J. Appl. Math. & Informatics Vol. **26***(2008), No. 5 - 6, pp. 1101 - 1121*
Website: http://www.kcam.biz

FOURIER'S TRANSFORM OF FRACTIONAL ORDER VIA MITTAG-LEFFLER FUNCTION AND MODIFIED RIEMANN-LIOUVILLE DERIVATIVE

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ABSTRACT. One proposes an approach to fractional Fourier's transform, or Fourier's transform of fractional order, which applies to functions which are fractional differentiable but are not necessarily differentiable, in such a manner that they cannot be analyzed by using the so-called Caputo-Djrbashian fractional derivative. Firstly, as a preliminary, one defines fractional sine and cosine functions, therefore one obtains Fourier's series of fractional order. Then one defines the fractional Fourier's transform. The main properties of this fractal transformation are exhibited, the Parseval equation is obtained as well as the fractional Fourier inversion theorem. The prospect of application for this new tool is the spectral density analysis of signals, in signal processing, and the nanalysis of some partial differential equations of fractional order.

AMS Mathematics Subject Classification: 44A10, 35A22, 33B99
Keywords and phrases: Fractional derivative, fractional Taylor's series, Mittag-Leffler function, fractional Fourier's series, fractional Fourier's transform, fractional partial differential equation.

1. Introduction

The Fourier's transform $\widehat{f}(\omega)$, $\omega \in \mathbb{R}$, of a $\mathbb{R} \to \mathbb{C}$ function $f(x)$ of the variable $x \in \mathbb{R}$ is given by the integral (the symbol $:=$ means that the left-side is defined by the right-side)

$$\widehat{f}(\omega) = \int_{\mathbb{R}} e^{i\omega x} f(x)dx \tag{1.1}$$

when it convergeswhen. It is very useful in applied mathematics, for instance for solving some differential equations and partial differential equations; but above all in physics and engineering where it has a practical meaning in terms of frequency distributions of signals. For instance $\widehat{f}(\omega)$ is directly related with

Received August 9, 2007. Revised February 21, 2008. Accepted February 25, 2008.

the spectral densities of stochastic processes, therefore its usefulness in signal processing.

The function $f(x)$ so involved is usually continuous and continuously differentiable (almost anywhere), and the question is what happens when it is continuous but with fractional derivative of order α, $0 < \alpha < 1$, only? Two instances can occur. In the first one, $f(x)$ has both a continuous derivative and a fractional derivative, in which case the expression (1.1) is quite meaningful. In the second case, $f(x)$ has a derivative of order α,$0<\alpha < 1$, but no derivative, and then (1.1) fails to apply; and as result we have to find out an alternative.

The purpose of the present contribution is exactly to provide a possible approach to this alternative, which is based on fractional calculus, and more especially on the modified Riemann-Liouvillle fractional derivative as we suggested it recently together with the fractional Taylor's series which results as a consequence. The key is to replace the exponential function by the Mittag-Leffler function on the one hand, and the integral with respect to dx by a fractional integral with respect to $(dx)^{\alpha}$, on the other hand.

The article is organized as follows. For the convenience of the reader, we shall give a brief background on the definition of fractional derivative as we use it (Section 2) and on fractional Taylor's series (Section 3). In Section 4, we shall give some results related to integral with respect to . In Section 5, we shall define fractional sine and cosine functions as a direct consequence of the Mittag-Leffler function applied to complex variables, and then, in the section 6, we shall arrive, in quite a natural way, to fractional Fourier's series. Lastly, in the sections 7 and 8, we shall successively define the fractional Fourier's transform and its inversion formula.

2. Background on Fractional Derivative (Revisited)

2.1. Fractional derivative via fractional difference.

Definition 2.1. Let $f : \mathbb{R} \to \mathbb{R}, x \to f(x)$, denote a continuous (but not necessarily differentiable) function, and let h denote a constant discretization span. Define the forward operator $FW(h)$ by the equality (the symbol := means that the left side is defined by the right side)

$$FW(h)f(x) := f(x+h) \ , \tag{2.1}$$

then the fractional difference of order α, $0 < \alpha < 1$, of $f(x)$ is defined by the expression [11, 14-19]

$$\Delta^{\alpha} f(x) = (fW-1)^{\alpha} f(x) \ ,$$

$$\Delta^{\alpha} f(x) = \sum_{k=0}^{\infty} (-1)^{k} \binom{c}{k} f\left(x + (\alpha - k)\, h\right), \tag{2.2}$$

and its fractional derivative of order α is defined by the limit

$$f^{(\alpha)}(x) = \lim_{h \to 0} \frac{\Delta^{\alpha} f(x)}{h^{\alpha}}. \tag{2.3}$$

In the following we shall propose an alternative which relates this definition with the modeling via the Riemann-Liouville integral.

2.2. Modified fractional Riemann-Liouville derivative (via integral).

An alternative to the Riemann-Liouville definition of fractional derivative

In order to circumvent some drawbacks involved in the classical Riemann-Liouville definition, we have proposed the following alternative referred to as modified Riemann-Liouville derivative [16].

Definition 2.2 (Riemann-Liouville definition revisited). Refer to the function $f(x)$ of Proposition 2.1.

(i) Assume that $f(x)$ is a constant K . Then its fractional derivative of order α is

$$D_x^\alpha K = \frac{k}{\Gamma(1-\alpha)} x^{-\alpha}, \quad \alpha \leq 0, \tag{2.4}$$

$$= 0, \quad \alpha > 0\,. \tag{2.5}$$

(ii) When $f(x)$ is not a constant, then one will set

$$f(x) = f(0) + (f(x) - f(0))\,,$$

and its fractional derivative will be defined by the expression

$$f^{(\alpha)}(x) := D_x^\alpha f(0) + D_x^\alpha (f(x) - f(0))\,,$$

in which, for negative α, one has

$$D_x^{(\alpha)} (f(x) - f(0)) := \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\xi)^{-\alpha-1} f(\xi) d\xi, \ \alpha < 0. \tag{2.6}$$

whilst for positive, one will set

$$D_x^\alpha (f(x) - f(0)) = D_x^\alpha f(x),$$

$$D_x \left(f^{(\alpha-1)}(x)\right), 0 < \alpha < 1,$$

$$:= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha} (f(\xi) - f(0))\, d\xi. \tag{2.7}$$

When $n \leq \alpha < n+1$, one will set

$$f^{(\alpha)}(x) := \left(f^{(\alpha-n)}(x)\right)^{(n)}, 0 < \alpha < 1. \tag{2.8}$$

We shall refer to this fractional derivative as to the modified Riemann Liouville derivative, and it is of order to point out that it is strictly equivalent to the definition 2.1, via the equation (2.2).

With this definition, the Laplace transform $L\{\}$of the fractional derivative is

$$L\{f^\alpha(x)\} = s^\alpha L\{f(x)\} - s^{\alpha-1} f(0), 0 < \alpha < 1. \tag{2.9}$$

2.3. Sequences of fractional derivatives. *On the order of cascaded derivatives.* Assume that we want to calculate $D^{\alpha+\theta}f(x)$, $0 < \alpha, \theta < 1$, by applying $D^{\alpha}$and D^{θ}in any order. At first glance, one could use either $D^{\alpha}D^{\theta}f(x)$ or $D^{\theta}D^{\alpha}f(x)$, but the results so obtained are sensibly different, since then in terms of Laplace's transform (see Equ. (2.9)) one has

$$L\left\{D^{\theta}D^{\alpha}f(x)\right\} = s^{\alpha+\theta}F(s) - s^{\alpha+\theta-1}f(0) - s^{\theta-1}f^{(\alpha)}(0), \tag{2.10}$$

and

$$L\left\{D^{\alpha}D^{\theta}f(x)\right\} = s^{\alpha+\theta}F(s) - s^{\alpha+\theta-1}f(0) - s^{\alpha-1}f^{(\theta)}(0). \tag{2.11}$$

The same problem occurs when θ, for instance, is a positive integer n, and here again one has $D^{\theta}D^{\alpha}f(x) \neq D^{\alpha}D^{\theta}f(x)$. For instance, when $f(x) = x^2$, $n = 3$ and $\alpha = 0.5$; one obtains

$$D^{0.5}D^{3}(x^2) = 0,$$

and

$$D^{3}D^{0.5}(x^2) = KD^{3}(x^{1.5}) = -1.5(0.5)^2Kx^{-1.5},$$

with K denoting a constant.

Once more, we are facing the same problem when we try to define $D^{\alpha}f(x)$ with $n < \alpha < n+1$, in which case we have to set either $D^{\alpha} := D^{n}D^{\alpha-n}$ or $D^{\alpha} := D^{\alpha-n}D^{n}$.

As a result, we have to choose a model, and we suggest the following.

Definition 2.3 (Principle of increasing order for derivatives). The fractional derivative of fractional order $D^{\alpha+\theta}$ expressed in terms of $D^{\alpha}$and D^{θ} is defined by the equality

$$D^{\alpha+\theta}f(x) = D^{\max(\alpha,\theta)}\left(D^{\min(\alpha,\theta)}f(x)\right). \tag{2.12}$$

On doing so, we merely follows the practical rule in accordance of which we increase the derivation order rather than the opposite. Or again, we start from low order derivative to define large order derivative.

On the decomposition of fractional derivatives

Let α be positive, and assume that $0 < 3\alpha < 1$. There are two different manners to obtain $D^{3\alpha}f(x)$. Either one can calculate $D^{\alpha}D^{\alpha}D^{\alpha}f(x)$ to obtain the Laplace's transform

$$L\{D^{\alpha}D^{\alpha}D^{\alpha}f(x)\} = s^{3\alpha}F(s) - s^{3\alpha-1}f(0) - s^{2\alpha-1}f^{(\alpha)}(0) - s^{\alpha-1}f^{(2\alpha)}(0),$$

or else directly calculate $D^{3\alpha}f(x)$ to obtain

$$L\left\{D^{3\alpha}f(x)\right\} = s^{3\alpha}F(s) - s^{3\alpha-1}f(0),$$

in such a manner that one will have

$$D^{\alpha}D^{\alpha}D^{\alpha}f(x) \neq D^{3\alpha}f(x), 0 < 3\alpha < 1.$$

For instance $f(x) = x^{2\alpha}$ yields

$$D^{\alpha}D^{\alpha}D^{\alpha}(x^{3\alpha}) = 0,$$

and

$$D^{3\alpha}(x^{2\alpha}) = \frac{\Gamma(1+\alpha)}{\Gamma(1-2\alpha)}x^{-2\alpha}.$$

This pitfall can be easily circumvented if we carefully define the framework. When the problem which we are dealing with involves D^α as the basic derivative, then we shall necessarily refer to $D^\alpha D^\alpha D^\alpha$. Otherwise, if the smaller derivative so involved in the problem is $D^{3\alpha}$, then we shall use the modified Riemann-Liouville expression for the later.

For further reading on fractional calculus, regarding history and complements, see for instance [3, 4, 20, 23, 25, 26, 29-31, 34-36].

3. Background on Taylor's Series of Fractional Order

3.1. Main definition. A generalized Taylor expansion of fractional order which applies to non-differentiable functions (F-Taylor series in the following) reads as follows [11, 14-19].

Proposition 3.1. *Assume that the continuous function has fractional derivative of order $k\alpha$, for any positive integer k and a given α, $0 < \alpha < 1$, then the following equality holds, which is*

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{(\alpha k)}}{\Gamma(1+\alpha k)} f^{(\alpha k)}(x), \; 0 < \alpha < 1 \tag{3.1}$$

where $f^{(\alpha k)}$ is the derivative of order αk of $f(x)$).

With the notation

$$\Gamma(1+\alpha k) =: (\alpha k)!,$$

one has the formula

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{(\alpha k)}}{(\alpha k)!} f^{(\alpha k)}(x), 0 < \alpha < 1, \tag{3.2}$$

which looks like the classical one.

Alternatively, in a more compact form, one can write

$$f(x+h) = E_\alpha(h^\alpha D^\alpha) f(x),$$

where D is the derivative operator with respect to x and $E_\alpha(y)$ denotes the Mittag-Leffler function defined as

$$E_\alpha(y) := \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(1+\alpha k)}.$$

Corollary 3.1 *Assume that $m < \alpha < m+1$, $m \in N - \{0\}$ and that $f(x)$ has derivatives of order k (integer), $1 < k < m$. Assume further that $f^m(x)$ has a fractional Taylor's series of order $\alpha - m := \beta$ provided by the expression*

$$f^{(m)}(x+h) = \sum_{k=0}^{\infty} \frac{h^{k(\alpha-m)}}{\Gamma(1+k(\alpha-m))} D^{k(\alpha-m)} f^{(m)}(x), \; m < \alpha < m+1. \tag{3.3}$$

Then, integrating this series with respect to h yields

$$f(x+h) = \sum_{k=0}^{m} \frac{h^k}{k!} f^{(k)}(x) + \sum_{k=1}^{\infty} \frac{h^{(k\beta+m)}}{\Gamma(k\beta+m+1)} f^{(k\beta+m)}(x), \quad \beta := \alpha - m. \tag{3.4}$$

In the special case when $m = 1$, one has

$$f(x+h) = f(x) + hf'(x) + \sum_{k=1}^{\infty} \frac{h^{k\beta+1}}{\Gamma(k\beta+2)} f^{(k\beta+1)}(x), \quad \beta := \alpha - 1. \tag{3.5}$$

The order of the derivation in $f^{(k\beta+m)}$ is of paramount importance and should be understood as $D^{k\beta} f^{(m)}(x)$, since we start with the fractional Taylor's series of $f^{(m)}(x)$.

Mc-Laurin series of fractional order

Let us make the substitution $x \to h$ and $0 \to x$ into (3.1), we so obtain the fractional Mc-Laurin series

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+\alpha k)} f^{(\alpha k)}(0), \quad 0 < \alpha < 1. \tag{3.6}$$

3.2. Fractional Taylor's series for multivariable functions. Let us point out that the fractional Taylor's series so obtained can be generalized in a straightforward way to multivariable functions to yield, for instance, for two variables,

$$f(x+h, y+l) = E_\alpha\left(h^\alpha D_x^\alpha\right) E_\alpha\left(l^\alpha D_y^\alpha\right) f(x,y), \quad 0 < \alpha < 1, \tag{3.7}$$

therefore the differential

$$df(x,y) = \Gamma^{-1}(1+\alpha)\left(f_x^{(\alpha)}(x,y)(dx)^\alpha + f_y^{(\alpha)}(x,y)(dy)^\alpha\right), \quad 0 < \alpha < 1. \tag{3.8}$$

For larger values of , one will have, for instance

$$df(x,y) = f_x dx + f_y dy + \Gamma^{-1}(1+\alpha)\left(f_x^{(\alpha)}(dx)^\alpha + f_y^{(\alpha)}(dy)^\alpha\right)$$
$$+\Gamma^{-2}(1+\alpha) f_{xy}^{(2\alpha)}, \quad 1 < \alpha < 2. \tag{3.9}$$

3.3. Some useful relations. The equation (3.1) provides the useful relation

$$d^\alpha f \simeq \Gamma(1+\alpha) df, \quad 0 < \alpha < 1. \tag{3.10}$$

or in a difference form, $\Delta^\alpha f \simeq \Gamma(1+\alpha)\Delta f$.

Corollary 3.2 *The following equalities hold, which are*

$$D^\alpha x^\gamma = \Gamma(\gamma+1)\Gamma^{-1}(\gamma+1-\alpha) x^{\gamma-\alpha}, \quad \gamma > 0, \tag{3.11}$$

or, what amounts to the same (we set $\alpha = n + \theta$)

$$D^{n+\theta} x^\gamma = \Gamma(\gamma+1)\Gamma^{-1}(\gamma+1-n-\theta) x^{\gamma-n-\theta}, \quad 0 < \theta < 1,$$

$$\left(u(x)v(x)\right)^{(\alpha)} = u^{(\alpha)}(x)v(x) + u(x)v^{(\alpha)}(x), \tag{3.12}$$

$$\left(f\left[u(x)\right]\right)^{(\alpha)} = \frac{df(u)}{du} u^{(\alpha)}(x), \tag{3.13}$$

$$= f_u^{(\alpha)}(u)\left(\frac{du}{dx}\right)^{\alpha}. \tag{3.14}$$

$u(x)$ is non-differentiable in (3.12) and (3.13) and differentiable in (3.14), $v(x)$ is non-differentiable, and $f(u)$ is differentiable in (3.13) and non-differentiable in (3.14).

Corollary 3.3. *Assume that $f(x)$ and $x(t)$ are two $\mathbb{R} \to \mathbb{R}$ functions which both have derivatives of order α, $0 < \alpha < 1$, then one has the chain rule*

$$f_x^{(\alpha)}(x(t)) = \Gamma(2-\alpha)x^{\alpha-1}f_x^{(\alpha)}(x)x^{(\alpha)}(t). \tag{3.15}$$

Proof. The α-th derivative of x provides the equality

$$d^{\alpha}x = \frac{1}{(1-\alpha)!}x^{1-\alpha}(dx)^{\alpha}, \tag{3.16}$$

which allows us to write successively

$$d^{\alpha}f = f_x^{(\alpha)}(dx)^{\alpha},$$

$$d^{\alpha}f = f_x^{(\alpha)}(x)(1-\alpha)!x^{\alpha-1}d^{\alpha}x,$$

whereby the result. □

3.4. Further results and remarks. On the suitable fractional derivative definition to be selected

(i) With the modified Riemann-Liouville derivative, the solution of the equation (and this is why we introduced this modified Riemann-Liouville fractional derivative!)

$$D_t^{\alpha}x(t) = -\lambda x(t), \quad x(0) = x_0, \tag{3.17}$$

is exactly the Mittag-Leffler function, and this can be obtained easily on looking for a solution in the form

$$x(t) = \sum_{k=0}^{\infty} x_k (t^{\alpha})^k. \tag{3.18}$$

An alternative is to take the Laplace transform of the equation (3.17) to obtain, with our modified Riemann-Liouville derivative, the equation

$$s^{\alpha}X(s) - s^{\alpha-1}x(0) = -\lambda X(s),$$

where $X(s)$, which is the Laplace transform of $x(t)$, yields the Mittag-Leffler function.

On the differentiability of $f(x)$

(ii) As it is obvious, the series (3.1) applies to nondifferentiable functions, whilst (3.5) refers to differentiable functions.

(iii) Assume that $\alpha = 1/N$, N integer, in the F-Taylor series (3.1); then when $k = N$, we come across the first derivative. Nevertheless this does not mean that there is some inconsistency somewhere, but rather it is the meaning of these equations which must be clarified. Indeed, because of the presence of h^{α}, h is restricted to be positive,$h > 0$; and as a result, *all the derivatives involved* *in the F-Taylor series (3.1), either they are fractional or not, are derivatives on the right..*

Modeling the irreversibility of time

Assume that $f(.)$ is a function of time $f(t)$; then according to the above comments, the fractional Taylor series of $f(t+\Delta t)$ holds for positive Δt only. This property can be thought of as a practical describing of the irreversibility of time.

Relation with previous results in the literature

(iv) Osler [33] has previously proposed a generalization of Taylor's series in the complex plane, in the form

$$f(z) = \alpha \sum_{k=-\infty}^{k=+\infty} \frac{f^{(\alpha k)}(z_0)}{\Gamma(1+\alpha k)} (z-z_0)^{\alpha k}, \tag{3.19}$$

which provides the fractional Mc Laurin's series

$$f(z) = \alpha \sum_{k=-\infty}^{k=+\infty} \frac{f^{(\alpha k)}(0)}{\Gamma(1+\alpha k)} z^{\alpha k}. \tag{3.20}$$

In order to enlighten the discrepancy between this series and our's, we proceed as follows. On taking the expression (3.20) in terms of modified Riemann-Liouville derivative on the one hand, and identifying $f(x)$ with $E_\alpha(x^\alpha)$ on the other hand, we would obtain the equality

$$E_\alpha(x^\alpha) = \alpha E_\alpha(x^\alpha) + \alpha \sum_{k=1}^{\infty} \frac{x^{-\alpha k}}{\Gamma(1-\alpha k)},$$

therefore

$$E_\alpha(x^\alpha) = \frac{\alpha}{1-\alpha} \sum_{k=1}^{\infty} \frac{x^{-\alpha k}}{\Gamma(1-\alpha k)},$$

in such a manner we would have $E_\alpha(0) = \infty$.

(vi) More recently Kolwankar and Gangal [21,22] proved the so-called "local fractional Taylor expansion" (or Rolle's formula)

$$f(x+h) = \sum_{k=0}^{m} \frac{h^k}{k!} f^{(k)}(x) + \frac{f^{(\alpha)}(x)}{\Gamma(1+\alpha)} h^\alpha + R_\alpha(h), \quad m < \alpha < m+1, \tag{3.21}$$

where $R_\alpha(h)$ is a reminder, which is negligible when compared with the other terms. This is exactly our series (3.4), but here we give an explicit expression for $R_\alpha(h)$, namely

$$R_\alpha(h) = \sum_{k=2}^{\infty} \frac{h^{(k\beta+m)}}{\Gamma(k\beta+m+1)} f^{(k\beta+m)}(x), \quad \beta := \alpha - m. \tag{3.22}$$

Nevertheless, it is relevant to point out that this author does not use the Riemann-Liouville expression of fractional derivative as we did it, but rather define the later as the limit of a quotient involving the increment of the function on the one hand, and a so-called coarse grained mass or α-mass of a subset which is generally fractal. Loosely speaking the function is fractal because it is defined on a set which itself is fractal.

4. Integration with respect to $(dx)^\alpha$

The integral with respect to $(dx)^\alpha$ is defined as the solution of the fractional differential equation

$$dy = f(x)(dx)^\alpha, \quad x \geqslant 0, \quad y(0) = 0, \tag{4.1}$$

which is provided by the following result:

Lemma 4.1. *Let $f(x)$ denote a continuous function, then the solution $y(x)$, $y(0) = 0$, of the equation (4.1) is defined by the equality*

$$y(x) = \int_0^x f(\xi)(d\xi)^\alpha = \alpha \int_0^x (x-\xi)^{\alpha-1} f(\xi) d\xi, \quad 0 < \alpha < 1. \tag{4.2}$$

Proof. On multiplying both sides of (4.1) by $\alpha!$, and on taking account of (3.10), we have the equality

$$y^{(\alpha)}(x) = \alpha! f(x),$$

which provides

$$y(x) = \alpha! D^{-\alpha} f(x) = \frac{\alpha!}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi) d\xi. \tag{4.3}$$

□

As a result, the integration of (4.1) can be written also in the form

$$D^{-\alpha} f(x) = \frac{1}{\alpha!} \int_0^x f(\xi)(d\xi)^\alpha, \tag{4.4}$$

or in a like manner

$$f(x) = \frac{1}{\alpha!} \frac{d^\alpha}{dx^\alpha} \int_0^x f(\xi)(d\xi)^\alpha. \tag{4.5}$$

As a direct consequence of (4.2), one has the equality

$$\begin{aligned} \int_0^x f(\xi)(d\xi)^{\alpha+\beta} &= \frac{\alpha+\beta}{\alpha} \int_0^x (x-\xi)^\beta f(\xi)(d\xi)^\alpha, \quad 0 < \alpha+\beta < 1 \\ &= \frac{\alpha+\beta}{\beta} \int_0^x (x-\xi)^\alpha f(\xi)(d\xi)^\beta, \quad 0 < \alpha+\beta < 1. \end{aligned} \tag{4.6}$$

The *fractional integration by part formula*

$$\int_a^b u^{(\alpha)}(x) v(x) (dx)^\alpha = \alpha! \left[u(x)v(x)\right]_a^b - \int_a^b u(x) v^{(\alpha)}(x)(dx)^\alpha. \tag{4.7}$$

can be obtained easily by combining (4.1) with (4.3).

Change of variable. Consider the variable transformation $y = g(x)$. If $g(x)$ is differentiable, then one has

$$\int f(y)(dy)^\alpha = \int f(g(x))\,(g'(x))^\alpha\,(dx)^\alpha,\ 0<\alpha<1,$$

and when $g(x)$ has a fractional derivative of order β, $0<\alpha,\beta<1$, one has

$$\int f(y)(dy)^\alpha = \Gamma^{-1}(1+\beta)\int f(g(x))\left(g^{(\beta)}(x)\right)^\alpha (dx)^{\alpha\beta},\ \ 0<\alpha,\beta<1.$$

Some examples.

(i) On making $f(x) = x^\gamma$ in (4.1) one obtains

$$\int_0^x \xi^\gamma (d\xi)^\alpha = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma},\ \ 0<\alpha<1, \tag{4.8}$$

and, more especially one has

$$\int_0^x (d\xi)^\alpha = x^\alpha,\ \ 0<\alpha<1. \tag{4.9}$$

(ii) Assume now that $f(x)$ is the Dirac delta generalized function $\delta(x)$, then one has

$$\int_0^x \delta(\xi)(d\xi)^\alpha = \alpha x^{\alpha-1},\ \ 0<\alpha<1. \tag{4.10}$$

Application to the fractional derivative of the Dirac delta function

On using the equation (3.12) on the one hand, and extending well known definition on the other hand, we shall define the fractional derivative of the Dirac delta function by the equality

$$\int \delta^{(\alpha)}(\xi)f(\xi)(d\xi)^\alpha = -\int \delta(\xi)f^{(\alpha)}(\xi)(d\xi)^\alpha,\ \ 0<\alpha<1, \tag{4.11}$$

and the equation (3.12), direct will yields

$$\int \delta^{(\alpha)}(\xi)f(\xi)(d\xi)^\alpha = -\ \alpha x^{\alpha-1}f^{(\alpha)}(0),\ \ 0<\alpha<1. \tag{4.12}$$

5. Sine and Cosine functions of fractional order

5.1. Complex-valued Mittag-Leffler function.

Lemma 5.1. *The following equality holds, which is*

$$E_\alpha(\lambda x^\alpha)\,E_\alpha(\lambda y^\alpha) = E_\alpha(\lambda(x+y)^\alpha),\ \ \lambda\in\mathbb{C}. \tag{5.1}$$

Proof. Let us look for a function $f(x)$ which complies with the condition

$$f(\lambda x^\alpha)\,f(\lambda y^\alpha) = f(\lambda(x+y)^\alpha),\ \ \lambda\in\mathbb{C}$$

On α-th differentiating w.r.t. x and y respectively, and on taking account of (3.13) and (3.14), one obtains the equality

$$f^{(\alpha)}(x^\alpha)\,f(y^\alpha) = f(x^\alpha)\,f^{(\alpha)}(y^\alpha),\ \ \lambda\in\mathbb{C},$$

therefore the equation

$$f^{(\alpha)}(x^\alpha) = \lambda f(x^\alpha),$$

of which the solution is $E_\alpha(x)$subject to the condition that the derivative is taken in the sense of the modified Riemann-Liouville derivative. □

As a result of (5.1), one has

$$E_\alpha(ix^\alpha) E_\alpha(iy^\alpha) = E_\alpha(i(x+y)^\alpha), \quad \lambda \in \mathbb{C}. \tag{5.2}$$

Therefore we conclude that the function is periodic with the period M_α defined as the solution of the equation

$$E_\alpha(i(M_\alpha)^\alpha) = 1. \tag{5.3}$$

5.2. Fractalization of sin x and cos x.

Definition 5.2. Analogously with the trigonometric function, we can define the sine and cosine functions of fractional order by the relation

$$E_\alpha(ix^\alpha) = \cos_\alpha x^\alpha + i \sin_\alpha x^\alpha, \tag{5.4}$$

with

$$\cos_\alpha x^\alpha := \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k\alpha}}{(2k\alpha!)}, \tag{5.5}$$

and

$$\sin_\alpha x^\alpha := \sum_{k=1}^{\infty} (-1)^k \frac{x^{(2k+1)\alpha}}{(2k\alpha+\alpha)!}. \tag{5.6}$$

These functions have the period M_α defined by (5.3)and are such that

$$\sin_\alpha(-x)^\alpha = (-1)^\alpha \sin_\alpha x^\alpha,$$

$$\cos_\alpha(-x)^\alpha = \cos_\alpha x^\alpha,$$
$$D^\alpha \cos_\alpha x^\alpha = -\sin_\alpha x^\alpha,$$
$$D^\alpha \sin_\alpha x^\alpha = \cos_\alpha x^\alpha.$$

In addition (5.1) provides the equalities

$$\cos_\alpha(x+y)^\alpha = \cos_\alpha x^\alpha \cos_\alpha y^\alpha - \sin_\alpha x^\alpha \sin_\alpha y^\alpha, \tag{5.7}$$

$$\sin_\alpha(x+y)^\alpha = \sin_\alpha x^\alpha \cos_\alpha y^\alpha + \cos_\alpha x^\alpha \sin_\alpha y^\alpha, \tag{5.8}$$

which yield

$$\cos_\alpha x^\alpha \cos_\alpha y^\alpha = (1/2)\left(\cos_\alpha(x+y)^\alpha + \cos_\alpha(x-y)^\alpha\right), \tag{5.9}$$

$$\sin_\alpha x^\alpha \sin_\alpha y^\alpha = (1/2)\left(\cos_\alpha(x-y)^\alpha - \cos_\alpha(x+y)^\alpha\right), \tag{5.10}$$

$$\sin_\alpha x^\alpha \cos_\alpha y^\alpha = (1/2)\left(\sin_\alpha(x+y)^\alpha + \sin_\alpha(x-y)^\alpha\right). \tag{5.11}$$

The following orthogonality conditions hold

$$\int_0^{M_\alpha} \cos_\alpha(mx)^\alpha \cos_\alpha(nx)^\alpha (dx)^\alpha = \frac{(M_\alpha)^\alpha}{2} \delta_{mn}, \tag{5.12}$$

$$\int_0^{M_\alpha} \sin_\alpha(mx)^\alpha \sin_\alpha(nx)^\alpha (dx)^\alpha = \frac{(M_\alpha)^\alpha}{2}\delta_{mn}, \tag{5.13}$$

$$\int_0^{M_\alpha} \sin_\alpha(mx)^\alpha \cos_\alpha(nx)^\alpha (dx)^\alpha = 0. \tag{5.14}$$

These orthogonality conditions are direct consequences of (5.7) and (5.8). For instance they provide the equality

$$\cos_\alpha(mx)^\alpha \cos_\alpha(nx)^\alpha = (1/2)\left(\cos_\alpha(m+n)^\alpha x^\alpha + \cos_\alpha(m-n)^\alpha x^\alpha\right),$$

but since M_α is a multiple period for both $\cos_\alpha(m+n)^\alpha x^\alpha$ and $\cos_\alpha(m-n)^\alpha x^\alpha$, the integral of $\cos_\alpha(mx)^\alpha \cos_\alpha(nx)^\alpha$ on the interval $[0, M_\alpha]$ will be different from zero when and only when $m = n$, in which case one has

$$\int_0^{M_\alpha} \sin^2_\alpha(nx)^\alpha (dx)^\alpha = \int_0^{M_\alpha} \cos^2_\alpha(nx)^\alpha (dx)^\alpha = \frac{1}{2}\int_0^{M_\alpha}(dx)^\alpha = \frac{(M_\alpha)^\alpha}{2}.$$

Conclusion. The functions 1, $\cos_\alpha x^\alpha$, $\sin_\alpha x^\alpha$, $\cos_\alpha(2x)^\alpha$, $\sin_\alpha(2x)^\alpha$,... are orthogonal on the interval $0 \le x \le M_\alpha$ with the unit weight function.

5.3. Fractalization of sin ωx and cos ωx. All this material can be duplicated step by step to the functions $\sin_\alpha(\omega x)^\alpha$ and $\cos_\alpha(\omega x)^\alpha$, $x \in \mathbb{R}$, $\omega \in \widehat{\mathbb{R}}$, (following standard mathematical practice, we identify the "space domain" $\mathbb{R}$ as distinct from the "frequency domain" $\widehat{\mathbb{R}}$ (even though each symbol represents real numbers) for which the period is $X_\alpha = M_\alpha/\omega$, and mainly we now have the orthogonality conditions

$$\int_0^{X_\alpha} \cos_\alpha(m\omega x)^\alpha \cos_\alpha(n\omega x)^\alpha (dx)^\alpha = \frac{(X_\alpha)^\alpha}{2}\delta_{mn}, \tag{5.15}$$

$$\int_0^{X_\alpha} \sin_\alpha(m\omega x)^\alpha \sin_\alpha(n\omega x)^\alpha (dx)^\alpha = \frac{(X_\alpha)^\alpha}{2}\delta_{mn}, \tag{5.16}$$

$$\int_0^{X_\alpha} \sin_\alpha(m\omega x)^\alpha \cos_\alpha(n\omega x)^\alpha (dx)^\alpha = 0, \tag{5.17}$$

which provide

$$\int_0^{X_\alpha} \sin^2_\alpha(n\omega x)^\alpha (dx)^\alpha = \int_0^{X_\alpha} \cos^2_\alpha(n\omega x)^\alpha (dx)^\alpha = \frac{(X)^\alpha}{2}. \tag{5.18}$$

Conclusion. The functions 1, $\cos_\alpha(\omega x)^\alpha$, $\sin_\alpha(\omega x)^\alpha$, $\cos_\alpha(2\omega x)^\alpha$, $\sin_\alpha(2\omega x)^\alpha$,... are orthogonal on the interval $0 \le x \le M_\alpha/\omega$.

6. Fourier's Series of Fractional Order

6.1. Expansion in terms of fractional sine and cosine. Assume that $f(x)$ is a periodic function with the period $X_\alpha = M_\alpha/\omega$, which is continuous but nowhere differentiable. All we can claim is that it has a derivative of fractional order α, $0 < \alpha < 1$. In such a case the standard expansion in Fourier's series does not work here, since $f(x)$ is not differentiable, and we shall rather look for a series in the form

$$f(x) \ = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos_\alpha(n\omega x)^\alpha + b_n \sin_\alpha(n\omega\ x)^\alpha. \tag{6.1}$$

On using by-now classical arguments, we find easily that and are provided by the expressions

$$a_n \ = \ 2\left(\frac{\omega}{M_\alpha}\right)^\alpha \int_0^{M_\alpha/\omega} f(x)\cos_\alpha(n\omega x)^\alpha (dx)^\alpha, \tag{6.2}$$

and

$$b_n \ = \ 2\left(\frac{\omega}{M_\alpha}\right)^\alpha \int_0^{M_\alpha/\omega} f(x)\sin_\alpha(n\omega x)^\alpha (dx)^\alpha. \tag{6.3}$$

The Parseval-Pythagore theorem now becomes

$$\int_0^{M_\alpha/\omega} f^2(x)(dx)^\alpha = \frac{1}{2}\left(\frac{M_\alpha}{\omega}\right)^\alpha \sum_{n=0}^{\infty}\left(a_n^2 + b_n^2\right),$$

and it holds when and only when

$$\lim_{N\to\infty} \int_0^{M_\alpha/\omega} \left(f - \sum_{n=1}^{N}(a_n \cos_\alpha(n\omega x)^\alpha + b_n \sin_\alpha(n\omega\ x)^\alpha)\right)^2 (dx)^\alpha \ = \ 0\ .$$

6.2. Expansion via complex-valued Mittag-Leffler function. *Main definition.*

Given the definition of the fractional sine and cosine functions (equations (5.4) and (5.5)), the series (6.1) can be re-written in the equivalent form

$$f(x) \ = \ c_0 + \sum_{n=1}^{\infty}\left[c_n E_\alpha\left(i(n\omega x)^\alpha\right) + c_{-n} E_\alpha\left(i(-n\omega x)^\alpha\right]\right. . \tag{6.4}$$

Multiplying both sides by $E_\alpha\left(i(-n\omega x)^\alpha\right.$, and integrating over $[0, X_\alpha]$, we obtain the equality

$$\int_0^{X_\alpha} f(x) E_\alpha\left(i(-n\omega x)^\alpha\right)(dx)^\alpha \ = \ c_n \int_0^{X_\alpha} (dx)^\alpha.$$

Therefore we have (via (4.9))

$$c_n \ = \ \left(\frac{\omega}{M_\alpha}\right)^\alpha \int_0^{M_\alpha/\omega} f(x) E_\alpha\left[i(-n\omega x)^\alpha\right](dx)^\alpha. \tag{6.5}$$

Parseval formula

Assume that another function $\widetilde{f}\,(x)$ can be expanded in the form

$$\widetilde{f}(x) \;=\; \widetilde{c}_0 + \sum_{n=1}^{\infty} \left[\widetilde{c}_n E_\alpha\left(i(n\omega x)^\alpha\right) + \widetilde{c}_{-n} E_\alpha\left(i(-n\omega x)^\alpha\right]\right., \tag{6.6}$$

then one has the Parseval equality

$$\left(\frac{\omega}{M_\alpha}\right)^\alpha \int_0^{M_\alpha/\omega} f(x)\widetilde{f}(x)(dx)^\alpha \;=\; c_0\widetilde{c}_0 + \sum_{n=1}^{\infty} \left(c_n\widetilde{c}_{-n} + c_{-n}\widetilde{c}_n\right),$$

which is obtained by multiplying the series (6.4) and (6.7) together, and then α-integrating the result on the interval $[0, X_\alpha]$.

7. Function Transformation of Fractional Order

7.1. Fourier's transform of fractional order.

Definition 7.1. The Fourier's transform $F_\alpha\{f(x)\}$ of order α (or its $\alpha - th$ fractional Fourier's transform) of the function f, $\mathbb{R} \to \mathbb{C}$, $x \to f(x)$, is defined by the integral

$$\widehat{f}_\alpha^F(\omega) \;:=\; \int_{-\infty}^{+\infty} E_\alpha\left(i\omega^\alpha x^\alpha\right) f(x)(dx)^\alpha, \;\; 0 < \alpha < 1, \tag{7.1}$$

when the latter converges.

A sufficient condition for convergence is

$$\int_{-\infty}^{+\infty} |f(x)|\,(dx)^\alpha \;<\; M \;<\; \infty.$$

Assume that $f(x)$ is continuous and piecewise $\alpha - th$ continuously differentiable, that the integral $\int_{\mathbb{R}} E_\alpha\left\{i\omega^\alpha x^\alpha\right\} f(x)(dx)^\alpha$ converges for each ω, and that

$$\lim_{x \to \pm\infty} f(x) \;=\; 0.$$

Then, on integrating by parts we have

$$F_\alpha\left\{f^{(\alpha)}(x)\right\} \;=\; -\,i\omega^\alpha F_\alpha\left\{f(x)\right\}.$$

In the same way, the following formulae are easy to be obtained, clearly

$$F_\alpha\left\{ix^\alpha f(x)\right\} \;=\; D_\omega^\alpha F_\alpha\left\{f(x)\right\}, \tag{7.2}$$

$$F_\alpha\left\{f(ax-b)\right\} \;=\; \frac{1}{|a|^\alpha} E_\alpha\left(i\frac{\omega^\alpha b^\alpha}{a^\alpha}\right)\widehat{f}\left(\frac{\omega}{a}\right). \tag{7.3}$$

7.2. Laplace's transform of fractional order.

Definition 7.2. Let $f(x)$ denote a function which vanishes for negative values of the variable x. It's Laplace's transform $L_\alpha\{f(x)\}$ of order α(or its $\alpha - th$ fractional Laplace's transform) is defined by the expression, when it is finite,

$$\widehat{f}_\alpha^L(s) \ := \ \int_0^{+\infty} E_\alpha\left(-s^\alpha x^\alpha\right) f(x)(dx)^\alpha, \ \ 0<\alpha<1, \tag{7.4}$$

where $s \in \mathbb{C}$.

And of course, a sufficient condition for this integral to be finite is that

$$\int_0^{+\infty} |f(x)|\,(dx)^\alpha \ < \ M \ < \ \infty.$$

Using the properties of the Mittag-Leffler function and the integration by parts, we find that

$$L_\alpha\left\{f^{(\alpha)}(x)\right\} \ = \ s^\alpha L_\alpha\{f(x)\} - \Gamma(1+\alpha)f(0). \tag{7.5}$$

The following operational formulae can be easily obtained

$$\begin{aligned}
L_\alpha\{x^\alpha f(x)\} \ &= \ -D_s^\alpha L_\alpha\{f(x)\}, \\
L_\alpha\{f(ax)\}_s \ &= \ (1/a)^\alpha L_\alpha\{f(x)\}_{s/a} \\
L_\alpha\{f(x-b)\} \ &= \ E_\alpha\left(-s^\alpha b^\alpha\right) L_\alpha\{f(x)\}, \\
L_\alpha\{E_\alpha\left(-c^\alpha x^\alpha\right) f(x)\}_s \ &= \ L_\alpha\{f(x)\}_{s+c}.
\end{aligned}$$

8. Inversion formula for fractional Fourier's transform

8.1. An approach via Parseval Theorem. With this definition together with the properties outlined in the preceding subsection 5.1, we can duplicate the Fourier's transform theory, and mainly one has the Parseval equality

$$\int_{\mathbb{R}} \widehat{f}_\alpha^F(\omega)\overline{\widehat{g}_\alpha^F(\omega)}(d\omega)^\alpha \ = \ (M_\alpha)^\alpha \int_{\mathbb{R}} f(x)\overline{g(x)}(dx)^\alpha, \tag{8.1}$$

from where we can obtain the inversion formula

$$f(x) \ = \ \frac{1}{(M_\alpha)^\alpha}\int_{-\infty}^{+\infty} E_\alpha\left(-i\omega^\alpha x^\alpha\right) \widehat{f}_\alpha^F(\omega)(d\omega)^\alpha, \ \ 0<\alpha<1, \tag{8.2}$$

as follows.

Shortly, let us define the function $g(x)$ as

$$g(x) \ = \ \begin{cases} 0, \ x<0 \\ 1, \ \ 0 \le x \le x_0 \\ 0, \ \ x > x_0 \end{cases},$$

then, the Parseval equality allows us to write

$$(M_\alpha)^\alpha \int_0^{x_0} f(x)\bar{1}(dx)^\alpha \ = \int_{-\infty}^{+\infty} f_\alpha^F(\omega)\frac{E_\alpha\left(-i\omega^\alpha x_0^\alpha\right)-1}{-i\omega^\alpha}(d\omega)^\alpha,$$

and on equating the derivative d^α/dx_0^α of both sides yields the result.

8.2. Background on classical inverse Fourier's transform. The (usual) Fourier's transform of a function $f(x)$ is defined by the expression

$$\widehat{f}(\omega) := \int_{-\infty}^{+\infty} e^{i\omega x} f(x)dx, \tag{8.3}$$

and the corresponding inverse transform is

$$f(x) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega x}\widehat{f}(\omega)d\omega. \tag{8.4}$$

One way to check this correspondence is to substitute (8.3) into (8.4) to have

$$f(x) := \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(y)dy \int_{-\infty}^{+\infty} e^{-i\omega(x-y)}d\omega. \tag{8.5}$$

But one has the equality

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega x}d\omega = \delta(x), \tag{8.6}$$

in such a manner that (8.5) turns to be

$$f(x) = \int_{\mathbb{R}} f(y)\delta(x-y)dy.$$

In the following, we shall show how this derivation can be extended to fractional Fourier's transform, to provide another approach to the equation (8.2).

8.3. Dirac's delta distribution of fractional order.

Definition 8.1. The Dirac's distribution, or generalized function, of fractional order $\delta_\alpha(x)$ is defined by the equality

$$\int_{\mathbb{R}} f(x)\delta_\alpha(x)(dx)^\alpha = \alpha f(0).$$

Lemma 8.1. *Define the function*

$$\delta_\alpha(x,\epsilon) = \begin{cases} 0, & x \notin [0,\varepsilon], \\ \varepsilon^{-\alpha}, & 0 < x \leq \varepsilon \end{cases} \tag{8.8}$$

then one has the limit

$$\lim_{\varepsilon\to 0} \delta_\alpha(x,\varepsilon) = \delta_\alpha(x). \tag{8.9}$$

Proof. One has

$$\begin{aligned} \int_{\mathbb{R}} f(x)\delta_\alpha(x,\varepsilon)(dx)^\alpha &= \alpha\int_0^\varepsilon .(\varepsilon-\xi)^{\alpha-1} f(\xi)\delta_\alpha(\xi,\varepsilon)d\xi \\ &= \alpha\int_0^\varepsilon .(\varepsilon-\xi)^{\alpha-1} f(0)\delta_\alpha(\xi,\varepsilon)d\xi \\ &= \alpha\varepsilon(\varepsilon^{\alpha-1})f(0)\delta_\alpha(\xi) \end{aligned}$$

therefore $\delta_a(x) = \varepsilon^{-\alpha}$. □

Remark. If instead of (8.7), we had selected the condition

$$\int_{\mathbb{R}} f(x)\delta_\alpha(x)(dx)^\alpha \;=\; \delta_\alpha(x),$$

then we would find that

$$\delta_\alpha(x,\varepsilon) \;=\; \alpha^{-1}\varepsilon^{-\alpha}. \tag{8.10}$$

8.4. Fractional Dirac's delta and Mittag-Leffler function. The relation between $\delta_\alpha(x)$ and $E_\alpha(x^\alpha)$ is enlightened by the following result which will be of basic help later when we shall deal with Fourier's transform of fractional order.

Lemma 8.2. *The following equality holds, which is*

$$\frac{\alpha}{(M_\alpha)^\alpha}\int_{-\infty}^{+\infty} E_\alpha\left[i(-\omega x)^\alpha\right](d\omega)^\alpha \;=\; \delta_\alpha(x). \tag{8.11}$$

where M_α is the period of the complex-valued Mittag-Leffler function defined by the equality $E_\alpha\left(i\left(M_\alpha\right)^\alpha\right)=1$. .

Proof. We check that (8.11) is consistent with

$$\alpha \;=\; \int_{\mathbb{R}} E_\alpha\left(i\omega^\alpha x^\alpha\right)\delta_\alpha(x)(dx)^\alpha,$$

and to this end, we replace $\delta_\alpha(x)$ in this expression by (8.11) to obtain

$$\begin{aligned}
\alpha &= \int_{\mathbb{R}}(dx)^\alpha\int_{\mathbb{R}}\frac{\alpha}{(M_\alpha)^\alpha}E_\alpha\left(ix^\alpha\left(\omega-u\right)^\alpha\right)(du)^\alpha\\
&= \int_{\mathbb{R}}(dx)^\alpha\int_{\mathbb{R}}\frac{\alpha}{(M_\alpha)^\alpha}E_\alpha\left(i\left(-vx\right)^\alpha\right)(dv)^\alpha\\
&= \int_{\mathbb{R}}\delta_\alpha(x)(dx)^\alpha\\
&= \alpha.
\end{aligned}$$

Remark that one has as well

$$\frac{\alpha}{(M_\alpha)^\alpha}\int_{-\infty}^{+\infty} E_\alpha\left[i(\omega x)^\alpha\right](d\omega)^\alpha \;=\; \delta_\alpha(x).$$

□

8.5. Fourier's transform inversion theorem.

Proposition 8.2. *Given the Fourier's transform (7.1) that we recall here for convenience.*

$$\widehat{f}^F_\alpha(\omega) \;:=\; \int_{-\infty}^{+\infty} E_\alpha\left(i\omega^\alpha x^\alpha\right)f(x)(dx)^\alpha,\quad 0<\alpha<1, \tag{8.13}$$

one has the inversion formula

$$f(x) \;:=\; \frac{1}{(M_\alpha)^\alpha}\int_{-\infty}^{+\infty} E_\alpha\left(i\left(-\omega x\right)^\alpha\right)\widehat{f}^F_\alpha(\omega)\;(d\omega)^\alpha,\quad 0<\alpha<1. \tag{8.14}$$

Proof. Substituting (8.14) into (8.13), we have successively

$$f(x) := \frac{1}{(M_\alpha)^\alpha} \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha \left(i\left(-\omega x\right)^\alpha\right) E_\alpha \left(i\left(\omega y\right)^\alpha\right) f(y)(dy)^\alpha \ (d\omega)^\alpha,$$

$$f(x) := \frac{1}{(M_\alpha)^\alpha} \int_{\mathbb{R}} \int_{\mathbb{R}} E_\alpha \left(i\omega^\alpha \left(y-x\right)^\alpha\right) f(y)(dy)^\alpha (d\omega)^\alpha.$$

We now refer to (8.7) to obtain the result. □

9. Application to fractional partial differential equation

Background on previous results

As an illustrative example of the kind of results one can so obtain in applying fractional Fourier's transform to fractional partial differential equation, let us consider the equation

$$D_t^\alpha u(x,t) = cD_x^\beta u(x,t), \tag{9.1}$$

$0 < \alpha, \beta < 1$, with the initial condition

$$u(x,t) = f(x). \tag{9.2}$$

This equation is a very simple case of diffusion equation [2], and in the frame-owrk of the modified Riemann-Liouville derivative, its general solution has been obtained in the form [18]

$$u(x,t) = \varphi\left((\beta!)^{-1}x^\beta + c(\alpha!)^{-1}t^\alpha\right). \tag{9.3}$$

On using the initial condition

$$\varphi\left(x^\beta/\beta!\right) = f(x), \tag{9.4}$$

one obtains

$$u(x,t) = f\left((\beta!)^{1/\beta}\left(\frac{x^\beta}{\beta!} + c\frac{t^\alpha}{\alpha!}\right)^{1/\beta}\right). \tag{9.5}$$

Application of fractional Fourier's transform

We refer to the fractional Fourier's transform

$$\hat{u}_\beta^F(\omega,t) := \int_{-\infty}^{+\infty} E_\alpha\left(i\omega^\beta x^\beta\right) u(x,t)(dx)^\beta, \quad 0 < \beta < 1, \tag{9.6}$$

Taking the Fourier's transform of the equation (9.1) yields the equation

$$D_t^\alpha \hat{u}_\beta(\omega,t) = c(-i\omega^\beta)\hat{u}_\beta(\omega,t), \tag{9.7}$$

of which the solution is

$$\hat{u}_\beta(\omega,t) = \hat{f}_\beta(\omega)E_\alpha\left(-ic\omega^\beta t^\alpha\right), \tag{9.8}$$

therefore

$$u(x,t) := \frac{1}{(M_\beta)^\beta} \int_{-\infty}^{+\infty} E_\beta\left(-i\omega^\beta x^\beta\right) E_\alpha\left(-ic\omega^\beta t^\alpha\right) \hat{f}_\beta^F(\omega) \ (d\omega)^\alpha. \tag{9.9}$$

10. Concluding Remarks

We think it is once more worth to point out that, in our point of view, the very reason of fractional calculus is to deal with functions which are not differentiable, in such a manner that, any formulation involving derivatives should be disqualified. In this way of thought, we believe that the so-called Caputo-Djrbashian derivative [5,7] (which defines fractional derivative of order lower than the unit by the derivative) is not quite always suitable (or relevant), and this is the reason why we introduced the modified Riemann-Liouville derivative.

This being the case, this new tool of fractional Fourier's transform is proposed for use in applied mathematics, and in this way of thought, it provides a direct definition for the fractional spectral density, which could be of help, for instance, to analyze fractal signals [1] with new points of view. A question which comes in mind is whether one can use Fractional Fourier's transform to check if a given signal is of fractal nature. On a general standpoint, one could once more consider various questions to see whether one can contribute further results.

The range of application of fractional calculus is increasing due to the fact that, if a particle moves in a space in which the point element has a "thickness", coarse-graining space, then its local trajectory is defined by the equation instead of . So we do expect that fractional Fourier's transform will be of help to get more insight in many physical problems [2,6,8,9,10,12,13,24,27, 28,32,37]

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