J. Appl. Math. & Informatics Vol. 26(2008), No. 5 - 6, pp. 1057 - 1069
Website: http://www.kcam.biz

AN ASYMPTOTIC FINITE ELEMENT METHOD FOR SINGULARLY PERTURBED HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS OF CONVECTION-DIFFUSION TYPE WITH DISCONTINUOUS SOURCE TERM

A. RAMESH BABU AND *N. RAMANUJAM

ABSTRACT. We consider singularly perturbed Boundary Value Problems (BVPs) for third and fourth order Ordinary Differential Equations(ODEs) of convection-diffusion type with discontinuous source term and a small positive parameter multiplying the highest derivative. Because of the type of Boundary Conditions(BCs) imposed on these equations these problems can be transformed into weakly coupled systems. In this system, the first equation does not have the small parameter but the second contains it. In this paper a computational method named as " An asymptotic finite element method " for solving these systems is presented. In this method we first find an zero order asymptotic approximation to the solution and then the system is decoupled by replacing the first component of the solution by this approximation in the second equation. Then the second equation is independently solved by a fitted mesh Finite Element Method (FEM). Numerical experiments support our theoritical results.

AMS Mathematics Subject Classification : 6510, CR G1.7
Key words and phrases :Singularly perturbed problem, Discontinuous source term, Third order differential equation, Fourth order differential equation, Asymptotic expansion approximation, Streamline-diffusion, Finite element method, Self-adjoint, Boundary value problem, Fitted mesh.

1. Introduction

Singularly Perturbed Differential Equations(SPDEs) appear in several branches of applied mathematics. Analytical and numerical treatment of these equations have drawn much attention of many researchers [?, ?, ?, ?, ?]. In general, classical numerical methods fail to produce good approximations for these equations. Hence one has to look for non-classical methods. A good number of articles have been appearing in the past three decades on non-classical methods which

Received July 30, 2007. Revised November 12, 2007. Accepted November 17, 2007.
*Corresponding author. The first author wishes to acknowledge Bharathidasan University, Tamilnadu, India for its financial support.

cover mostly second order equations. But only a few authors have developed numerical methods for higher-order differential equations [?], [?]-[?].

Singularly perturbed higher-order problems are classified on the basis that how the order of the original differential equation is affected if one sets $\varepsilon = 0$. Here ε is a small positive parameter multiplying the highest derivative of the differential equation. We say that a Singular Perturbation Problem (SPP) is of convection- diffusion type if the order of the DE is reduced by 1, whereas it is called reaction-diffusion type if the order is reduced by 2. In this paper the first type is considered.

For the analytical treatment of SPBVPs for the higher-order non-linear Ordinary Differential Equations(ODEs) which have important applications in fluid dynamics, one may refer [?, ?]. For the higher-order problem if the order of the equation is even, then FEM based on standared C^{m-1} splines on a shishkin mesh is reported in [?].

In this paper, we consider the following two problems: *Fourth Order Singularly Perturbed Boundary Value Problem* [?].

Find $y \in C^2(\bar{\Omega}) \cap C^3(\Omega) \cap C^4(\Omega^- \cup \Omega^+)$ such that

$$-\varepsilon y^{iv}(x) + a(x)y'''(x) + b(x)y''(x) - c(x)y(x) = -f(x), x \in (\Omega^- \cup \Omega^+) \quad (1.1)$$

$$y(0) = p, \quad y(1) = q, \quad y''(0) = -r, \quad y''(1) = -s, \quad (1.2)$$

where $a(x), b(x)$ and $c(x)$ are sufficiently smooth functions satisfying the following conditions

$$a(x) \geq \alpha, \quad \alpha > 0, \quad (1.3)$$

$$b(x) \geq 0, \quad (1.4)$$

$$0 \geq c(x) \geq -\gamma, \quad \gamma > 0, \quad (1.5)$$

$$\alpha - \theta\gamma \geq \eta > 0, \theta > 1 \text{ is arbitraryly close to 1, for some } \eta. \quad (1.6)$$

Third order Singularly Perturbed Boundary Value Problem [?].

Find $y \in C^1(\bar{\Omega}) \cap C^2(\Omega) \cap C^3(\Omega^- \cup \Omega^+)$ such that

$$-\varepsilon y'''(x) + a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x), x \in (\Omega^- \cup \Omega^+), \quad (1.7)$$

$$y(0) = p, \quad y'(0) = q, \quad y'(1) = r, \quad (1.8)$$

where $a(x), b(x)$ and $c(x)$ are sufficiently smooth functions on $\bar{\Omega}$ satisfying the following conditions

$$a(x) \geq \alpha, \quad \alpha > 0, \quad (1.9)$$

$$b(x) \geq 0, \quad (1.10)$$

$$0 \geq c(x) \geq -\gamma, \quad \gamma > 0, \quad (1.11)$$

$$\alpha - \theta\gamma \geq \eta > 0, \quad \text{for some } \theta \text{ arbitrarily close to 2, for some } \eta. \quad (1.12)$$

For both the problems defined above, $\Omega^- = (0, d), \Omega^+ = (d, 1), \Omega = (0, 1)$ and ε is a small positive parameter. It is assumed that f is sufficiently smooth on $\Omega \setminus \{d\}$. Further it is assumed that $f(x)$ and its derivatives have right and left limits at the point d. It is convenient to introduce the notation for jump at d for any function w as $[w](d) = w(d+) - w(d-)$.

Motivated by the papers [?], [?], a computational method is suggested for the above problems. Because of the type of the boundary conditions imposed, one can transform the problem in to a weakly coupled system of differential equations. Then one obtains a zero order asymptotic expansion approximation for solution of the problem. Then the first component of the solution appearing in the second equation is replaced by its zero order asymptotic expansion approximation. Then the system gets decoupled. Then the second equation can be solved independently. Infact, the second equation was solved by earlier authors [?]-[?] by using FMM (Fitted Mesh Method) on Shishkin mesh and the order of convergence obtained by them is of $O(\varepsilon + N^{-1}\ln N)$. In the present paper, we apply FEM on Shishkin and Bakhavlov-Shishkin meshes and obtain higher-order convergence for small values of parameter ε.

Through out this paper, C denotes a generic constant that is independent of the parameter ε and N, the dimension of the discrete problem.

2. Asymptotic expansion approximation

As mentioned above zero-order asymptotic expansions for the solutions of the problems (??-??) and (??-??) are obtained. The SPBVP (??-??) can be transformed into an equivalent problem of the form

$$\begin{cases} -y_1''(x) - y_2(x) = 0, \quad x \in \Omega, \\ -\varepsilon y_2''(x) + a(x)y_2'(x) + b(x)y_2(x) + c(x)y_1(x) = f(x), x \in \Omega^- \cup \Omega^+, \end{cases} \tag{2.1}$$

$$y_1(0) = p, \quad y_1(1) = q, \quad y_2(0) = r, \quad y_2(1) = s, \tag{2.2}$$

where $\bar{y} = (y_1, y_2)^T$, $y_1 \in C^2(\bar{\Omega}) \cap C^3(\Omega) \cap C^4(\Omega^- \cup \Omega^+)$ and $y_2 \in C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$.

Similarly the SPBVP (??-??) can be transformed into

$$\begin{cases} y_1'(x) - y_2(x) = 0, \quad x \in \Omega \cup \{1\}, \\ -\varepsilon y_2''(x) + a(x)y_2'(x) + b(x)y_2(x) + c(x)y_1(x) = f(x), x \in \Omega^- \cup \Omega^+, \end{cases} \tag{2.3}$$

$$y_1(0) = p, \quad y_2(0) = q, \quad y_2(1) = r, \tag{2.4}$$

where $\bar{y} = (y_1, y_2)^T$, $y_1 \in C^1(\bar{\Omega}) \cap C^2(\Omega) \cap C^3(\Omega^- \cup \Omega^+)$ and $y_2 \in C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$.

Remark 2.1. Here after, the above systems are only considered instead of BVPs (??-??) and (??-??). The conditions (??) and (??) ensure that the above systems are quasi-monotone (refer [?] and [?]). The conditions (??) and (??) are sufficient to establish maximum principle for problems (??-??) and (??-??) respectively. This, in turn, can be used to derive stability result, error estimates etc., Motivated by [?] we can construct an asymptotic expansion approximation for solution of (??-??).

Find $u_{01}(x), u_{02}(x)$ such that

$$\begin{cases} -u_{01}''(x) - u_{02}(x) = 0, \\ a(x)u_{02}'(x) + b(x)u_{02}(x) + c(x)u_{01}(x) = f(x), \quad x \in \Omega^- \cup \Omega^+, \\ u_{01}(0) = p, \quad u_{01}(d-) = u_{01}(d+), \quad u_{02}(0) = r, \\ u_{01}'(d-) = u_{01}'(d+), \quad u_{02}(d-) = u_{02}(d+), \quad u_{01}(1) = q. \end{cases} \tag{2.5}$$

That is, in particular find u_{01} on $\bar{\Omega}$ such that

$$\begin{cases} a(x)u_{01}'''(x) + b(x)u_{01}''(x) - c(x)u_{01}(x) = -f(x), \quad x \in \Omega^- \cup \Omega^+, \\ u_{01}(0) = p, \quad u_{01}(d-) = u_{01}(d+), \quad u_{01}''(0) = r, \\ u_{01}'(d-) = u_{01}'(d+), \quad u_{01}''(d-) = u_{01}''(d+), \quad u_{01}(1) = q. \end{cases} \tag{2.6}$$

Then $(y_{1,as}, y_{2,as})$ is defined by

$$y_{1,as}(x) = u_{01}(x) + \begin{cases} v_{L01}(x), x \in \Omega^- \cup \{0\}, \\ v_{R01}(x), x \in \Omega^+ \cup \{1\}, \\ v_{L01}(d-) = v_{R01}(d+), \end{cases}$$

and

$$y_{2,as}(x) = u_{02}(x) + \begin{cases} v_{L02}(x), x \in \Omega^- \cup \{0\}, \\ v_{R02}(x), x \in \Omega^+ \cup \{1\}, \\ v_{L02}(d-) = v_{R02}(d+), \end{cases}$$

where the terms $v_{L01}, v_{R01}, v_{L02}$ and v_{R02} can be defined following ideas presented in [?]. Similarly one can construct an asymtotic expansion for the solution of the BVP (?? - ??).

Find $u_{01}(x), u_{02}(x)$ such that

$$\begin{cases} u_{01}'(x) - u_{02}(x) = 0, \quad x \in (\Omega \cup \{1\}), \\ a(x)u_{02}'(x) + b(x)u_{02}(x) + c(x)u_{01}(x) = f(x), x \in \Omega^- \cup (\Omega^+ \cup \{1\}), \\ u_{01}(0) = p, u_{01}(d-) = u_{01}(d+), u_{02}(d-) = u_{02}(d+), u_{02}(0) = q. \end{cases} \tag{2.7}$$

That is, in particular find u_{01} on $\bar{\Omega}$ such that

$$\begin{cases} a(x)u_{01}''(x) + b(x)u_{01}'(x) + c(x)u_{01}(x) = f(x), x \in \Omega^- \cup (\Omega^+ \cup \{1\}), \\ \quad u_{01}(0) = p, u_{01}'(0) = q, u_{01}'(d-) = u_{01}'(d+), u_{01}(d-) = u_{01}(d+). \end{cases} \tag{2.8}$$

That is,

$$\begin{cases} a(x)u_{02}'(x) + b(x)u_{02}(x) = f(x) - c(x)u_{01}(x), x \in \Omega^- \cup (\Omega^+ \cup \{0\}), \\ u_{02}(0) = q, \ u_{02}(d-) = u_{02}(d+). \end{cases} \tag{2.9}$$

Define $(y_{1,as}, y_{2,as})$ as

$$y_{1,as}(x) = u_{01}(x) + \begin{cases} v_{L01}(x), & x \in \Omega^- \cup \{0\}, \\ v_{R01}(x), & x \in \Omega^+ \cup \{1\}, \\ v_{L01}(d-) = v_{R01}(d+), \end{cases}$$

and

$$y_{2,as}(x) = u_{02}(x) + \begin{cases} v_{L02}(x), & x \in \Omega^- \cup \{0\}, \\ v_{R02}(x), & x \in \Omega^+ \cup \{1\}, \\ v_{L02}(d-) = v_{R02}(d+), \end{cases}$$

where u_{01}, u_{02} are defined above and the boundary layer terms $v_{L01}, v_{R01}, v_{L02}$ and v_{R02} can be defined. Using the maximum principle and hence the stability result, the authors [?, ?] proved the following result.

Remark 2.2. One can see that there is a strong layer at $x = 1$ and a weak layer at $x = d$ for the solution component y_2

Theorem 2.3. *The zero order asymptotic expansion* $(y_{1,as}, y_{2,as})$ *defined above for the solution* (y_1, y_2) *of the BVP (?? - ??) satisfies the inequality*

$$| y_i(x) - y_{i,as}(x) | \leq C\varepsilon, x \in \bar{\Omega}, \quad i = 1, 2.$$

In particular, we have

$$| y_1(x) - u_{01}(x) | \leq C\varepsilon.$$

Remark 2.4. Similar statements are true for the BVP (?? - ??) [?]. As we do not need the above boundary layer terms, we have not given their explicit forms here.

3. Some analytical and numerical results for SPBVP for second order convection-diffusion equation with a discontinuous source term

We consider the BVP

$$\begin{cases} -\varepsilon y_2^{*''}(x) + a(x) y_2^{*'}(x) + b(x) y_2^*(x) = f^*(x), & x \in (\Omega^- \cup \Omega^+), \\ y_2^*(0) = q, \quad y_2^*(1) = r, \end{cases} \tag{3.1}$$

where $f^*(x) = f(x) - c(x)u_{01}(x)$ and u_{01} is the solution of the initial value problem (??).

3.1. Analytical Result.

Theorem 3.1. *If* (y_1, y_2) *and* y_2^* *are solutions of the BVPs (??-??) and (??) respectively and* u_{01} *is the solution of the reduced problem (??), then* [?]

$$|y_2(x) - y_2^*(x)| \leq C\varepsilon, \quad x \in \bar{\Omega}.$$

Remark 3.2. A similar statement can be made for the BVP (??-??).

3.2. Numerical Results. There is a good literature in the area numerical solution for SPPs for second order ODEs with non-smooth data [?]. Consider the convection - diffusion problem (??)

$$\begin{cases} -\varepsilon y_2^{*''}(x) + a(x)y_2^{*'}(x) + b(x)y_2^*(x) = f^*(x), x \in (\Omega^- \cup \Omega^+), \\ y_2^*(0) = y_2^*(1) = 0, \end{cases} \tag{3.2}$$

where $f^*(x) = f(x) - c(x)u_{01}(x)$. For the sake of completeness we now describe the Streamline-Diffusion Finite Element Method (SDFEM), as presented in [?]. A standard weak formulation of (??) is: Find $y_2^* \in V$ such that

$$B(y_2^*, v) := (\varepsilon y_2^{*'}, v') + (ay_2^{*'}, v) + (by_2^*, v) = (f^*, v), \text{ for all } v \in V \tag{3.3}$$

where $V = H_0^1(\Omega)$ denotes the usual Sobolev space and $(.,.)$ is the inner product on $L_2(\Omega)$. As in [?], we shall consider $\Omega_\varepsilon^N = \{x_0, x_1, \cdots, x_N\}$ to be the set of mesh points x_i, for some positive integer N. For $i \in \{1, 2, \cdots, N\}$ we set $h_i = x_i - x_{i-1}$ to be the local mesh step size, and for $i \in \{1, 2, \cdots, N\}$ let $\bar{h}_i = (h_i + h_{i+1})/2$. For the discretization of (??) we use streamline-diffusion method, we get

$$\begin{aligned} B_h(y_2^*, v) &:= (\varepsilon y_2^{*'}, v') + (ay_2^{*'}, v) + \\ & \sum_{i=1}^{N-1} \bar{h}_i b_i y_{2,i}^* v_i + \sum_{i=1}^{N-1} \int_{x_{i-1}}^{x_i} \delta_i(-\varepsilon y_2^{*''} + ay_2^{*'} + by_2^*)av', \\ f_h^*(v) &:= (f^*, v) + \textstyle\sum_{i=1}^{N-1} \int_{x_{i-1}}^{x_i} \delta_i f^* av' \end{aligned}$$

and we choose $d = x_{N/2}$ and take, $f^*(d) = f_{N/2}^* = (f_{(N/2)-1}^* + f_{(N/2)+1}^*)/2$. The parameter δ_i is called the streamline-diffusion parameter and will be determined later. Now the discrete problem is: find $y_{2h}^* \in V^h$ such that

$$B_h(y_{2h}^*, v_h) = f_h^*(v_h) \quad \forall v_h \in V^h \tag{3.4}$$

and basis functions of V^h are given by

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_i}, & x \in [x_{i-1}, x_i], \\ \frac{x_{i+1} - x}{h_{i+1}}, & x \in [x_i, x_{i+1}], \\ 0, & x \notin [x_{i-1}, x_{i+1}]. \end{cases} \tag{3.5}$$

The solution y_{2h}^* of (??) will represent an approximation to the exact solution y_2^* of the problem (??). The corresponding difference scheme is

$$L^N y_{2,i}^* := \begin{cases} -\varepsilon\left(\frac{y_{2,i+1}^* - y_{2,i}^*}{h_{i+1}} - \frac{y_{2,i}^* - y_{2,i-1}^*}{h_i}\right) + \alpha_i\left(\frac{y_{2,i+1}^* - y_{2,i}^*}{h_{i+1}}\right) + \\ \beta_i\left(\frac{y_{2,i}^* - y_{2,i-1}^*}{h_i}\right) + \gamma_i y_{2,i}^* = f_h^*(\phi_i), \end{cases} \tag{3.6}$$

$$y_{2,0}^* = y_{2,N}^* = 0, \tag{3.7}$$

where $y^*_{2,i} = y^*_{2h}(x_i), \quad i = 1, 2, ..., N-1$ and

$$\alpha_i = h_{i+1} \int_{x_i}^{xi+1} (a\phi'_{i+1}\phi_i + \delta_{i+1}a^2\phi'_{i+1}\phi'_i + \delta_{i+1}ab\phi_{i+1}\phi'_i),$$
$$\beta_i = -h_i \int_{x_{i-1}}^{xi} (a\phi'_{i-1}\phi_i + \delta_i a^2\phi'_{i-1}\phi'_i + \delta_i ab\phi_{i-1}\phi'_i),$$
$$\gamma_i = \bar{h}_i\widehat{b}_i + \int_{x_{i-1}}^{x_i} \delta_i ab\phi'_i + \int_{x_i}^{xi+1} \delta_{i+1}ab\phi'_i.$$

The choice of the parameter δ_i is determined by the structure of the coefficient matrix of the scheme (??) and $\widehat{b}_i = \frac{\bar{a}^2}{\alpha^2}\|b\|_{L_\infty[x_i,x_{i+1}]}$, $\bar{a} = \|a\|_{L_\infty}$. Naturally, if the local mesh step is small enough, then the standard Galerkin method can be applied, so it is possible to choose $\delta_i = 0$. In other case, we shall choose δ_i from the condition $\alpha_{i-1} = 0$. Finally we have

$$\delta_i = \begin{cases} 0, & h_i \le \dfrac{2\varepsilon}{\| a \|_\infty}, \\ -\displaystyle\int_{x_{i-1}}^{x_i} a\phi'_i\phi_{i-1}\Big(\int_{x_{i-1}}^{x_i} (a^2\phi'_i\phi'_{i-1} + ab\phi'_i\phi'_{i-1})\Big)^{-1}, & h_i > \dfrac{2\varepsilon}{\| a \|_\infty}. \end{cases} \tag{3.8}$$

For the discretization described above we shall use a mesh of the general type introduced in [?], but here adapted for the layers at $x = d$ and $x = 1$.

Let $N > 4$ be a positive even integer and

$$\sigma_1 = \min\left\{\frac{d}{2}, \frac{\varepsilon}{\beta}\tau_0 \ln N\right\}, \quad \sigma_2 = \min\left\{\frac{1-d}{2}, \frac{\varepsilon}{\beta}\tau_0 \ln N\right\}, \quad \tau_0 \ge 3.$$

Remark 3.3. To acheive the higher order convergence the proof of the theorem demands that $\tau_0 \ge 3$.

Our mesh will be equidistant on $\bar{\Omega}_s$, where

$$\Omega_s = (0, d - \sigma_1) \cup (d, 1 - \sigma_2)$$

and graded on $\bar{\Omega}_0$ where

$$\Omega_0 = (d - \sigma_1, d) \cup (1 - \sigma_2, 1).$$

First we shall assume $\sigma_1 = \sigma_2 = \tau_0\varepsilon/\beta \ln N$ as otherwise N^{-1} is exponentially small compared to ε. We choose the transition points to be

$$x_{N/4} = d - \sigma_1, \quad x_{N/2} = d, \quad x_{3N/4} = 1 - \sigma_2.$$

Because of the specific layers, here we have to use two mesh generating functions φ_1 and φ_2 which are both continuous and piecewise continuously differentiable and monotonically decreasing functions and

$$\varphi_1(1/4) = \ln N, \qquad \varphi_1(1/2) = 0$$
$$\varphi_2(3/4) = \ln N, \qquad \varphi_2(1) = 0.$$

The mesh points are

$$x_i = \begin{cases} u\frac{4i}{N}(d-\sigma_1), & i = 0, ..., N/4, \\ d - \frac{\tau_0}{\beta}\varepsilon\varphi_1(t_i), & i = N/4+1, ..., N/2, \\ d + \frac{4}{N}(1-d-\sigma_2)(i-N/2), & i = N/2+1, ..., 3N/4, \\ 1 - \frac{\tau_0}{\beta}\varepsilon\varphi_2(t_i), & i = 3N/4+1, ..., N, \end{cases}$$

where $t_i = i/N$. We define new functions ψ_1 and ψ_2 by

$$\varphi_i = -\ln\psi_i, \quad i = 1, 2.$$

There are several mesh-characterizing functions ψ in the literature, but we shall use only those which correspond to Shishkin mesh and Bakhvalov-Shishkin mesh:

- Shishkin mesh

$$\psi_1(t) = e^{-2(1-2t)lnN}, \qquad \psi_2(t) = e^{-4(1-t)lnN},$$

- Bakhvalov-Shishkin mesh

$$\psi_1(t) = 1 - 2(1-N^{-1})(1-2t), \quad \psi_2(t) = 1 - 4(1-N^{-1})(1-t).$$

The set of interior mesh points is denoted by $\Omega_\varepsilon^N = \bar{\Omega}_\varepsilon^N \setminus \{x_{N/2}\}$. Also, for the both meshes, on the coarse part Ω_s we have

$$h_i \le CN^{-1}.$$

It is well known that on the layer part of the Shishkin mesh [?]

$$h_i \le C\varepsilon N^{-1}\ln N$$

and of the Bakhvalov-Shishkin mesh we have

$$h_i \le \begin{cases} \frac{\tau_0}{\beta}\varepsilon N^{-1}\max|\,\psi_1'\,|\exp\left(\frac{\beta}{\tau_0\varepsilon}(d-x_{i-1})\right), & i = N/4+1, ..., N/2, \\ \frac{\tau_0}{\beta}\varepsilon N^{-1}\max|\,\psi_2'\,|\exp\left(\frac{\beta}{\tau_0\varepsilon}(1-x_{i-1})\right), & i = 3N/4+1, ..., N \end{cases}$$

and

$$\frac{h_i}{\varepsilon} \le CN^{-1}\max|\,\varphi'\,| \le C.$$

Theorem 3.4. *If y_2^* and $y_{2h}^*(x_i)$ are the solutions of (??) and (??) respectively, then*

$$|y_2^*(x_i) - y_{2h}^*(x_i)| \le \begin{cases} CN^{-2}\ln^2 N, & \textit{for Shishkin mesh} \\ CN^{-2}, & \textit{for Bakhvalov-Shishkin mesh}, \end{cases} \quad i = 1, 2, ..., N.$$

Proof. Refer [?]. □

Remark 3.5. By suitable transformation a BVP of the type (??) with non-homogeneous BCs can be transformed into another BVP with homogeneous BCs [?]. Hence before applying the above FEM to any BVP having non-homogeneous BCs, the problem should be transformed into a BVP with homogeneous BCs.

4. Error Estimate

Theorem 4.1. *Let (y_1, y_2) be the solution of (??-??). Further let $y^*_{2h}(x_i)$ be the numerical solution of (??) after applying the FEM to it. Then*

$$| y_2(x_i) - y^*_{2h}(x_i) | \leq \begin{cases} C(\varepsilon + N^{-2}\ln^2 N), \textit{for Shishkin mesh} \\ C(\varepsilon + N^{-2}), \textit{for Bakhvalov-Shishkin mesh}, i = 1, 2, ..., N. \end{cases} \tag{4.1}$$

Proof.

$$\begin{aligned} & | y_2(x_i) - y^*_{2h}(x_i) | \\ = \; & | y_2(x_i) - y^*_2(x_i) + y^*_2(x_i) - y^*_{2h}(x_i) | \\ \leq \; & | y_2(x_i) - y^*_2(x_i) | + | y^*_2(x_i) - y^*_{2h}(x_i) | \\ \leq \; & \begin{cases} C(\varepsilon + N^{-2}\ln^2 N), \text{for Shishkin mesh} \\ C(\varepsilon + N^{-2}), \text{for Bakhvalov-Shishkin mesh}, i = 1, 2, ..., N, \end{cases} \end{aligned}$$

by Theorem ?? and Theorem ??. □

Remark 4.2. A similar statement is true for the BVP (?? - ??). In [?, ?] the authors applied FMM on Shishkin mesh and obtained an error estimate of order $O(\varepsilon + N^{-1}\ln N)$. From the above result, it is obvious that the present method has improved the earlier results.

5. Numerical Experiments

In this section we experimentally verify our theoretical results proved in the previous section.

Example 5.1. Consider the BVP

$$-\varepsilon y'''(x) + 2y''(x) + 4y'(x) - 2y(x) = \begin{cases} 0.7, & x \leq 0.5, \\ -0.6, & x > 0.5, \end{cases} \qquad x \in (0, 1), \tag{5.1}$$

$$y(0) = 1, \quad y'(0) = 0, \quad y'(1) = 0, \tag{5.2}$$

and the corresponing system

$$-y_1'(x) + y_2(x) = 0, \; -\varepsilon y_2''(x) + 2y_2'(x) + 4y_2(x) - 2y_1(x) = \begin{cases} 0.7, & x \leq 0.5, \\ -0.6, & x > 0.5, \end{cases} \tag{5.3}$$

$$y_1(0) = 1, \quad y_2(0) = 0, \quad y_2(1) = 0. \tag{5.4}$$

For our tests, we take $\varepsilon = 10^{-16}$, which is sufficiently small choice to bring out the singularly perturbed nature of the problem. We measure the accuracy in the discrete maximum norm $\| \cdot \|_\infty$. The rates of convergence r^N are computed using the following formula:

$$r^N = \log_2 \left(\frac{E^N}{E^{2N}} \right),$$

where $E^N = \| u_h - u_{2h}^I \|_\infty$ and u_h^I denotes the piecewise linear interpolant of u. In Table ??, we present values of E^N, r^N for the first derivative of the solution of the BVP (?? - ??) for Shishkin and Bakhavlov-Shishkin meshes. The Figure ?? depicts the first derivative of the numerical solution of the BVP (?? - ??) for Shishkin mesh.

We compare the values of E^N, r^N for the first derivative of the solution of the same BVP (?? - ??) for Shishkin mesh using the standard upwind scheme adopted in [?]. From these tables, it is obvious that the scheme presented in this paper performs very well.

Example 5.2. Consider the BVP

$$-\varepsilon y^{iv}(x) + 4y'''(x) + 4y''(x) = \begin{cases} -0.7, & x \leq 0.5 \\ 0.6, & x > 0.5, \end{cases} \tag{5.5}$$

$$y(0) = y(1) = 1, \quad y''(0) = y''(1) = 0, \tag{5.6}$$

and the corresponing system

$$-y_1''(x) = y_2(x), \, -\varepsilon y_2''(x) + 4y_2'(x) + 4y_2(x) = \begin{cases} 0.7, & x \leq 0.5 \\ -0.6, & x > 0.5, \end{cases} \tag{5.7}$$

$$y_1(0) = y_1(1) = 1, \quad y_2(0) = y_2(1) = 0. \tag{5.8}$$

The Table ?? presents the values of E^N, r^N for the second derivative of the solution of the BVP (?? - ??). The graph of the second derivative of the numerical solution of the BVP (?? - ??) is given Figure ??. The numerical results are clear illustrations of the convergence estimates of both type of meshes.

TABLE 1. Values of E^N and r^N for the first derivative of the solution of the BVP (?? - ??).

N	Shishkin mesh		Bakhavlov-Shishkin mesh	
	E^N	r^N	E^N	r^N
32	6.6712e-03	1.0516	5.4925e-03	0.7709
64	3.2184e-03	0.8904	3.2187e-03	0.8903
128	1.7361e-03	0.9463	1.7364e-03	0.9463
256	9.0093e-04	0.9734	9.0111e-04	0.9734
512	4.5882e-04	0.9868	4.5893e-04	0.9868
1024	2.3151e-04	0.9926	2.3157e-04	0.9934
2048	1.1632e-04	-	1.1392e-04	-

TABLE 2. Values of E^N and r^N for the second derivative of the solution of the BVP (?? - ??).

N	Shishkin mesh		Bakhavlov-Shishkin mesh	
	E^N	r^N	E^N	r^N
32	2.4380e-03	1.0252	2.2738e-03	0.9246
64	1.1979e-03	0.9704	1.1979e-03	0.9700
128	6.1134e-04	0.9871	6.1135e-04	0.9875
256	3.0840e-04	0.9940	3.0841e-04	0.9940
512	1.5484e-04	0.9971	1.5484e-04	0.9971
1024	7.7572e-05	0.9986	7.7572e-05	0.9986
2048	3.8823e-05	-	3.8823e-05	-

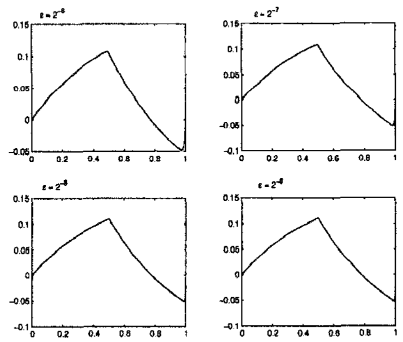

FIGURE 1. Graphs of the first derivative of the numerical solution of the BVP (?? - ??) for various values of ε with $N = 512$.

REFERENCES

1. P. A. Farrell, A. F. Hegarty, J. J. H. Miller, E. O'Riordan, G. I. Shishkin, *Robust computational techniques for boundary layers*, Chapman Hall/ CRC, Boca Raton, 2000.
2. E.P. Doolan, J.J.H.Miller, W. H. A. Schilders, *Uniform numerical methods for problems with initial and boundary layers*, Boole, Dublin, 1980.
3. F. A. Howes, *Differential inequalites of higher order and the asymptotic solution of nonlinear boundary value problems*, SIAM J.Math.anal. **13(1)** (1982) 61-80.
4. H-G. Roos, T. Linß, *Sufficient conditions for uniform convergence on layer-adapted grids*, Computing, **63** (1999) 27-45.
5. H-G. Roos, M. Stynes, L. Tobiska, *Numerical methods for singularly perturbed differential equations*, Volume 24 of Springer series in Computational Mathematics, Springer-Verlag, Berlin, 1996.
6. H-G. Roos, Helena Zarin, *The streamline-diffusion method for a convection-diffusion problem with a point source*,J. Comp. Appl. Math.,**10(4)**(2002) 275-289.
7. A. H. Nayfeh, *Introduction to Perturbation Methods*, Wiley, New York, 1981.

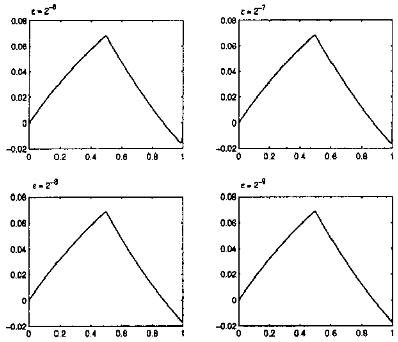

FIGURE 2. Graphs of the second derivative of the numerical solution of the BVP (?? - ??) for various values of ε with $N = 512$.

8. S. Valarmathi, N. Ramanujam, *A computational method for solving boundary value problems for singularly perturbed third order ordinary differencial equations.*, Applied mathematics and Computation, **129** (2002) 345-373.
9. R. E. O'Malley, *Singular perturbation methods for ordinary differential equations*, Springer, New York, 1990.
10. G. Sun, M. Stynes, *Finite element methods for singularly perturbed higher order elliptic two point boundary value problem II: convection-diffusion type*, IMA J.Numer.Anal.**15** (1995) 197-219.
11. V. Shanthi, N. Ramanujam, *Asymptotic numerical method for boundary value problems for singularly perturbed fourth-order ordinary differential equations of convection-diffusion type,* Journal of Applied Mathematics and Computation, **133** (2002) 559-579.
12. V. Shanthi, N. Ramanujam, *Asymptotic numerical method for singularly perturbed fourth-order ordinary differential equation with a weak interior layer,* Journal of Applied Mathematics and Computation, **172** (2006) 252-266.
13. T. Valanarasu, N. Ramanujam, *Asymptotic numerical method for singularly perturbed third order ordinary differential equations with a weak interior layer,* International Journal of Computer Mathematics **84** (2007) 333-346.
14. S. Valarmathi, *Numerical methods for singularly perturbed boundary value problems for third order ordinary differencial equations*, Ph.D., Thesis, Bharathidasan University, Tiruchirappalli 620 024, India, 2002.

A. Ramesh Babu recieved his M.Sc.,(Mathematics) degree from Department of Mathematics, Bharathidasn University. He worked as a lecturer in Mathematics at Chidambarampillai College for Women, Mannachanallur, Tiruchirappalli. He has been awarded University Research Studentship from Bharathidasan University. Currently he is doing his Ph.D program under the direction of Dr. N. Ramanujam at Bharathidasan University. His areas of interest are Differential Equations and Numerical Analysis.

Department of Mathematics, School of Mathematics and Computer Science, Bharathidasn University, Tiruchirappalli, Tamilnadu, India.
`e-mail: matramesh2k5@yahoo.co.in`

N. Ramanujam is working as a Professor and Head, Department of Mathematics and Co-ordinator, School of Mathematics and Computer Science, Bharathidasn University.

His areas of interests are Differential Equations, Differential Inequalities and Numerical Analysis.

Department of Mathematics, School of Mathematics and Computer Science, Bharathidasn University, Tiruchirappalli, Tamilnadu, India.
e-mail: matram@bdu.ac.in, Website: http://bdu.ac.in