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ON DUALITY THEOREMS FOR CONVEXIFIABLE OPTIMIZATION PROBLEMS

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ABSTRACT. In this paper, we consider of a convexifiable programming problem with bounds on variables. We obtain Mond-Weir type duality theorems for the convexifiable programming problems. Moreover, we give a numerical example to illustrate our duality.

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1. Introduction and Preliminaries

Optimality conditions and duality in single objective or multiobjective programs have been of much interest in the recent past ([2, 5, 6, 7, 11]). Duality theorems, sufficient optimality conditions for optimization problems are closely related to convexity of their involving functions. In 1981, Hanson [1] introduced an invex differentiable function, which is an important generalization of a convex differentiable function, and established the Kuhn-Tucker sufficient optimality criteria, the weak duality and the strong duality for a nonlinear optimization problem involving differentiable invex functions. Until now, the invexity conception was extended to the nondifferentiable cases by many authors ([4, 8, 9]).

Recently, Jeyakumar et al. ([3]) established that Kuhn-Tucker necessary optimality condition is sufficient for global optimality of the class of convexifiable programming problems with bounds on variables for which a local minimizer is global.

In this paper, we obtain Mond-Weir type duality results for the convexifiable programming problems with bounds on variables. Moreover, we give a numerical example which illustrates the result.

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In this paper, we consider the following programming problem with bounds on variables:

$$\text{(P)} \qquad \begin{aligned} \text{Minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_j(x) \leq 0, \ j = 1, \cdots, m, \\ & u_i \leq x_i \leq v_i, \ i = 1, \cdots, n, \end{aligned}$$

where $u_i < v_i$ and $f_j : \mathbb{R}^n \to \mathbb{R}, \ j = 0, 1, \cdots, m$ are continuously differentiable functions.

Let $T : \mathbb{R} \to \mathbb{R}$ be a function and let $t : \mathbb{R}^n \to \mathbb{R}^n$ be a separable strictly monotone mapping, i.e., $t(y) = (t_1(y_1), \cdots, t_n(y_n))^T$ and t_i is strictly monotone for $y = (y_1, \cdots, y_n)^T$ and $i = 1, \cdots, n$. Let $X_0 = \{x \in \mathbb{R}^n \mid u_i \leq x_i \leq v_i, \ i = 1, \cdots, n\}$. We also assume that $X_0 \subseteq t(\mathbb{R}^n)$ and t^{-1} is continuous. The derivative of a function T of one variable at a is denoted by $T'(a)$, and the derivative of a function t of several variables at a is denoted by $\nabla t(a)$. Let

$$\begin{aligned} Y_0 \quad := \quad & \Big\{ y \in \mathbb{R}^n \mid y_i = t_i^{-1}(x_i), \ i = 1, \cdots, n, \ x \in X_0 \Big\} \\ = \quad & \Big\{ y \in \mathbb{R}^n \mid t_i^{-1}(u_i) \leq y_i \leq t_i^{-1}(v_i) \text{ if } t_i(y_i) \text{ is strictly increasing} \Big\} \\ = \quad & \Big\{ y \in \mathbb{R}^n \mid t_i^{-1}(v_i) \leq y_i \leq t_i^{-1}(u_i) \text{ if } t_i(y_i) \text{ is strictly decreasing} \Big\}. \end{aligned}$$

Then, clearly Y_0 is also a box and since t is one-to-one, $Y_0 = t^{-1}(X_0)$ and $t(Y_0) = X_0$.

Following the notion of convexifiability, given in [3] and [10], we define the following:

Definition 1.1. (Convexifiable functions) A function $h : \mathbb{R}^n \to \mathbb{R}$ is said to be (strictly) convexifiable over X_0 if the composite function $T \circ h \circ t$ is (strictly) convex over the box $Y_0 = t^{-1}(X_0)$ for some $t : \mathbb{R}^n \to \mathbb{R}^n$, which is separable, strictly monotone and continuously differentiable with $X_0 \subseteq t(\mathbb{R}^n)$ and t^{-1} is continuously differentiable, and for some $T : \mathbb{R} \to \mathbb{R}$ which is strictly increasing and continuously differentiable with $T'(x) > 0, \ \forall x \in h(X_0)$.

Definition 1.2. (Convexifiable Programming) The problem (P) is called a (strictly) convexifiable programming problem if for each $j = 0, 1, \cdots, m$, the functions f_j is convexifiable over X_0 with the same t. That is, there exist $T_j : \mathbb{R} \to \mathbb{R}, \ j = 0, 1, \cdots, m$ and $t : \mathbb{R}^n \to \mathbb{R}^n$ such that $T_j \circ f_j \circ t$ is (strictly) convex over $t^{-1}(X_0)$ for each $j = 0, 1, \cdots, m$, where t is separable, strictly monotone and continuously differentiable with $X_0 \subseteq t(\mathbb{R}^n)$ and t^{-1} is differentiable and T_j is strictly increasing and differentiable with $T_j'(x) > 0, \ \forall x \in f_j(X_0), \ j = 0, 1, \cdots, m$.

If $\bar{x} = (\bar{x}_1, \cdots, \bar{x}_n)^T \in X_0$ is a local minimizer of (P) and if a certain constraint qualification holds then the following Kuhn-Tucker conditions at $\bar{x}$ hold [1]:

$$\text{(KT)} \qquad (\exists \lambda \in \mathbb{R}^m_+)\ \lambda_j f_j(\bar{x}) = 0,\ j = 1, \cdots, m \ \text{ and}$$
$$\left[\nabla f_0(\bar{x}) + \sum_{j=1}^{m} \lambda_j \nabla f_j(\bar{x})\right]^T (x - \bar{x}) \geq 0, \quad \forall x \in X_0.$$

2. Mond-Weir type duality

Now we formulate the Mond-Weir type for convexifiable programming problem as follows, and we establish duality theorems.

$$\text{(D)} \qquad \text{Maximize} \quad f_0(u)$$
$$\text{subject to} \quad \left[\nabla f_0(u) + \sum_{j=1}^{m} \lambda_j \nabla f_j(u)\right]^T (x - u) \geq 0 \ \ \forall x \in X_0, \quad (1)$$
$$\lambda_j f_j(u) \geq 0,\ j = 1, \cdots, m, \quad (2)$$
$$\lambda \geq 0. \quad (3)$$

Theorem 2.1. (Weak Duality) *Let x be a feasible solution of (P) and $(\bar{x}, \bar{\lambda})$ be a feasible solution of (D). If (P) is a convexifiable programming problem, then*

$$f_0(x) \geq f_0(\bar{x}).$$

Proof. By convexifiablity of (P), we find $\phi : \mathbb{R}^n \to \mathbb{R}^n$ and $\psi_j : \mathbb{R} \to \mathbb{R},\ j = 0, 1, \cdots, m$ such that $p_j = \psi_j \circ f_j \circ \phi,\ j = 0, 1, \cdots, m$ are convex over $\phi^{-1}(X_0)$, where ϕ is separable strictly monotone and continuously differentiable with ϕ^{-1} is differentiable and such that $X_0 \subseteq \phi(\mathbb{R}^n)$ and ψ_j is strictly increasing and continuously differentiable with $\psi'_j(x) > 0,\ \forall x \in f_j(X_0),\ j = 0, 1, \cdots, m$. Let us denote ϕ^{-1} by t and ψ_j^{-1} by $T_j,\ j = 0, 1, \cdots, m$. Then

$$f_j(x) = T_j(p_j(t(x))), \quad x \in X_0,\ j = 0, 1, \cdots, m.$$

Let x be a feasible solution of (P) and $(\bar{x}, \bar{\lambda})$ be a feasible solution of (D). By (1), for each $i = 1, \cdots, n$,

$$T'_0(p_0(t(\bar{x})))(\nabla p_0(t(\bar{x})))_i t'_i(\bar{x}_i)(x_1 - \bar{x}_i) \geq$$
$$- \sum_{j=1}^{m} \bar{\lambda}_j T_j(p_j(t(\bar{x})))(\nabla p_j(t(\bar{x})))_i t'_i(\bar{x}_i)(x_1 - \bar{x}_i).$$

By following the method in [3], we can obtain the following:

$$\begin{aligned}
&f_0(x) - f_0(\bar{x}) \\
&= T_0\Big(p_0(t(x))) - T_0(p_0(t(\bar{x}))\Big) \\
&= T_0'(\xi)\Big(p_0(t(x)) - p_0(t(\bar{x}))\Big) \quad (\xi \text{ lies between } p_0(t(x)) \text{ and } p_0(t(\bar{x}))) \\
&\geq T_0'(\xi)\nabla p_0(t(\bar{x}))^T(t(x) - t(\bar{x})) \quad (\text{by the convexity of } p_0) \\
&= \frac{T_0'(\xi)}{T_0'(p_0(t(\bar{x})))} T_0'(p_0(t(\bar{x})))\nabla p_0(t(\bar{x}))^T(t(x) - t(\bar{x})) \\
&\geq 0.
\end{aligned}$$

Therefore, $f_0(x) \geq f_0(\bar{x})$. □

Theorem 2.2. (Strong Duality) *If $\bar{x}$ is an optimal solution of (P) at which a constraint qualification is satisfied, then there exists $\bar{\lambda} \in \mathbb{R}^m_+$ such that $(\bar{x}, \bar{\lambda})$ is feasible for (D) and their objective values are equal. Furthermore, if the hypothesis of Theorem 2.1 are satisfied for all feasible solutions of (P) and (D), then $\bar{x}$ and $(\bar{x}, \bar{\lambda})$ are optimal solutions of (P) and (D), respectively.*

Proof. Since $\bar{x}$ is an optimal solution of (P), by the Kuhn-Tucker necessary conditions there exists $\bar{\lambda} \in \mathbb{R}^m_+$ such that

$$\left[\nabla f_0(\bar{x}) + \sum_{j=1}^{m} \lambda_j \nabla f_j(\bar{x})\right]^T (x - \bar{x}) \geq 0 \quad \forall x \in X_0,$$
$$\lambda_j f_j(\bar{x}) = 0, \ j = 1, \cdots, m.$$

Thus $(\bar{x}, \bar{\lambda})$ is feasible for (D) and the objective values of (P) and (D) are equal. By Theorem 2.1, $f_0(\bar{x}) \geq f_0(u)$ for any feasible solution (u, λ) of (D). Since $(\bar{x}, \bar{\lambda})$ is a feasible solution of (D), $(\bar{x}, \bar{\lambda})$ is an optimal solution of (D). Hence the result holds. □

Example 2.1. Consider the following minimization problem:

$$\begin{aligned}
\text{(P)} \qquad & \text{Minimize} \quad f_0(x) = -x_1^2 - x_2^2 - x_3^2 \\
& \text{subject to} \quad f_1(x) = e^{x_1} - e^{x_2} - e^{x_3} \leq 0, \\
& \qquad\qquad\quad x \in X_0 = [1,2] \times [1,2] \times [1,2].
\end{aligned}$$

Clearly, (P) is a convexifiable programming problem. Then the Mond-Weir type dual problem of (P) is the following:

$$\begin{aligned}
\text{(D)} \qquad & \text{Maximize} \quad -u_1^2 - u_2^2 - u_3^2 \\
& \text{subject to} \quad (-2u_1 + \lambda e^{u_1})(x_1 - u_1) - (2u_2 + \lambda e^{u_2})(x_2 - u_2) \\
& \qquad\qquad\quad -(2u_3 + \lambda e^{u_3})(x_3 - u_3) \geq 0 \quad \forall x \in X_0, \\
& \qquad\qquad\quad \lambda(e^{u_1} - e^{u_2}) \geq 0, \\
& \qquad\qquad\quad \lambda \geq 0.
\end{aligned}$$

Let $\bar{\lambda} = 0$. Then $\bar{x} = (2,2,2)$ is a feasible solution for (P) and $(\bar{x}, \bar{\lambda}) = (2,2,2,0)$ is a feasible solution for (D) and $\bar{\lambda}(e^{\bar{x}_1} - e^{\bar{x}_2}) = 0$. Since the weak duality holds between (P) and (D), $\bar{x} = (2,2,2)$ is an optimal solution of (P) and $(\bar{x}, \bar{\lambda}) = (2,2,2,0)$ is an optimal solution of (D) and their objective values are equal. □

Remark 2.1. In Example 2.1, we can easily check that f_0 and f_1 are not η-invex at $\bar{x} = (1,2,2)$ with respect to same η and f_1 is not quasiconvex at $\bar{x} = (2,1,1)$.

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