction. $\square$

Lemma 2.2. *Assume* $0 < A < \beta + 1, \beta > 0, B > 0$. *Let* $\{x_n\}$ *be a positive solution of the difference equation* (1). *Then*

$$\limsup_{n\to\infty} x_n > 0.$$

Proof. To obtain a contradiction we assume that

$$\lim_{n\to\infty} x_n = 0.$$

Since $0 < A < \beta + 1$, we can choose $\epsilon > 0$ such that

$$0 < \frac{A + (B+1)\epsilon}{\beta + 1} < 1.$$

Since $\lim_{n\to\infty} x_n = 0$, there exists N such that we have

$$x_{N+k} < \epsilon$$

for all $k = 1, 2, \ldots$. Let $m := \dfrac{A + (B+1)\epsilon}{\beta + 1} < 1$. Let $m_k := \min\{x_{N+k}, x_{N+k-2}\}$ for all $k = 2, 3, \ldots$. Then from $x_{N+k+1} = \dfrac{\beta x_{N+k} + x_{N+k-2}}{A + Bx_{N+k} + x_{N+k-2}} < \epsilon$, we obtains

$$\frac{(\beta+1)m_k}{A + (B+1)\epsilon} < \frac{(\beta+1)m_k}{A + Bx_{N+k} + x_{N+k-2}} < \epsilon.$$

Therefore

$$m_k < m\epsilon \quad \text{for all } k = 2, 3, \ldots.$$

If $m_k = x_{N+k}$, then $m_{k-1} < m^2\epsilon$ from (1). On the other hand if $m_k = x_{N+k-2}$, then $m_{k-3} < m^2\epsilon$. Similarly $m_{k-2} < m^3\epsilon$ or $m_{k-4} < m^3\epsilon$ or $m_{k-6} < m^3\epsilon$. Since k is arbitrary and $m < 1$, by backward repetitions we have the contradiction. □

Theorem 2.3. *Assume* $1 < A < \beta + 1, B \geq \beta > 0$. *Let* $\{x_n\}$ *be a positive solution of the difference equation* (1). *Then*

$$\lim_{n\to\infty} x_n = \frac{\beta + 1 - A}{B + 1}.$$

Proof. Let $S := \limsup_{n\to\infty} x_n \geq 0, I := \liminf_{n\to\infty} x_n \geq 0$. By Lemma 2.1 and 2.2,

$$0 < S \leq \frac{\beta}{B}.$$

Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$x_{n_i+1} \to S, \quad x_{n_i} \to l_0, x_{n_i-1} \to l_{-1}, x_{n_i-2} \to l_{-2}.$$

From the difference equation (1)

$$S = \frac{\beta l_0 + l_{-2}}{A + Bl_0 + l_{-2}}.$$

Since $l_2 \leq S \leq \dfrac{\beta}{B}$ and $A \geq 1$,

$$S \leq \frac{\beta S + l_{-2}}{A + \beta S + l_{-2}}.$$

Since $B \geq \beta > 0$,

$$S \leq \frac{\beta S + S}{A + \beta S + S}.$$

Since $S > 0$,

$$S \leq \frac{\beta + 1 - A}{B + 1}.$$

We claim that $I := \liminf_{n\to\infty} x_n > 0$. Indeed we can choose $m > 0$ such that

$$S \le \frac{\beta+1-A}{B+1} < (\beta+1-A) - Bm.$$

So we can choose N such that

$$x_n < (\beta+1-A) - Bm \quad \text{for all } n \ge N-2.$$

Furthermore we choose $\delta > 0$ such that

$$\delta = \min\{x_N, x_{N-1}, x_{N-2}, m\}.$$

Then

$$\begin{aligned} x_{N+1} &= \frac{\beta x_N + x_{N-2}}{A + Bx_N + x_{N-2}} \\ &\ge \frac{\beta\delta + x_{N-2}}{A + B\delta + x_{N-2}} \\ &\ge \frac{\beta\delta + \delta}{A + B\delta + (\beta+1-A) - B\delta} \\ &\ge \frac{(\beta+1)\delta}{\beta+1} \\ &\ge \delta. \end{aligned}$$

Inductively $x_n \ge \delta$ for all $n \ge N+1$. Therefore $I > 0$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$x_{n_i+1} \to I, \quad x_{n_i} \to l_0, x_{n_i-1} \to l_{-1}, x_{n_i-2} \to l_{-2}.$$

By the difference equation (1)

$$I = \frac{\beta l_0 + l_{-2}}{A + Bl_0 + l_{-2}}.$$

Since $I > 0$, $I \ge \dfrac{\beta+1-A}{B+1}$. We conclude that $S = I$ and

$$\lim_{n\to\infty} x_n = \frac{\beta+1-A}{B+1}.$$

□

Remark 2.4. We test the results with the following simple Mathematica Code.

```
nst1[ui_,vi_,wi_,p_,q_,r_,n_]:=Module[{l},
df1[w_,u_]:=(r w+ u)/(p+q w+u);
l=(r+1-p)/(q+1);
Print[l];
phi[x_List]:={x[[2]],x[[3]],df1[x[[3]],x[[1]]]};
NestList[phi,{ui,vi,wi},n]
]
```

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