J. Appl. Math. & Informatics Vol. 26(2008), No. 5 - 6, pp. 1161 - 1168
Website: http://www.kcam.biz

CONTROLLABILITY IN DIFFERENTIAL INCLUSIONS

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ABSTRACT. We prove a theorem that there exists at least a solution reaching the prescribed target in autonomous differential inclusion. A weak invariance theorem is obtained from this theorem as its corollary. To deduce the conclusion, we assume that the target satisfies inward pointing condition. This condition will be given by proximal normal cone.

AMS Mathematics Subject Classification : 34A60, 49J24
Key words and phrases : Weak invariance, proximal normal, controllability, differential inclusion

1. Introduction

We consider the following autonomous differential inclusion:

$$\begin{aligned} &\dot{x}(t) \in F(x(t)) \ \text{ a.e. } \ t \in [S,T] \\ &x(S) \in C_S \\ &x(T) \in C_T \end{aligned} \tag{1}$$

the data for which comprise an interval

$$[S,T] \subset \mathbf{R}$$

a set-valued function (also called multifunction)

$$F : \mathbf{R}^n \rightsquigarrow \mathbf{R}^n$$

and two set

$$C_S \subset \mathbf{R}^n, \quad C_T \subset \mathbf{R}^n$$

We would like to find a solution to the above system, i.e., an absolutely continuous function satisfying the differential inclusion $\dot{x}(t) \in F(x(t))$ and steering the initial state $x(S) \in C_S$ to the target $x(T) \in C_T$. This problem is called the controllability in optimal control. We will prove the existence of solutions to this problem and a weak invariance theorem as its corollary. Invariance theorem

Received December 27, 2007. Accepted June 4, 2008. *Corresponding author.

concern solutions to a differential inclusion, that satisfy a specified constraint. Theorem giving conditions for existence of at least one solution satisfying the constraint are called weak invariance theorems or viability theorems. Those asserting that all solutions satisfy the constraint are called strong invariance theorems. The invariance theorems play an important role in viscosity solution theory of optimal control (see [1] and [6]). In this article we will focus only on controllability and weak invariance.

2. Some examples

We give some examples in which there exists at least a solution to (1).

Example 1. Fix $C_T \subset \mathbf{R}^n$. We define $\tilde{F}(x) = -F(x)$ and

$$C_S = \{\tilde{x}(T) : \tilde{x} \text{ is a solution to } (2)\}$$

where

$$\begin{array}{l} \dot{\tilde{x}}(t) \in \tilde{F}(\tilde{x}(t)) \text{ a.e. } t \in [S,T] \\ \tilde{x}(S) \in C_T \end{array} \tag{2}$$

Set $x(t) = \tilde{x}(T+S-t)$. Then since

$$\dot{x}(t) = -\dot{\tilde{x}}(T+S-t)$$

and

$$x(T) = \tilde{x}_S$$

x is a solution to (1). In other words, for all $x_S \in C_S$, there exists a solution to (1) satisfying $x(S) = x_S$.

Example 2. Fix C_S and $C_T \subset \mathbf{R}^n$. Assume that

$$\frac{C_T - C_S}{T-S} \subset F(x) \quad \forall x$$

If we set for some $x_S \in C_S$ and $x_T \in C_T$

$$x(t) = \frac{x_T - x_S}{T-S}(t-S) + x_S,$$

then

$$\begin{array}{l} \dot{x}(t) = \dfrac{x_T - x_S}{T-S}(t-S) \in F(x(t)) \\ x(S) \in C_S \\ x(T) \in C_T \end{array}$$

In other words, for all $x(S) \in C_S$ and $x(T) \in C_T$, there exists a solution to (1) satisfying $x(S) = x_S$ and $x(T) = x_T$.

Example 3. (Completely controllability) Consider the following linear controlled system:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t) \text{ a.e. } t \in [S,T] \\ &x(S) \in C_S \\ &x(T) \in C_T \end{aligned} \tag{3}$$

where $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times k}$ and $u : [S,T] \to \mathbf{R}^k$ is measurable. It is well known that the above system is equivalent to (1) if we set

$$F(x) = \{Ax + Bu : u \in \mathbf{R}^k\}$$

Assume that Kalman's rank condition

$$rank(B, AB, ..., A^{n-1}B) = n$$

holds. Then, for any $S < T$ and $(x_S, x_T) \in C_S \times C_T$, there exist a solution to (1). See [5] for detail.

Example 4. Some results for weak invariance for time-varying(nonautonomous) differential inclusion are found in [2] and [3].

3. Existence of solutions reaching target

We assume that

(I) C_S and C_T are closed in $\mathbf{R}^n$

(II) $\text{Gr}F := \{(x,y) \in \mathbf{R}^n \times \mathbf{R}^n : y \in F(x)\}$ is closed

(III) $F(x)$ is a nonempty convex set for each $x \in \mathbf{R}^n$

(IV) there exists $K > 0$ such that for all $x \in \mathbf{R}^n$

$$\sup_{v \in F(x)} \|v\| \leq K$$

(V) there exists $\delta \geq 0$ satisfying that, for every $x \in C_T$ such that $N^P_{C_T}$ is nonempty,

$$\min_{v \in F(x)} \xi \cdot v \leq -\delta, \quad \forall \xi \in N^P_{C_T}(x)$$

(this condition is called inward pointing condition)

(VI) for every $x \in C_S$

$$\sqrt{2\delta(T-S)} \geq d(x, C_T)$$

See [4] and [6] for the definition and the properties of proximal normal cone $N^P_D(x)$ for some closed set D.

Theorem 3.1. *Under the assumptions (I)-(VI), given any $x_S \in C_S$, there exists an absolutely continuous function x satisfying $x(S) = x_S$ and the following differential inclusion system:*

$$\begin{aligned} &\dot{x}(t) \in F(x(t)) \;\; a.e. \;\; t \in [S, T] \\ &x(S) \in C_S \\ &x(T) \in C_T \end{aligned}$$

Proof. Fix an integer $m > 0$. Let $\{t_0 = S, t_1^m, t_2^m, ..., t_m^m = T\}$ be an uniform partition of $[S, T]$. Set $h_m = (T - S)/m$.

We define sequences $\{x_i^m\}_{i=1}^m$, $\{v_i^m\}_{i=1}^{m-1}$ and $\{y_i^m\}_{i=1}^{m-1}$ recursively as follows. Take

$$x_0^m = x_S.$$

Since C_T is closed, we can choose, for some $i \in \{0, 1, ..., m-1\}$, $y_i^m \in C_T$ to satisfy

$$d(x_i^m, C_T) = |x_i^m - y_i^m|$$

and

$$x_i^m - y_i^m \in N_{C_T}^P(y_i^m).$$

Under the assumption (V), there exists $v_i^m \in F(y_i^m)$ such that

$$(x_i^m - y_i^m) \cdot v_i^m \leq -\delta.$$

Finally, set

$$x_{i+1}^m = x_i^m + h_m v_i^m.$$

Now define, for $i = 0, 1, ..., m-1$ and

$$z^m(t) = x_i^m + (t - t_i^m) v_i^m \quad \text{for all} \quad t \in [t_i^m, t_{i+1}^m].$$

Note that $|v_i^m| \leq K$ by assumption (IV) and

$$z^m(T) = z^m(t_m^m) = x_{m-1}^m + (T - t_{m-1}^m) v_{m-1}^m$$

Since $y_i^m \in C_T$, we have

$$\begin{aligned} d^2(x_1^m, C_T) &\leq \Big(|x_1^m - x_0^m| + |x_0^m - y_0^m|\Big)^2 \\ &= \Big(h_m |v_0^m| + |x_0^m - y_0^m|\Big)^2 \end{aligned}$$

and

$$\begin{aligned}
d^2(x_i^m, C_T) &\le (|x_i^m - y_{i-1}^m|)^2 \\
&\le |x_{i-1}^m - y_{i-1}^m|^2 + |x_i^m - x_{i-1}^m|^2 + 2(x_{i-1}^m - y_{i-1}^m)\cdot(x_i^m - x_{i-1}^m) \\
&= d^2(x_{i-1}^m, C_T) + h_m^2|v_{i-1}^m|^2 + 2h_m(x_{i-1}^m - y_{i-1}^m)\cdot v_{i-1}^m \\
&\le d^2(x_{i-1}^m, C_T) + h_m^2K^2 - 2h_m\delta \\
&\le (d^2\Big(x_{i-2}^m, C_T) + h_m^2K^2 - 2h_m\delta\Big) + h_m^2K^2 - 2h_m\delta \\
&= d^2(x_{i-2}^m, C_T) + 2h_m^2K^2 - 4h_m\delta)
\end{aligned}$$

Therefore

$$\begin{aligned}
d^2(x_i^m, C_T) &\le d^2(x_1^m, C_T) + (i-1)h_m^2K^2 - 2(i-1)h_m\delta) \\
&\le \Big(h_m|v_0^m| + |x_0^m - y_0^m|\Big)^2 + (i-1)h_m(K^2h_m - 2\delta) \\
&= \Big(h_mK + d(x_S, C_T)\Big)^2 + (i-1)h_m(K^2h_m - 2\delta)
\end{aligned}$$

and

$$\begin{aligned}
d^2(x_{m-1}^m, C_T) &\le (h_mK + d(x_S, C_T))^2 + (m-2)h_m(K^2h_m - 2\delta) \\
&\le \left(\frac{T-S}{m}K + \sqrt{2\delta(T-S)}\right)^2 \\
&\qquad + (m-2)\frac{T-S}{m}\left(K^2\frac{T-S}{m} - 2\delta\right)
\end{aligned}$$

Taking the limiting process,

$$\lim_{m\to\infty} d^2(x_{m-1}^m, C_T) = 0$$

So we can deduce from the fact that $|v_i^m| \le K$ and $t_{m-1}^m \to T$

$$\begin{aligned}
&\lim_{m\to\infty} d^2(z^m(T), C_T) \\
&= \lim_{m\to\infty} d^2\Big(x_{m-1}^m + (T - t_{m-1}^m)v_{m-1}^m, C_T\Big) \\
&= \lim_{m\to\infty} d^2(x_{m-1}^m, C_T) \\
&= 0
\end{aligned}$$

Now when $t \in [t_i^m, t_{i+1}^m]$, we have

$$|\dot{z}^m(t) = |v_1^m| \leq K$$

This inequality is valid for all $t \in [S, T]$. Since

$$z^m(t) = z^m(S) + \int_S^t \dot{z}^m(\tau) d\tau,$$

we have

$$\begin{aligned} |z^m(t)| &\leq |z^m(S)| + \int_S^t |\dot{z}^m(\tau)| d\tau \\ &\leq |x_S| + K(T - S) \end{aligned}$$

and for all t and s in $[S, T]$

$$|z^m(t) - z^m(s)| \leq \int_s^t |\dot{z}^m(\tau)| d\tau \leq K|t - s|$$

Thus the sequence $\{z^m\}_{m=1}^{\infty}$ are uniformly bounded and equicontinuous. By the theorem of Arzela-Ascoli, the sequence $\{z^m\}$ has an uniformly convergent subsequence. Furthermore since $|\dot{z}^m(t)| \leq K$, by the theorem of Dunford-Pettis we have by passing to further subsequence if necessary,

$$\dot{z}^m \rightarrow v \text{ weakly in } L^1 \text{ for some } v$$

Consider the arc

$$z(t) = x_S + \int_S^t v(\tau) d\tau$$

By the definition of weak convergence, we have

$$z^m \rightarrow z \text{ uniformly.}$$

Therefore

$$d^2(z(T), C_T) = \lim_{m \to \infty} d^2(z^m(T), C_T) = 0$$

Since C_T is closed,

$$z(T) \in C_T$$

We have proved the existence of arc reaching the target C_T.

It remains to check that z is a $F-$trajectory, i.e .,

$$\dot{z}(t) \in F(z(t)) \text{ a.e. } t \in [S, T]$$

We can see this in the proof of 2D.5 of [4]. But for the readers' convenience, we reproduce the proof. Define

$$H(x, p) = \sup\{p \cdot v : v \in F(x)\}$$

Fix $p \in L^\infty([S, T]; \mathbf{R}^n)$. It follows from Corollary 2D.2 of [4]

$$p(t) \cdot \dot{z}_m(t) \leq H(z_m(t), p(t)) \text{ a.e. } t \in [S, T]$$

Note that H is upper semicontinuous with repect to x and

$$H(z_m(t), p(t)) \leq K|p(t)|.$$

We apply Fatou's Lemma to obtain

$$\begin{aligned}\int_S^T p(t)\cdot \dot{z}(t)dt &= \limsup_{k\to\infty}\int_S^T p(t)\cdot \dot{z}_m(t)dt \\ &\leq \limsup_{k\to\infty}\int_S^T H(z_m(t),p(t))dt \\ &\leq \int_S^T \limsup_{k\to\infty} H(z_m(t),p(t))dt \\ &\leq \int_S^T H(z(t),p(t))dt. \end{aligned} \tag{4}$$

Pick any $q \in Q^n$. Let τ be any Lebesgue point for the integrable functions $\dot{z}$ and $H(z(\cdot),q)$. For any small $h > 0$, let

$$p(t) = \frac{q}{h} \text{ for } t \in [\tau, \tau + h].$$

Then (4) gives

$$\frac{1}{h}\int_\tau^{\tau+h}(q\cdot \dot{z}(t) - H(z(t),q))dt \leq 0.$$

In the limit as $h \to 0^+$ we get

$$q\cdot \dot{z}(\tau) - H(z(\tau),q) \leq 0.$$

This inequality holds for all τ outside a Lebesgue null set which may depend on q . But since the set $\mathbf{Q}^n$ is countable, the union of all these null set remains negligible. Thus

$$q\cdot \dot{z}(t) \leq H(z(t),q) \ \ \forall q \in \mathbf{Q}^n, \ \text{ a.e. } \ t \in [S,T]$$

Since both sides of this inequality are continuous as functions of q,

$$p\cdot \dot{z}(t) \leq H(z(t),p) \ \ \forall p \in \mathbf{R}^n, \ \text{ a.e. } \ t \in [S,T]$$

This implies

$$\dot{z}(t) \in F(z(t)) \ \text{ a.e. } \ t \in [S,T]$$

□

When $C_S = C_T = C$, we obtain a weak invariance theorem as a corollary.

Corollary 3.2. Assume that

$$\min_{v\in F(x)} \xi\cdot v \leq 0 \quad \forall \xi \in N_C^P(x).$$

Then for any $x_S \in C$ there exists a solution to:

$$\begin{aligned}&\dot{x}(t) \in F(x(t)) \ \text{ a.e. } \ t \in [S,T],\\ &x(S) = x_S,\\ &x(t) \in C \ \ \forall t \in [S,T].\end{aligned}$$

Proof. First, in Theorem ??, if we set $C_S = C_T = C$ and $\delta = 0$, then there exists a solution z to

$$\begin{aligned}
&\dot{x}(t) \in F(x(t)) \text{ a.e. } t \in [S,T],\\
&x(S) = x_S \in C,\\
&x(T) \in C.
\end{aligned}$$

Next, we must have

$$z(t) \in C \text{ for all } t \in (S,T)$$

For this, in the proof of the theorem, we set $\delta = 0$ and $C_S = C_T = C$. Then

$$d^2(z(t), C) = \lim_{m\to\infty} d^2(z_m(t), C) = \lim_{m\to\infty} d^2(x_{m-1}^m, C) = 0$$

□

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