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ON GLOBAL EXPONENTIAL STABILITY FOR CELLULAR NEURAL NETWORKS WITH TIME-VARYING DELAYS

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ABSTRACT. In this paper, we consider the global exponential stability of cellular neural networks with time-varying delays. Based on the Lyapunov function method and convex optimization approach, a novel delay-dependent criterion of the system is derived in terms of LMI (linear matrix inequality). In order to solve effectively the LMI convex optimization problem, the interior point algorithm is utilized in this work. Two numerical examples are given to show the effectiveness of our results.

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1. Introduction

Cellular neural networks (CNNs) have been investigated extensively during the recent decades because CNN can be applied in various fields such as pattern recognition, associative memories, signal processing, fixed-point computation and etc. For more details, see [1]-[5] and references therein. On the other hand, time-delay is a natural phenomenon in many applications due to the finite switching speed of amplifiers in electronic networks or finite speed for signal propagation in biological networks. Moreover the delay is frequently a source of instability and oscillation. Therefore, many researchers have focused on the study for the stability analysis of delayed cellular neural networks (DCNNs) ([6]-[22]). In determining the speed of neural computation for real-time computation, the property of exponential convergence rate is often used to derive the fast behaviors of a system. Hence, the global exponential stability for DCNNs has also been investigated in very recent years ([16]-[22]).

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In this paper, we study the global exponential stability analysis of DCNNs. The described DCNNs have time-varying delays, which is more general cases than time-invariant ones. By employing a suitable Lyapunov-Krasovskii functionals method, a new exponential stability criterion is proposed. The derived criterion is delay-dependent one which is less conservative than delay-independent one when the size of delays is small [23]. Also, the proposed stability criterion is of the form of LMIs which can be solved efficiently by using the interior-point algorithms [25]. In this work, the popular model transformation technique, which leads an additional dynamics, is not used. Instead, in order to derive a less conservative results, note that a new integral inequality lemma is proposed. Finally, two numerical examples are included to show that our results are less conservative than those of the existing ones.

Notation: $\mathcal{R}^n$ is the n-dimensional Euclidean space, $\mathcal{R}^{m\times n}$ denotes the set of $m \times n$ real matrix. $\|\cdot\|$ refers to the Euclidean vector norm and the induced matrix norm. For symmetric matrices X and Y, the notation $X > Y$ (respectively, $X \geq Y$) means that the matrix $X - Y$ is positive definite, (respectively, nonnegative). $diag\{\cdots\}$ denotes the block diagonal matrix. $\star$ represents the elements below the main diagonal of a symmetric matrix. $\lambda_M(\cdot)$ and $\lambda_m(\cdot)$ mean the largest and smallest eigenvalue of given square matrix, respectively.

2. Problem Statements

Consider the following neural networks with time-varying delays:

$$\dot{y}_i(t) = -a_i y_i(t) + \sum_{j=1}^{n} w_{ij} f_j(y_j(t)) + \sum_{j=1}^{n} w_{ij}^1 f_j(y_j(t-h(t))) + b_i, \quad (1)$$

or equivalently

$$\dot{y}(t) = -Ay(t) + Wf(y(t)) + W_1 f(y(t-h(t))) + b, \quad (2)$$

where $i = 1, ..., n$, n denotes the number of neurons in a neural network, $y(t) = [y_1(t), ..., y_n(t)]^T \in \mathcal{R}^n$ is the neuron state vector, $f(y(t)) \in \mathcal{R}^n$ denotes the activation functions, $f(y(t-h(t))) \in \mathcal{R}^n$, $b = [b_1, ..., b_n]^T$ means a constant input vector, $A = diag\{a_i\}$ is a positive diagonal matrix, $W = (w_{ij})_{n\times n}$ and $W_1 = (w_{ij}^1)_{n\times n}$ are the interconnection matrices representing the weight coefficients of the neurons. The delay, $h(t)$, is a time-varying continuous function that satisfies

$$0 \leq h(t) \leq \bar{h}, \ \dot{h}(t) \leq \mu, \quad (3)$$

where $\bar{h}$ and μ are positive constants.

The activation functions, $f_i(y_i(t)), i = 1, ..., n$, are assumed to be nondecreasing, bounded and globally Lipschiz; that is,

$$0 \leq \frac{f_i(\xi_1) - f_j(\xi_2)}{\xi_1 - \xi_2} \leq l_i, \ \xi_i, \ \xi_j \in \mathcal{R}, \ \xi_1 \neq \xi_2, \ i = 1, ..., n. \quad (4)$$

Note that by using the Brouwer's fixed-point theorem [6], it can be easily proven that there exists at least one equilibrium point for Eq. (2).
For simplicity, in stability analysis of the system (2), the equilibrium point $y^* = [y_1^*, ..., y_n^*]^T$ is shifted to the origin by utilizing the transformation $x(\cdot) = y(\cdot) - y^*$, which leads the system (2) to the following form:

$$\dot{x}(t) = -Ax(t) + Wg(x(t)) + W_1 g(x(t - h(t)) \tag{5}$$

where $x(t) = [x_1(t), ..., x_n(t)]^T \in \mathcal{R}^n$ is the state vector of the transformed system, $g(x(t)) = [g_1(x(t)), ..., g_n(x(t))]^T$ and $g_j(x_j(t)) = f_j(x_j(t)+y_j^*)-f_j(y_j^*)$ with $g_j(0) = 0(j = 1, ..., n)$. It is noted from (4) that $g_j(\cdot)$ satisfies the following condition:

$$0 \leq \frac{g_j(\xi_j)}{\xi_j} \leq l_j, \forall \xi_j \neq 0, \ j = 1, ..., n \tag{6}$$

which is equivalent to $g_j(\xi_j)\,[g_j(\xi_j) - l_j\xi_j] \leq 0, \ g_j(0) = 0, \ j = 1, ..., n.$

Here, as a mathematical tool for our analysis, the following zero equation is introduce:

$$Gx(t) - Gx(t - h(t)) - G\int_{t-h(t)}^{t} \dot{x}(s)ds = 0.$$

Then, we can represent the system (2) as

$$\begin{aligned} \dot{x}(t) &= (-A + G)x(t) - Gx(t - h(t)) - G\int_{t-h(t)}^{t} \dot{x}(s)ds \\ &\quad + Wg(x(t)) + W_1 g(x(t - h(t)) \end{aligned} \tag{7}$$

where $G \in \mathcal{R}^{n\times n}$ will be chosen later.

Before deriving our main results, we state the following facts, definition and lemma.

Fact 1. (Schur complement) *Given constant symmetric matrices $\sum_1, \sum_2, \sum_3$ where $\sum_1 = \sum_1^T$ and $0 < \sum_2 = \sum_2^T$, then $\sum_1 + \sum_3^T \sum_2^{-1} \sum_3 < 0$ if and only if*

$$\begin{bmatrix} \sum_1 & \sum_3^T \\ \sum_3 & -\sum_2 \end{bmatrix} < 0, \text{ or } \begin{bmatrix} -\sum_2 & \sum_3 \\ \sum_3^T & \sum_1 \end{bmatrix} < 0.$$

Fact 2. *For any real vectors a, b and any matrix $Q > 0$ with appropriate dimensions, it follows that:*

$$2a^T b \leq a^T Qa + b^T Q^{-1} b.$$

Definition 1. For system defined by (1), if there exist the positive constants k and $\gamma > 1$ such that

$$\|x(t)\| \leq \gamma e^{-kt} \sup_{-\bar{h}\leq\theta\leq 0} \|x(\theta)\| \ \forall t > 0,$$

then, the trivial solution of the system (1) is exponentially stable where k is called the *convergence rate (or degree) of exponential stability.*

Lemma 1. *For a positive matrix $Q > 0$, any matrices $F_i(i = 1, ..., 6)$, and scalar $\bar{h} \geq 0$, the following inequality holds:*

$$-\int_{t-h(t)}^{t} \dot{x}^T(s)Q\dot{x}(s) \leq \zeta^T(t)\widetilde{F}\zeta(t) + \bar{h}\zeta(t)^T F^T Q^{-1} F\zeta(t)$$

where $\zeta^T = \left[x^T \; x^T(t-h(t)) \; \left(\int_{t-h(t)}^{t} \dot{x}(s)ds\right)^T \; \dot{x}^T \; g^T(x) \; g^T(x(t-h(t))) \right]^T$,
$F = \begin{bmatrix} F_1 & F_2 & F_3 & F_4 & F_5 & F_6 \end{bmatrix}$, *and*

$$\widetilde{F} = \begin{bmatrix} 0 & 0 & F_1^T & 0 & 0 & 0 \\ \star & 0 & F_2^T & 0 & 0 & 0 \\ \star & \star & F_3^T + F_3 & F_4 & F_5 & F_6 \\ \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & 0 \end{bmatrix}. \tag{8}$$

Proof. Utilizing Fact 2, we have

$$\begin{aligned} - \int_{t-h(t)}^{t} \dot{x}^T(s)Q\dot{x}(s)ds &\leq 2\left(\int_{t-h(t)}^{t} \dot{x}(s)ds\right)^T F\zeta + \int_{t-h(t)}^{t} \zeta^T F^T Q^{-1} F\zeta ds \\ &\leq 2\zeta^T \bar{I} F\zeta + \bar{h}\zeta^T F^T Q^{-1} F\zeta = \zeta^T \widetilde{F}\zeta + \bar{h}\zeta^T F^T Q^{-1} F\zeta. \end{aligned}$$

where $\bar{I} = [0\; 0\; I\; 0\; 0\; 0]^T$. □

Lemma 2. [24] Suppose that (4) holds, then

$$\int_{v}^{u} [g_i(s) - g_i(v)]ds \leq [u-v][g_i(u) - g_i(v)], i = 1, 2, ..., n.$$

3. Main results

In this section, we propose a new exponential stability criterion for neural networks with time-varying delays (7). Now, we have the following main results.

Theorem 1. *For given $0 \leq h(t) \leq \bar{h}$, $\dot{h}(t) \leq \mu$ and $L = diag\{l_1, \; l_2, ..., \; l_n\}$, the equilibrium point of (1) is globally exponentially stable with convergence rate k if there exist positive definite matrices P, $R_i(i = 1, 2, 3)$ and positive* *diagonal matrices* $D = diag\{d_1, d_2, ..., d_n\}$, $H_i (i = 1, 2)$ *and any matrices* Y_1, F_i, M_i, N_i $(i = 1, ..., 6)$ *satisfying the following LMI:*

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & \Sigma_{15} & \Sigma_{16} & \bar{h}F_1^T \\ \star & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} & \Sigma_{25} & \Sigma_{26} & \bar{h}F_2^T \\ \star & \star & \Sigma_{33} & \Sigma_{34} & \Sigma_{35} & \Sigma_{36} & \bar{h}F_3^T \\ \star & \star & \star & \Sigma_{44} & \Sigma_{45} & \Sigma_{46} & \bar{h}F_4^T \\ \star & \star & \star & \star & \Sigma_{55} & \Sigma_{56} & \bar{h}F_5^T \\ \star & \star & \star & \star & \star & \Sigma_{66} & \bar{h}F_6^T \\ \star & \star & \star & \star & \star & \star & -\bar{h}e^{2k\bar{h}}R_3 \end{bmatrix} < 0, \tag{9}$$

where

$$\begin{aligned}
\Sigma_{11} &= 2kP - PA - A^TP + Y_1 + Y_1^T + R_2 - N_1A - A^TN_1^T + M_1 + M_1^T, \\
\Sigma_{12} &= -Y_1 - A^TN_2^T - M_1 + M_2^T, \\
\Sigma_{13} &= -Y_1 + e^{-2k\bar{h}}F_1^T - A^TN_3^T - M_1 + M_3^T, \\
\Sigma_{14} &= -N_1 - A^TN_4^T + M_4^T, \\
\Sigma_{15} &= PW + 2kD + LH_1 + N_1W - A^TN_5^T + M_5^T, \\
\Sigma_{16} &= PW_1 + N_1W - A^TN_6^T + M_6^T, \\
\Sigma_{22} &= -(1-\mu)e^{-2k\bar{h}}R_2 - M_2 - M_2^T, \\
\Sigma_{23} &= e^{-2k\bar{h}}F_2^T - M_2 - M_3^T, \qquad \Sigma_{24} = -N_2 - M_4^T, \\
\Sigma_{25} &= N_2W - M_5^T, \qquad \Sigma_{26} = LH_2 + N_2W_1 - M_6^T, \\
\Sigma_{33} &= e^{-2k\bar{h}}F_3 + e^{-2k\bar{h}}F_3^T - M_3 - M_3^T, \\
\Sigma_{34} &= e^{-2k\bar{h}}F_4 - N_3 - M_4^T, \qquad \Sigma_{35} = e^{-2k\bar{h}}F_5 + N_3W - M_5^T, \\
\Sigma_{36} &= e^{-2k\bar{h}}F_6 + N_3W_1 - M_6^T, \qquad \Sigma_{44} = \bar{h}R_3 - N_4 - N_4^T, \\
\Sigma_{45} &= D + N_4W - N_5^T, \qquad \Sigma_{46} = N_4W_1 - N_6^T, \\
\Sigma_{55} &= R_1 - 2H_1 + N_5W + W^TN_5^T, \qquad \Sigma_{56} = N_5W_1 + W^TN_6^T, \\
\Sigma_{66} &= -(1-\mu)e^{-2k\bar{h}}R_1 - 2H_2 + N_6W_1 + W_1^TN_6^T.
\end{aligned}$$

Proof. For positive definite matrices P, $D = diag\{d_1, ..., d_n\}$, and $R_i (i = 1, ..., 3)$, let us consider the Lyapunov-Krasovskii functional candidate:

$$V = V_1 + V_2 + V_3 + V_4 + V_5 \tag{10}$$

where

$$\begin{aligned}
V_1 &= e^{2kt}x^T(t)Px(t), V_2 = 2\sum_{i=1}^{n} d_i e^{2kt}\int_0^{x_i(t)} g_i(s)ds, \\
V_3 &= \int_{t-h(t)}^{t} e^{2ks}g^T(x(s))R_1g(x(s))ds, V_4 = \int_{t-h(t)}^{t} e^{2ks}x^T(s)R_2x(s)ds, \\
V_5 &= \int_{t-\bar{h}}^{t}\int_s^t e^{2ku}\dot{x}^T(u)R_3\dot{x}(u)duds.
\end{aligned}$$

From Eq. (7), differentiating V_1 leads to

$$\dot{V}_1 = e^{2kt}\Big[x^T(2kP - PA - A^TP + PG + G^TP)x - 2x^TPGx(t-h(t))$$
$$-2x^TPG\int_{t-h(t)}^{t}\dot{x}(s)ds + 2x^TPWg(x) + 2x^TPW_1g(x(t-h(t)))\Big]. \quad (11)$$

By differentiating V_2, V_3 and V_4, respectively, we have

$$\begin{aligned}
\dot{V}_2 &= 4k\sum_{i=1}^{n} d_i e^{2kt}\int_0^{x_i(t)} g_i(s)ds + 2\sum_{i=1}^{n} d_i e^{2kt} g_i(x_i(t))\dot{x}_i(t) \\
&\le e^{2kt}\left[4kg^T(x(t))Dx(t) + 2g^T(x(t))D\dot{x}(t)\right], \\
\dot{V}_3 &\le e^{2kt}\left[g^T(x)R_1g(x) - (1-\mu)e^{-2k\bar{h}}g^T(x(t-h(t)))R_1g(x(t-h(t)))\right] \\
\dot{V}_4 &= e^{2kt}x^TR_2x - (1-\dot{h}(t))e^{2k(t-h(t))}x^T(t-h(t))R_2x(t-h(t)) \\
&\le e^{2kt}\left[x^TR_2x - (1-\mu)e^{-2k\bar{h}}x^T(t-h(t))R_2x(t-h(t))\right], \qquad (12)
\end{aligned}$$

where Lemma 2 was utilized in obtaining an upper bound of $\dot{V}_2$.
The time-derivatives of V_5 is obtained as

$$\begin{aligned}
\dot{V}_5 &= e^{2kt}\left[\bar{h}\dot{x}^T(t)R_3\dot{x}(t) - \int_{t-\bar{h}}^{t} e^{2k(s-t)}\dot{x}^T(s)R_3\dot{x}(s)ds\right] \\
&\le e^{2kt}\left[\bar{h}\dot{x}^T(t)R_3\dot{x}(t) - e^{-2k\bar{h}}\int_{t-\bar{h}}^{t}\dot{x}^T(s)R_3\dot{x}(s)ds\right]. \qquad (13)
\end{aligned}$$

Here, by utilizing Lemma 1, we obtain

$$\begin{aligned}
&-\int_{t-\bar{h}}^{t}\dot{x}^T(s)R_3\dot{x}(s)ds \le -\int_{t-h(t)}^{t}\dot{x}^T(s)R_3\dot{x}(s)ds \\
&\le \zeta^T\widetilde{F}\zeta + \bar{h}\zeta^TF^TR_3^{-1}F\zeta, \qquad (14)
\end{aligned}$$

where ζ is defined in (8).
Thus, we have a new upper bound of V_5 as follows:

$$\dot{V}_5 \le e^{2kt}\left[\bar{h}\dot{x}^T(t)R_3\dot{x}(t) + e^{-2k\bar{h}}\zeta^T\widetilde{F}\zeta + e^{-2k\bar{h}}\bar{h}\zeta^TF^TR_3^{-1}F\zeta\right]. \quad (15)$$

As a tool of deriving a less conservative stability criterion, we add the following two zero equation with any matrices $N_i (i = 1, ..., 6)$ and $M_i (i = 1, ..., 6)$ to be chosen as

$$2e^{2kt}\Big[x^TN_1 + x^T(t-h(t))N_2 + \left(\int_{t-h(t)}^{t}\dot{x}(s)ds\right)^T N_3 + \dot{x}^TN_4 + g^T(x)N_5$$
$$+g^T(x(t-h(t)))N_6\Big] \times [-\dot{x} + Ax + Wg(x(t)) + W_1g(x(t-h(t)))] = 0,$$

$$2e^{2kt}\Big[x^TM_1 + x^T(t-h(t))M_2 + \left(\int_{t-h(t)}^{t}\dot{x}(s)ds\right)^T M_3 + \dot{x}^TM_4 + g^T(x)$$
$$\times M_5 + g^T(x(t-h(t)))N_6\Big] \times \left[x - x(t-h(t)) - \int_{t-h(t)}^{t}\dot{x}(s)ds\right] = 0. \quad (16)$$

This can be represented as

$$e^{2kt}\zeta^T(\Xi_1 + \Xi_2)\zeta(t) = 0, \quad (17)$$

where

$$\Xi_1 = \begin{bmatrix} -N_1A - AN_1^T & -A^TN_2^T & -A^TN_3^T & -N_1 - A^TN_4^T & N_1W - A^TN_5^T & N_1W_1 - A^TN_6^T \\ \star & 0 & 0 & -N_2 & N_2W & N_2W_1 \\ \star & \star & 0 & -N_3 & N_3W & N_3W_1 \\ \star & \star & \star & -N_4 - N_4^T & N_4W - N_5^T & N_4W_1 - N_6^T \\ \star & \star & \star & \star & N_5W + W^TN_5^T & N_5W_1 + WN_6^T \\ \star & \star & \star & \star & \star & N_6W_1 + W_1^TN_6^T \end{bmatrix},$$

and

$$\Xi_2 = \begin{bmatrix} M_1 + M_1^T & -M_1 + M_2^T & -M_1 + M_3^T & M_4^T & M_5^T & M_6^T \\ \star & -M_2 - M_2^T & -M_2 - M_3^T & -M_4^T & -M_5^T & M_6^T \\ \star & \star & -M_3 - M_3^T & -M_4^T & -M_5^T & M_6^T \\ \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & 0 \end{bmatrix}.$$

Eq. (6) means that

$$g_j(x_j(t))\Big[g_j(x_j(t) - l_jx_j(t)\Big] \le 0 \ (j = 1, ..., n), \quad (18)$$

and

$$g_j(x_j(t-h(t)))\Big[g_j(x_j(t-h(t)) - l_jx_j(t-h(t))\Big] \le 0 \ (j = 1, ..., n). \quad (19)$$

From the above two inequalities (18) and (19), for any diagonal positive matrices $H_1 = diag\{h_{11}, ..., h_{1n}\}$ and $H_2 = diag\{h_{21}, ..., h_{2n}\}$, the following inequalities hold

$$0 \le -2e^{2kt}\sum_{j=1}^{n} h_{1j}g_j(x_j(t))\Big[g_j(x_j(t)) - l_jx_j(t)\Big]$$

$$
\begin{aligned}
& -2e^{2kt}\sum_{j=1}^{n} h_{2j}g_j(x_j(t-h(t)))\left[g_j(x_j(t-h(t))) - l_j x_j(t-h(t))\right] \\
= \;& 2e^{2kt}\Big[x^T L H_1 g(x) - g^T(x) H_1 g(x) + x^T(t-h(t)) L H_2 g(x(t-h(t))) \\
& -g^T(x(t-h(t))) H_2 g(x(t-h(t)))\Big]. \qquad (20)
\end{aligned}
$$

From (11)-(15) and adding (17) and (20), the time derivative of V has a new upper bound as

$$
\begin{aligned}
& \dot{V}(t) \le e^{2kt}\Big[x^T(2kP - PA - A^T P + PG + G^T P)x - 2x^T PGx(t-h(t)) \\
& -2x^T PG\int_{t-h(t)}^{t}\dot{x}(s)ds + 2x^T PWg(x) + 2x^T PW_1 g(x(t-h(t)))\Big] \\
& +e^{2kt}\left[4kg^T(x)Dx + 2g^T(x)D\dot{x}(t)\right] + x^T(t-h(t))LH_2 g(x(t-h(t))) \\
& +e^{2kt}\Big[g^T(x)R_1 g(x) - (1-\mu)e^{-2k\bar{h}}g^T(x(t-h(t)))R_1 g(x(t-h(t)))\Big] \\
& +e^{2kt}\Big[x^T R_2 x - (1-\mu)e^{-2k\bar{h}}x^T(t-h(t))R_2 x(t-h(t))\Big] \\
& +e^{2kt}\Big[\bar{h}\dot{x}^T R_3 \dot{x} + e^{-2k\bar{h}}\zeta^T\widetilde{F}\zeta + e^{-2k\bar{h}}\bar{h}\zeta^T F^T R_3^{-1}F\zeta\Big] \\
& +e^{2kt}\zeta^T(\Xi_1 + \Xi_2)\zeta(t) + 2e^{2kt}\Big[x^T(t)LH_1 g(x(t)) - g^T(x)H_1 g(x) \\
& -g^T(x(t-h(t)))H_2 g(x(t-h(t)))\Big] \\
= \;& e^{2kt}\zeta^T\Big(\Omega + e^{-2k\bar{h}}\bar{h}\zeta^T F^T R_3^{-1}F\Big)\zeta, \qquad (21)
\end{aligned}
$$

where

$$
\Omega = \begin{bmatrix}
(1,1) & (1,2) & (1,3) & \Sigma_{14} & \Sigma_{15} & \Sigma_{16} \\
\star & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} & \Sigma_{25} & \Sigma_{26} \\
\star & \star & \Sigma_{33} & \Sigma_{34} & \Sigma_{35} & \Sigma_{36} \\
\star & \star & \star & \Sigma_{44} & \Sigma_{45} & \Sigma_{46} \\
\star & \star & \star & \star & \Sigma_{55} & \Sigma_{56} \\
\star & \star & \star & \star & \star & \Sigma_{66}
\end{bmatrix}
$$

and

$$
\begin{aligned}
(1,1) &= 2kP + P(-A+G) + (-A+G)^T P + R_2 - N_1 A - A^T N_1^T + M_1 + M_1^T, \\
(1,2) &= -PG - A^T N_2^T - M_1 + M_2^T, \\
(1,3) &= -PG + e^{-2k\bar{h}}F_1^T - A^T N_3^T - M_1 + M_3^T.
\end{aligned}
$$

By defining $Y_1 = PG$ and using Fact 1, the inequality $\Omega + e^{-2k\bar{h}}\bar{h}F^T R_3^{-1}F < 0$, which guarantees the stability of the system (1) by the Lyapunov stability theory, is equivalent to the LMI (9). Since the matrix Σ given in Theorem 1 is the negative definite matrix, we have $\dot{V} \le 0$. This result leads $V \le V(0)$.

Then, we have the followings:

$$V(0) = x^T(0)Px(0) + 2\sum_{i=1}^{n} d_i \int_0^{x_i(0)} g_i(s)ds + \int_{-h(0)}^{0} e^{2ks} g^T(x(s))R_1$$
$$\cdot g(x(s))ds + \int_{-h(0)}^{0} e^{2\alpha s} x^T(s)R_2 x(s)ds + \int_{-\bar{h}}^{0}\int_s^0 e^{2\alpha s}\dot{x}^T(u)R_3\dot{x}(u)duds.$$

Further we have

$$\begin{aligned}
V(0) &\le \lambda_M(P)\|\phi\|^2 + 2d_M L_M\|\phi\|^2 + \lambda_M(R_1)L_M^2 \int_{-\bar{h}}^{0} e^{2\alpha s}x^T(s)x(s)ds \\
&\quad +\lambda_M(R_2)\int_{-\bar{h}}^{0} e^{2\alpha s}x^T(s)x(s)ds + \lambda_M(R_3)\int_{-\bar{h}}^{0}\int_s^0 e^{2\alpha s}\dot{x}^T(u)\dot{x}(u)duds \\
&\le \lambda_M(P)\|\phi\|^2 + 2d_M L_M\|\phi\|^2 + \lambda_M(R_1)L_M^2\|\phi\|^2 \int_{-\bar{h}}^{0} e^{2\alpha s}ds \\
&\quad +\lambda_M(R_2)\|\phi\|^2 \int_{-\bar{h}}^{0} e^{2\alpha s}ds + \lambda_M(R_3)\int_{-\bar{h}}^{0}\int_s^0 e^{2\alpha s}\dot{x}^T(u)\dot{x}(u)duds \\
&= \lambda_M(P)\|\phi\|^2 + 2d_M L_M\|\phi\|^2 + (\lambda_M(R_1)L_M^2 + \lambda_M(R_2))\|\phi\|^2 \frac{1-e^{2k\bar{h}}}{2k} \\
&\quad +\lambda_M(R_3)\int_{-\bar{h}}^{0}\int_s^0 e^{2\alpha s}\dot{x}^T(u)\dot{x}(u)duds, \qquad (22)
\end{aligned}$$

where $d_M = \max_i(d_i)$, $L_M = \max_i(L_i)$, and $\|\phi\| = \sup_{-\bar{h}\le\theta\le 0}\|x(\theta)\|$.
It follows from Fact 1 that

$$\begin{aligned}
\dot{x}^T(u)\dot{x}(u) &\le 3x^T(u)^T A^T Ax(u) + 3g^T(x(u))W^T Wg(x(u)) \\
&\quad + 3g^T(x(t-h(u)))W_1^T W_1 g(x(t-h(u))) \\
&\le 3\lambda_M(A^T A)\|\phi\|^2 + 3\lambda_M(W^T W)L_M^2\|\phi\|^2 \\
&\quad +3\lambda_M(W_1^T W_1)L_M^2\|\phi\|^2. \qquad (23)
\end{aligned}$$

From the relationship (23) and simple calculation, we further have

$$\begin{aligned}
V(0) \le & \Bigg[\lambda_M(P) + 2d_M L_M + \big(\lambda_M(R_1)L_M^2 + \lambda_M(R_2)\big)\frac{1-e^{2k\bar{h}}}{2\alpha} \\
& +\Big(3\lambda_M(A^T A) + 3\lambda_M(W^T W)L_M^2 + 3\lambda_M(W_1^T W_1)L_M^2\Big) \\
& \cdot\lambda_M(R_3)\frac{1-e^{2k\bar{h}}}{2\alpha}\Bigg]\|\phi\|^2 \equiv \gamma_1\|\phi\|^2.
\end{aligned}$$

Furthermore, we have $V \ge e^{2kt}\lambda_m(P)\|x(t)\|^2$. Then, we can easily obtain

$$e^{2kt}\lambda_m(P)\|x(t)\|^2 \le V(0) \le \gamma_1\|\phi\|^2 \qquad (24)$$

which leads to

$$\|x(t)\| \leq \sqrt{\frac{\gamma_1}{\lambda_m(P)}} e^{-kt}\|\phi\| \equiv \gamma_2 e^{-kt}\|\phi\|, \tag{25}$$

with the definition of $\gamma_2 \geq 1$. Thus by Definition 1, system (1) is exponentially stable with the exponential convergence rate k. This completes our proof. □

Remark 1. Since the LMIs (9) in Theorem 1 can be easily solved by various efficient convex algorithms. In this paper, we utilize Matlab's LMI Control Toolbox [26] which implements the interior-point algorithm. This algorithm is significantly faster than classical convex optimization algorithms [25].

Remark 2. By iteratively solving the LMIs given in Theorem 1 with respect to $\bar{h}$ for fixed exponential decay rate k, one can find the maximum upper bound of time delay $\bar{h}$ for guaranteeing asymptotic stability of system (1).

Remark 3. In [17]-[19], the additional condition $\dot{h}(t) \leq \mu < 1$ is required to guarantee the stability of DCNNs with time-varying delays. However, our criterion do not need this condition, which is more general cases than the previous in other literaure.

Remark 4. When the bound of time-delay derivative μ is unknown, we can obtain a delay-dependent stability criterion using similar method in Theorem 1. With the Lyapunov functional candidate,

$$\begin{aligned} V &= e^{2kt}x^T(t)Px(t) + 2\sum_{i=1}^{n} d_i e^{2kt}\int_0^{x_i(t)} g_i(s)ds \\ &\quad + \int_{t-\bar{h}}^{t}\int_s^t e^{2ku}\dot{x}^T(u)R_3\dot{x}(u)duds, \end{aligned}$$

the delay-dependent stability criterion can be obtained by letting $R_1 = R_2 = 0$ in Theorem 1.

4. Numerical Example

Example 1. Consider following cellular neural networks with time-varying delays [21]

$$\dot{y}(t) = -Ay(t) + Wf(y(t)) + W_1 f(y(t-h(t))),$$

where $A = diag\{0.7, 0.7\}$ and

$$W = \begin{bmatrix} -0.3 & 0.3 \\ 0.1 & -0.1 \end{bmatrix},\ W_1 = \begin{bmatrix} 0.1 & 0.1 \\ 0.3 & 0.3 \end{bmatrix},\ L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In [21], the exponential stability of the above system with unknown time-delay derivative was investigated. The obtained maximum admissible time delay for stability was $h = 0.29145$ in [21], while, by applying Theorem 1 and Remark

4 in this paper, we have $\bar{h} = 2.1570$. Moreover, the exponential decay rate in [21] was $k = 0.04884$. However, if we fix the delay bound $\bar{h}$ as 1, our stability criterion presents $k = 0.1342$, which means that our criterion gives larger bounds of time-delay and exponential decay rate.

Example 2. Consider the system (1) with the following system parameters:

$$A = diag\{4, 4\},\ W = \begin{bmatrix} 0.7 & -0.8 \\ 0.6 & 0.5 \end{bmatrix},\ W_1 = \begin{bmatrix} 0.4 & -0.6 \\ -0.4 & 0.4 \end{bmatrix},\ L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Table 1 shows a comparison of our result with other ones. From Table 1, we can see our obtained maximum allowable time-delay bounds which guarantees the asymptotic stability of the above system 1 with the same exponential decay rate $k = 0.7764$ is larger than results in other literature.

TABLE 1. Stability bounds of time-delay with $\mu = 0.001$ and $k = 0.7764$

	[22]	[20]	Our result
Stability bounds	1	1.32	1.48

5. Conclusion

In this paper, the problems of exponential stability and exponential convergence rate criterion for cellular neural networks with time-varying delays have been studied. By constructing a suitable Lyapunov-Krasovskii functionals and based on LMI framework, the delay-dependent exponential stability criterion are derived. Two numerical examples are included to show that our proposed method provides a larger time-delay bound and convergence rates than other results.

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