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ON THE MOMENTS OF BINARY SEQUENCES AND AUTOCORRELATIONS OF THEIR GENERATING POLYNOMIALS

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ABSTRACT. In this paper we focus on a type of Unimodular polynomial pair used for digital systems and present some new properties of them which lead us to estimation of their autocorrelation coefficients and the moments of a Rudin-Shapiro polynomial product. Some new results on the Rudin-shapiro sequences will be presented in the last section.
Main Facts: For positive integers M and n with $M < 2^n - 1$, consider the $2^n - M$ numbers ϵ_k $(M \leq k \leq 2^n - 1)$ which form a collection of Rudin-Shapiro coefficients. We verify that $|\sum_{k=M}^{2^n-1} \epsilon_k e^{ikt}|$ is dominated by $(2+\sqrt{2})\sqrt{2^n - M} - \sqrt{2}$.

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1. Introduction

We Shall start very briefly by mentioning the "infrared spectrometry". On the instrumental optics (multislit infrared spectrometry) [3, 4], Marcel Goley introduced pairs as follows:

Given finite sequence of the same length $(a_0, a_1, \cdots, a_d)$ and $(b_0, b_1, \cdots, b_d)$, suppose that A and B in $\mathbf{C}[z]$ are their generating polynomials, that is

$$A(z) = \sum_{k=0}^{d} a_k z^k, \qquad A(z) = \sum_{k=0}^{d} a_k z^k.$$

If $|A(z)|^2 + |B(z)|^2 = 2(d+1)$ for all $|z| = 1$, then we say A and B form it Golay complementary polynomial pair.

The most important type of the Golay complementary polynomial pairs are the it Rudin-Shapiro polynomials, [7], which has been studied extensively by

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telecommunication engineers [1, 8]. Setting $p_0 = q_0 = 1$, we define them inductively as follows.

$$p_n = p_{n-1} + e^{i2^{n-1}t} q_{n-1}, \qquad q_n = p_{n-1} - e^{i2^{n-1}t} q_{n-1} \quad (n \in \mathbf{N}). \tag{1}$$

Letting $l_n = 2^n$ for every n, in what followed we will show that these polynomials form a Golay complementary pair. Using the definition (2), we write

$$\begin{aligned} |p_n(z)|^2 &= z^{l_{n-1}} \bar{p}_{n-1}(z) q_{n-1}(z) + (|p_{n-1}(z)|^2 + |q_{n-1}(z)|^2) \\ &\quad + \bar{z}^{l_{n-1}} p_{n-1}(z) \bar{q}_{n-1}(z) \\ &= (|p_{n-1}(z)|^2 + |q_{n-1}(z)|^2) + 2Re(z^{l_{n-1}} \bar{p}_{n-1}(z) q_{n-1}(z)) \end{aligned}$$

and

$$\begin{aligned} |q_n(z)|^2 &= -z^{l_{n-1}} \bar{p}_{n-1}(z) q_{n-1}(z) + (|p_{n-1}(z)|^2 \\ &\quad + |q_{n-1}(z)|^2) - \bar{z}^{l_{n-1}} p_{n-1}(z) \bar{q}_{n-1}(z) \\ &= (|p_{n-1}(z)|^2 + |q_{n-1}(z)|^2) - 2Re(z^{l_{n-1}} \bar{p}_{n-1}(z) q_{n-1}(z)). \end{aligned}$$

Hence

$$\begin{aligned} |p_n(z)|^2 + |q_n(z)|^2 &= 2(|p_{n-1}(z)|^2 + |q_{n-1}(z)|^2) \\ &= 2^2(|p_{n-2}(z)|^2 + |q_{n-2}(z)|^2) \\ &\ \ \vdots \\ &= 2^n(|p_0(z)|^2 + |q_0(z)|^2) = 2^n(1+1). \end{aligned}$$

Thus if $|z| = 1$, then

$$|p_n(z)|^2 + |q_n(z)|^2 = 2^{n+1} = l_{n+1}. \tag{2}$$

Thus the Rudin-Shapiro polynomials are in fact are of Golay complementary type.

For a complex polynomial p and a positive real number q, define $||p||_q$ by

$$||p||_q = \left(\frac{1}{2\pi} \int_0^{2\pi} |p(e^{it})|^q dt \right)^{1/q}.$$

Letting $f(t) = |p_n(e^{it})|^2$, we can write

$$f(t) = \sum_{k=1-2^n}^{2^n-1} c_k e^{ikt},$$

on which the central coefficient c_0 (called *the central frequency of* f) is 2^n. One can easily verify that $||p_n||_2^2 = c_0$ and $c_k = c_{-k}$. Each c_k is called an *autocorrelation coefficient of* p_n. Finding the "best value" for $\gamma_n := \max_{1 \le k \le 2^n - 1} |c_k|$ is an old problem.

Theorem 1. *There exists an absolute constant* C *such that* $\gamma_n > C 2^{\frac{1}{2}n}$.

Proof. Since $c_k = c_{-k}$ for all k and $c_0 = 2^n$, we have

$$\|f\|_2^2 = c_0^2 + 2\sum_{k=1}^{2^n-1} |c_k|^2 = (2^n)^2 + 2\sum_{k=1}^{2^n-1} |c_k|^2. \tag{3}$$

On the other hand

$$\|f\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |p_n(e^{it})|^4 dt = \|p_n\|_4^4. \tag{4}$$

One can easily show (using induction on n) that

$$\frac{4}{3}(2^{2n}) - 2^{n-1} < \|p_n\|_4^4 < \frac{4}{3}(2^{2n}) + 2^{n-1},$$

so that $\frac{4}{3} - 2^{-n-1} < \frac{\|p_n\|_4^4}{(2^{2n})} < \frac{4}{3} + 2^{-n-1}$. Thus $\|p_n\|_4^4$ is asymptotic to $\frac{4}{3}(2^{2n})$ written $\|p_n\|_4^4 \sim \frac{4}{3}(2^{2n})$ (the $\sim$ symbol means that the ratio of the left and right hand sides converges to 1 as $n \to \infty$). So, by (3) and (4)

$$\sum_{k=1}^{2^n-1} |c_k|^2 = \frac{1}{2}(\|f\|_2^2 - 2^{2n}) \sim \frac{1}{2}\left(\frac{4}{3}(2^{2n}) - 2^{2n}\right) = \frac{1}{6}(2^n)^2.$$

Therefore,

$$\begin{aligned} \gamma_n^2 &= \max_{1\le k\le 2^n-1} |c_k|^2 \ge \frac{1}{2^n-1}\sum_{k=1}^{2^n-1} |c_k|^2 \\ &\sim \frac{1}{2^n-1}\left(\frac{1}{6}(2^n)^2\right) > \frac{1}{2^n}\left(\frac{1}{6}(2^n)^2\right) \\ &= \frac{1}{6}(2^n). \end{aligned}$$

Hence $\gamma_n > \frac{1}{\sqrt{6}}(2^{\frac{1}{2}n})$. □

It is known that $\gamma_n \le C2^{\frac{3}{4}n}$ for some absolute constant C and it was thought that the correct answer should be $\gamma_n \le C_\epsilon L^{\frac{1}{2}+\epsilon}$ with ϵ being any positive number and C_ϵ is a constant depending only on ϵ. But a counter example was given in [9] that provides a particular k and a universal constant D so that $|c_k| > DL^{0.73}$. We showed in [10] that 0.73 is optimal in upper bound case.

Next we define the class $\mathcal{K}_\backslash$ as the collection of all (complex) unimodular polynomials of degree $n \ge 1$ so that if $a_n \in \mathcal{K}_\backslash$, then $a_n(z) = \sum_0^n c_k z^k$ with each c_k a complex number and $|c_k| = 1$. One can easily check that by Parseval's formula, $\int_0^{2\pi} |a_n(e^{it})|^2 dt = 2\pi(n+1)$ and so $\min |a_n(z)| \le \sqrt{n+1} \le \max |a_n(z)|$, where both min and max are taken over all z with $|z| = 1$.

We also define the class $\mathcal{L}_\backslash$ as the collection of all (real) unimodular polynomials of degree $n \ge 1$. This class is called the set of *Littlewood polynomials* of degree n. Note that the Rudin-Shapiro polynomials p_n and q_n are in $\mathcal{L}_\backslash$. In

1966, Littlewood, [6] conjectured the existence of universal positive constants c_1 and c_2 and arbitrary large integer n such that

$$c_1\sqrt{n} \le |a_n(z)| \le c_2\sqrt{n} \qquad (|z|=1),$$

for some $a_n \in \mathcal{L}_\backslash$. The Rudin-Shapiro polynomilas satisfy the upper bound in this condition with $c_1 = \sqrt{2}$, but no sequence is known to satisfy the lower bound. In fact, the best known result here is some 30 years old (see [2]), used the Barker sequence of length 13 to show that for sufficiently large n there exist polynomials $a_n \in \mathcal{L}_\backslash$ with $|a_n(z)| > n^{0.431}$ on $|z| = 1$. letting $c = \sup_{t>0} \sin^2 t/t \approx 0.73$, $c_1 = \sqrt{1-c}$ and $c_2 = \sqrt{1+c}$ we have

$$c_1 + O\left(\frac{1}{n}\right) \le \frac{|a_n(z)|}{\sqrt{n}} \le c_2 + O\left(\frac{1}{n}\right).$$

The question that how close $a_n \in \mathcal{K}_\backslash$ or $a_n \in \mathcal{L}_\backslash$ can come to satisfy $|a_n(z)| = \sqrt{n+1}$ obviously is impossible if $n \ge 1$. There are various ways of seeking such an "approximate situation". One way suggested by Littlewood in [6] that, conceivably, there might exist a sequence $\{a_n\}$ of polynomials $a_n \in \mathcal{K}_\backslash$ (possibly even $a_n \in \mathcal{L}_\backslash$) such that $(n+1)^{-1/2}|a_n(e^{it})|$ converge to 1 uniformly in $t \in \mathbf{R}$. Such sequences of unimodular polynomials are called *ultraflat.* More precisely,

$$\lim_{n\to\infty} \max_{|z|=1} |(n+1)^{\frac{-1}{2}} |a_n(z)| - 1| = 0.$$

Erdelyi proved the 1996 conjecture that for $f_n(t) := Re(a_n(e^{it}))$ and $0 < q < \infty$ we have

$$\|f_n\|_q \sim \left(\frac{\Gamma(\frac{q+1}{2})}{\Gamma(\frac{q}{2}+1)\sqrt{\pi}}\right)^{\frac{1}{q}} \sqrt{n}$$

and

$$\|f_n'\|_q \sim \left(\frac{\Gamma(\frac{q+1}{2})}{(q+1)\Gamma(\frac{q}{2}+1)\sqrt{\pi}}\right)^{\frac{1}{q}} \sqrt{n^3}$$

where Γ denotes the usual gamma function.

Let $p \in \mathcal{K}_\backslash$ and write

$$p(z) = \sum_{k=0}^{n} a_k z^k,$$

where $a_k = \pm 1$ for all k. Define *the conjugate reciprocal polynomial of* p by $p^*(z) := z^n \bar{p}(1/z)$. One can easily verify that $p^*(z) = \sum_{k=0}^{n} \bar{a}_{n-k} z^k$ and moreover

$$\int_{\mathbf{T}} |p(z) - p^*(z)|^2 |dz| = 2n + \circ(n),$$

where $\circ(n)$ denotes a quality for which $\lim_{n\to\infty} \circ(n)/n = 0$.

Now if we define $\check{p}(z) = p(-z)$, then $(\check{p})^* = (-1)^{deg(p)}(\check{p^*})$.

Observation. Let $q \geq 2$ be a fixed integer and consider a polynomial of the form

$$p(z) = A_1(z)A_2(z)\cdots A_q(z),$$

where $A_k \in \{p_n, \check{p_n}, p_n^*, \check{p_n^*}\}$ and where p_n is the n^{th} Rudin-Shapiro polynomial of degree $2^n - 1$. Hence

$$p(z) = \sum_{k=0}^{q(2^n-1)} b_k z^k$$

is a unimodular polynomial. Moreover $\|p\|_1 \leq cL^{\frac{q}{2}}$ (the proof is similar to that $\|p_n\|_{\mathbf{C}} \leq \sqrt{2}L^{\frac{1}{2}}$ shown in [5]).

Case 1: Suppose that
(i) *q is an even integer* and
(ii) $A \in \{p_n, \check{p_n}, p_n^*, \check{p_n^*}\}$ implies that $A^* \in \{p_n, \check{p_n}, p_n^*, \check{p_n^*}\}$.
The central coefficient of p here is the same order as $\|p\|$ and for every k there exist $\delta_q > 0$ and c_q both depended only on q so that

$$|b_k| \leq c_q 2^{n(\frac{q}{2}-\delta_q)}). \tag{5}$$

Of course the central coefficient $k = 0$ case excluded in (5).

Case 2: Suppose that niehter (i) nor (ii) in case 1 hold. Then again for every k there exist $\delta_q > 0$ and c_q both depended only on q so that

$$\max_{0 \leq k \leq q(2^n-1)} |b_k| \leq c_q 2^{n(\frac{q}{2}-\delta_q)}.$$

2. The main Result

In what follows by ϵ_s $(s \leq 2^n - 1)$ we mean the s^{th} coefficient of any Rudin-Shapiro polynomial p_n. Since by (1), the first half part of p_n is indeed p_{n-1}, it make sense to consider a fixed sequence $\{\epsilon_1, \epsilon_2, \cdots\} \subset \{+1, -1\}$ called *the p-Rudin-Shapiro sequence.* Similarly $\{\delta_1, \delta_2, \cdots\}$ is called *the q-Rudin-Shapiro sequence.*

Theorem 2. *Suppose that $m \leq n$ be positive integers, $N = \omega_m 2^m + \cdots + \omega_0 2^0 \neq 0$, where $\omega_k = 0$ or 1 and let $0 \leq M \leq 2^n$. Suppose that $\{\epsilon_1, \epsilon_2, \cdots\}$ and $\{\delta_1, \delta_2, \cdots\}$ are p & q-Rudin-Shapiro sequence. Then*

(a). $$\left|\sum_{k=0}^{N} \epsilon_k e^{ikt}\right| \leq 1 + \sum_{l=0}^{m} \omega_l |p_l| \leq (\sqrt{2}+1)\sqrt{N} - 1.$$

(b). $$\left|\sum_{k=0}^{N} \delta_k e^{ikt}\right| \leq 1 + \sum_{l=0}^{m} \omega_l |p_l|.$$

(c). $$\left|\sum_{k=M}^{2^n-1} \epsilon_k e^{ikt}\right| \leq (2+\sqrt{2})\sqrt{2^n - M} - \sqrt{2}.$$

Lemma 1. *Let m be a positive integer and suppose that $\omega_0, \omega_1, \cdots, \omega_m$ not all zeros take values of* 0 *and* 1*. Then*

$$\sum_{k=0}^{m} \omega_k 2^{\frac{k}{2}} \leq (\sqrt{2}+1)\sqrt{\sum_{k=0}^{m} \omega_k 2^k} - \sqrt{2}.$$

Proof. Define $M = \sum_{k=0}^{m} \omega_k 2^{\frac{k}{2}}$ and $N = \sum_{k=0}^{m} \omega_k 2^k$ and note that we may assume $\omega_m = 1$. If $\omega_k = 1$ for all k, then $M = (\sqrt{2}+1)(2^{(m+1)/2} - 1)$ and $N = 2^{m+1} - 1$. Therefore

$$\begin{aligned} M &= (\sqrt{2}+1)(\sqrt{N+1}-1) = (\sqrt{2}+1)(\sqrt{N} + \sqrt{N+1} - \sqrt{N} - 1) \\ &\leq (\sqrt{2}+1)\sqrt{N} + (\sqrt{2}+1)(\sqrt{2}-2) \\ &= (\sqrt{2}+1)\sqrt{N} - \sqrt{2}. \end{aligned}$$

So, assume that $\omega_k = 0$ for some $k < m$.

Case 1. $\omega_m = \omega_{m-1} = 1$. In this case, let $r \geq 1$ be so that $\omega_m = \omega_{m-1} = \cdots = \omega_r = 1$ and $\omega_{r-1} = 0$. Then

$$M \leq (\sqrt{2}+1)\left[(2^{\frac{l}{2}} - 1)2^{\frac{r}{2}} + 2^{\frac{r-1}{2}} - 1\right], \qquad N \geq (2^l - 1)2^r,$$

where $l = m - r + 1 \geq 2$ so that

$$\frac{M + \sqrt{2} + 1}{\sqrt{N}} < (\sqrt{2}+1) \times \frac{2^{\frac{l}{2}} - 1 + 2^{\frac{-l}{2}}}{\sqrt{2^l - 1}}.$$

Setting $x = 2^{\frac{l}{2}} \geq 2$, we have $x - 1 + \frac{-l}{2} < \sqrt{x^2 - 1}$ which means $x^2 - (2 - \sqrt{2})x + (3 - 2\sqrt{2})/2 < x^2 - 1$ or $(5 - 2\sqrt{2}) < 2(2 - \sqrt{2})x$, which is the case since $x \geq 2$. Thus $M + \sqrt{2} + 1 < (\sqrt{2}+1)\sqrt{N}$.

Case 2. $\omega_m = 1$, $\omega_{m-1} = \omega_{m-2} = 0$. In this case, we of course assume $m \geq 2$. Thus

$$\begin{aligned} M &\leq 2^{\frac{m}{2}} + (\sqrt{2}+1)(2^{\frac{m-1}{2}} - 1) \\ &= [1 + (\sqrt{2}+1)/2]2^{\frac{m}{2}} - (\sqrt{2}+1) \\ &< (\sqrt{2}+1)\sqrt{N} - (\sqrt{2}+1). \end{aligned}$$

Case 3. $\omega_m = 1$, $\omega_{m-1} = 0$, $\omega_{m-2} = 1$ and $\omega_{m-3} = 0$. So if $M \geq 3$, then

$$M \leq 2^{\frac{m}{2}} + 2^{\frac{m-2}{2}} + (\sqrt{2}+1)(2^{\frac{m-3}{2}} - 1),$$

$$N \geq 2^m + 2^{m-2} = (8+2)2^{m-3},$$

$$\frac{M + \sqrt{2} + 1}{\sqrt{N}} \leq \frac{2\sqrt{2} + \sqrt{2} + (\sqrt{2}+1)}{\sqrt{10}} = \frac{4\sqrt{2}+1}{\sqrt{10}} < \sqrt{2} + 1.$$

Case 4. $\omega_m = 1$, $\omega_{m-1} = 0$, $\omega_{m-2} = \omega_{m-3} = 1$. Then

$$\frac{M+\sqrt{2}+1}{\sqrt{N}} < \frac{2^{\frac{m}{2}} + (\sqrt{2}+1)2^{\frac{m-1}{2}}}{\sqrt{2^m + 2^{m-2} + 2^{m-3}}} = \frac{4\sqrt{2}+2}{\sqrt{11}} < \sqrt{2}+1.$$

Case 5. $\omega_m = 1$ *and* $\omega_k = 0$ *for all* $k < m$. Then

$$M = 2^{\frac{m}{2}} \le (\sqrt{2}+1)2^{\frac{m}{2}} - \sqrt{2} = (\sqrt{2}+1)\sqrt{N} - \sqrt{2}.$$

So the proof of lemma is complete. □

Proof of theorem. Using the complementary condition (2) and lemma 1, we first prove the second inequality in **(a)** as follows:

$$\begin{aligned} 1+\sum_{l=0}^{m} \omega_l |p_l| &\le 1+\sqrt{2}\sum_{l=0}^{m}\omega_l 2^{l/2} \\ &\le 1+\sqrt{2}\left[(\sqrt{2}+1)\sqrt{N}-\sqrt{2}\right] \\ &= \sqrt{2}(\sqrt{2}+1)\sqrt{N}-1. \end{aligned}$$

Now the first inequality in (a) is trivial for $N = 0, 1$. Suppose $N \ge 1$ and the result is true with N replaced by $N-1$. Let ω_l be as above, where $\omega_m = 1$. Let the positive integer r be the least one with $\omega_m = \omega_{m-1} = \cdots = \omega_r = 1$. Noting $2^m \le N < 2^{m+1}$, we have

$$\begin{aligned} \left|\sum_{k=0}^{N} \epsilon_k e^{ikt}\right| &\le \left|\sum_{k=0}^{2^m-1} \epsilon_k e^{ikt}\right| + \left|\sum_{k=2^m}^{N} \epsilon_k e^{ikt}\right| \\ &= |p_m| + \left|\sum_{k=0}^{N-2^m} \delta_k e^{ikt}\right| \\ &\le |p_m| + |p_{m-1}| + \left|\sum_{k=2^{m-1}}^{N-2^m} \delta_k e^{ikt}\right| \\ &= |p_m| + |p_{m-1}| + \left|\sum_{k=0}^{N-2^m-2^{m-1}} \delta_k e^{ikt}\right| \\ &\le \cdots \\ &\le |p_m| + |p_{m-1}| + \cdots + |p_r| + \left|\sum_{k=0}^{N-2^m-\cdots-2^r} \delta_k e^{ikt}\right| \\ &= \sum_{l=r}^{m} |p_l| + \left|\sum_{k=0}^{N-2^m-\cdots-2^r} \epsilon_k e^{ikt}\right|, \end{aligned}$$

since $N - 2^m - \cdots - 2^r < 2^{r-1}$. By induction, we get the required inequality.

To show (b), note that if $\omega_m = 0$ (that is $N < 2^m$), then

$$\sum_{k=0}^{N} \delta_k e^{ikt} = \sum_{k=0}^{N} \epsilon_k e^{ikt}.$$

Thus because of (a) we may assume $\omega_m = 1$. Therefore, $N \geq 2^m$ and (1) together with (2) imply

$$\begin{aligned} \left|\sum_{k=0}^{N} \delta_k e^{ikt}\right| &\leq \left|\sum_{k=0}^{2^m-1} \epsilon_k e^{ikt}\right| + \left|\sum_{k=2^m}^{N} \delta_k e^{ikt}\right| \\ &= |p_m| + \left|\sum_{k=0}^{N-2^m} \delta_k e^{ikt}\right|. \end{aligned}$$

Thus (b) follows from the proof of (a). To show (c), note that by the first paragraph of the proof, it suffices to show

$$\left|\sum_{k=M}^{2^n-1} \epsilon_k e^{ikt}\right| \leq 1 + \sum_{l=0}^{m} \omega_l |q_l|, \tag{6}$$

where $\omega_l = 0$ or 1 are so that $2^n - 1 - M = \omega_m 2^m + \cdots + \omega_0 2^0$. Suppose that (6) is true if m is replaced by $m-1$. If $M = 0$, then $2^n - 1 - M = 2^{n-1} + \cdots + 1$ and

$$\begin{aligned} \left|\sum_{k=0}^{2^n-1} \epsilon_k e^{ikt}\right| &= |p_n| \\ &\leq |p_{n-1}| + |q_{n-1}| \leq |p_{n-2}| + |q_{n-2}| + |q_{n-1}| \\ &\leq \cdots \leq 1 + \sum_{l=0}^{n-1} |q_l|. \end{aligned}$$

So assume $M \geq 1$ and $\omega_m = 1$. Let $0 \leq p \leq q$ be so that $M = 2^q + 2^{q-1} + \cdots + 2^p$. Then

$$\begin{aligned} \left|\sum_{k=M}^{2^n-1} \epsilon_k e^{ikt}\right| &\leq \left|\sum_{k=2^{q+1}}^{2^n-1} \epsilon_k e^{ikt}\right| + \left|\sum_{k=M}^{2^{q+1}-1} \epsilon_k e^{ikt}\right| \\ &\leq \sum_{l=q+1}^{n-1} |q_l| + \left|\sum_{k=M-2^q}^{2^q-1} \delta_k e^{ikt}\right| \\ &= \sum_{l=q+1}^{n-1} |q_l| + \left|\sum_{k=M_1}^{2^p-1} \delta_k e^{ikt}\right|, \end{aligned}$$

where $M_1 = M - 2^q - \cdots - 2^p < 2^{p-1}$. Note that the second inequality above follows from (1), (2) and

$$\left|\sum_{k=2^m}^{2^n-1} \epsilon_k e^{ikt}\right| \leq \sum_{l=m}^{n-1} |q_l|.$$

We now have

$$p = 0 \Longrightarrow \left|\sum_{k=M}^{2^n-1} \epsilon_k e^{ikt}\right| \leq \sum_{l=q+1}^{n-1} |q_l| + 1,$$

and

$$p \geq 1 \Longrightarrow \left|\sum_{k=M}^{2^n-1} \epsilon_k e^{ikt}\right| \leq \sum_{l=q+1}^{n-1} |q_l| + |q_{p-1}| + \left|\sum_{k=M_1}^{2^{p-1}-1} \epsilon_k e^{ikt}\right|.$$

Since $2^n - 1 - M = (2^{n-1} + \cdots + 2^q + \cdots + 2^p + \cdots + 1) - (2^q + \cdots + 2^p + 0 \times 2^{p-1} + \cdots)$, we have $p = 0 \Longrightarrow 2^n - 1 - M = \sum_{l=q+1}^{n-1} 2^l$, and $p \geq 1 \Longrightarrow 2^n - 1 - M = \sum_{l=q+1}^{n-1} 2^l + 2^p - 1 - M_1$, which complete the proof.

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