J. Appl. Math. & Informatics Vol. 26(2008), No. 5 - 6, pp. 945 - 959
Website: http://www.kcam.biz

n-ARY P-H_v-GROUPS

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ABSTRACT. n-ary H_v-structures is a generalisation of both n-ary structures and H_v-structures. A wide class of n-ary H_v-groups is the n-ary P-H_v-groups that is concidered in this paper. In this paper the notion of a normal subgroup of an n-ary P-H_v-group is introduced and the isomorphism theorems for n-ary P-H_v-groups are stated and proved. Also some examples and related properties are investigated.

AMS Mathematics Subject Classification : 20N20, 20N15.
Key words and phrases : n-ary H_v-group, n-ary P-H_v-group, n-ary P-H_v-normal subgroup, quotient n-ary P-H_v-group, isomorphism theorems.

1. Introduction

Since Marty [8] had introduced the basic concepts of hyperstructures several authors have studied about them. H_v-structures which is a larger class of hyperstructures introduced by T. Vougiouklis in [13]. P-H_v-structures which is a subclasses of H_v-structures studied in [10] by S. Spartalis and T. Vougiouklis. The reader will found in [9, 11, 12] a deep discussion on P-hyperstructures theory. The notion of an n-ary group was introduced by Dornte in [3], which is a natural generalization of the notion of a group. Since then many papers concerning various n-ary algebra have appeared in the literature, for example see [3-6]. The notion of n-ary hypergroups is defined and considered by Davvaz and Vougiouklis in [2], which is a generalization of hypergroups in the sense of Marty and a generalization of n-ary groups. It is introduced the notion of an n-ary H_v-group in [7].

This paper deals with a certain algebraic system as a subclass of n-ary H_v groups which is called n-ary P-H_v-groups. The notion of a normal subgroup of an n-ary P-H_v-group is introduced and the isomorphism theorems for n-ary P-H_v-groups are stated and proved. Also some examples and related properties are investigated.

2. n-ary P-H_v-Groups

Received July 14, 2007. Accepted January 29, 2008. *Corresponding author.

Let H be a non-empty set and H^n be the Cartesian product $H \times \cdots \times H$ where H appears n times. An element of H^n will be denoted by $(x_1, \cdots, x_n)$. In general, a mapping

$$f : H^n \to \mathcal{P}^*(H)$$

is called an n-ary hyperoperation and n is called the arity of the hyperoperation, where $\mathcal{P}^*(H)$ is the set of all non-empty subsets of H. Let f be an n-ary hyperoperation on H. For $A_1, \cdots, A_n$ subsets of H, it is defined

$$f(A_1, \cdots, A_n) = \bigcup_{x_i \in A_i} f(x_1, \cdots, x_n).$$

We shall use the following abbreviated notation:

The sequence $x_i, x_{i+1}, \cdots, x_j$ will be denoted by x_i^j. For $j < i$, x_i^j is the empty set. In this convention $f(x_1, \cdots, x_i, y_{i+1}, \cdots, y_j, z_{j+1}, \cdots, z_n)$ will be written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$.

If $m = k(n-1)+1$, the m-ary hyperoperation h given by

$$h(x_1^{k(n-1)+1}) = \underbrace{f\Big(f(\cdots f(}_{k}f(x_1^n), x_{n+1}^{2n-1}), \cdots, x_{(k-1)(n-1)+2}^{k(n-1)+1})\Big)$$

will be denoted by $f_{(k)}$. In certain situations, when the arity n does not play a crucial role, or when it will differ depending on additional assumptions, we write $f_{(.)}$ to mean $f_{(k)}$ for some $k = 1, 2, \cdots$.

According to [2] a non-empty set H with an n-ary hyperoperation $f : H^n \to \mathcal{P}^*(H)$ is an n-ary semihypergroup if the associativity is valid, i.e. for all $x_1, \cdots, x_{2n-1} \in H$ and $i, j \in \{1, \cdots, n\}$

$$f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n+i-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1}).$$

Moreover, if for all $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n, b \in H$ and $1 \le i \le n$, there exists $x_i \in H$ such that

$$b \in f(a_1^{i-1}, x_i, a_{i+1}^n)$$

then (H, f) is called an n-ary hypergroup. This axiom is called reproduction axiom.

An element e of H is called identity(neutral) element if

$$f(\underbrace{e, \cdots, e}_{i-1}, x, \underbrace{e, \cdots, e}_{n-1})$$

includes x, for any $x \in H$ and $1 \le i \le n$.

The n-ary hyperstructure (H, f) will be called an n-ary H_v-semigroup[7] if the weak associativity is valid, i.e. for all $x_1, \cdots, x_{2n-1} \in H$,

$$\bigcap_{1 \le i \le n} f\Big(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}\Big) \neq \phi.$$

And if the reproduction axiom is valid then (H, f) is called an n-ary H_v-group. The reproduction axiom can be formulated by $f(a_1^{i-1}, H, a_{i+1}^n) = H$.

Let (H, f) be an n-ary H_v-group(semigroup) and $B \subseteq H$, then B is called an H_v-subgroup(subsemigroup) of H if and only if (B, f) is an n-ary H_v-group(semigroup).

Lemma 2.1. [7] *Let B be a subset of the n-ary H_v-group (H, f); then*

(i) B is an H_v-subsemigroup of H if and only if $f(a_1^{i-1}, H, a_{i+1}^n) \subseteq H$,

(ii) B is an H_v-subgroup of H if and only if $f(a_1^{i-1}, H, a_{i+1}^n) = H$,

for all $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n \in H$, and $1 \leq i \leq n$.

Definition 2.2. Consider the semigroup $(H, \cdot)$ with center $Z(H)$ and suppose $\phi \neq P \subseteq H$. For $k = 0, 1, \cdots, n$ it is defined the map $P_k^* : H^n \to \mathcal{P}^*(H)$ as follows:

$$\begin{aligned} P_0^*(x_1^n) &= Px_1 \cdots x_n, \\ P_k^*(x_1^n) &= x_1 \cdots x_k P x_{k+1} \cdots x_n, \text{ for } 1 \leq k \leq n, \end{aligned}$$

for $x_1, \cdots, x_n \in H$. It is clear that P_k^* is an n-ary hyperoperation which is called an *n-ary P-hyperoperation* and if $P \subseteq Z(H)$ then

$$P_k^* = P_s^*, \text{ for } k, s \in \{0, 1, \cdots, n\}.$$

We denote the P-hyperoperation P_0^* by P^*. If algebraic structure (H, P_k^*) is an n-ary H_v-group(semigroup), then it is called an n-ary P-H_v-group(semigroup) over $(H, \cdot)$.

Example 2.3. Let $(H, \cdot)$ be a group with identity element e. If $P = \{e\}$ then for $P^*(= e^*$ say) we have

$$e^*(x_1^n) = \{x_1 \cdots x_n\}.$$

So, (H, e^*) is an n-ary P-H_v-group and also n-ary group, which is called the simple n-ary group over $(H, \cdot)$.

Lemma 2.4. *For $k = 0, 1, \cdots, n$ and semigroup $(H, \cdot)$, consider the algebraic structure (H, P_k^*),*

(i) if $P \subseteq Z(H)$ then it is an n-ary semihypergroup,

(ii) if $P \cap Z(H) \neq \phi$ then it is an n-ary P-H_v-group if and only if $(H, \cdot)$ is a group.

Proof. (*i*) Suppose $x_1, \cdots, x_{2n-1} \in H$ and $1 \leq i, j \leq n$. Then

$$\begin{aligned} P_k^*(x_1^{i-1}, P_k^*(x_i^{n+i-1}), x_{n+i}^{2n-1}) &= PPx_1 \cdots x_{2n-1}, \text{ since } P \subseteq Z(H) \\ &= P_k^*(x_1^{j-1}, P_k^*(x_j^{n+j-1}), x_{n+j}^{2n-1}). \end{aligned}$$

This shows that P_k^* is associative and (H, P_k^*) is an n-ary semihypergroup.

(*ii*) Suppose (H, P_k^*) is an n-ary P-H_v-group. Then by reproduction axiom for every $x_3, \cdots, x_n, h \in H$ we have

$$H = P_k^*(x_3^n, H, h) = x_3 \cdots x_{k+2} P x_{k+3} \cdots x_n H h \subseteq Hh.$$

Therefore $Hh = H$ and we can get that $(H, \cdot)$ is a group.

Conversely; suppose $(H,\cdot)$ is a group. If $a \in P \cap Z(H)$ then for sequence x_1^{2n-1} in H we have

$$aax_1\cdots x_{2n-1} \in \bigcap_{1\leq i\leq n} P_k^*(x_1^{i-1}, P_k^*(x_i^{n+i-1}), x_{n+i}^{2n-1}).$$

So (H, P_k^*) is an n-ary P-H_v-semigroup. Since $(H,\cdot)$ is a group and $P \subseteq H$ we have

$$x_1\cdots x_k P x_{k+1}\cdots x_{i-1} H x_{i+1}\cdots x_n = H.$$

i.e., $P_k^*(x_1^i, H, x_{i+1}^n) = H$. Therefore (H, P_k^*) is an n-ary P-H_v-group. □

Corollary 2.5. *Suppose $(H,\cdot)$ is a group. For $k = 0, 1, \cdots, n$,*
(i) if $e \in P$, then (H, P_k^*) is an n-ary P-H_v-group,
(ii) if $P \subseteq Z(H)$, then (H, P_k^*) is an n-ary hypergroup.

Proof. (i) Since $e \in P$, $P \cap Z(H) \neq \phi$, by (ii) of Lemma 2.3 (H, P_k^*) is an n-ary H_v-group.

(ii) By (i) of Lemma 2.3 (H, P_k^*) is an n-ary semihypergroup. The proof will be completed as the proof of (ii) of Lemma 2.4. □

Theorem 2.6 [7]. *Consider the P-H_v-group (H, P^*) over the group $(H,\cdot)$. Suppose $e \in P$ and $B \subseteq H$. Then (B, P^*) is an n-ary P-H_v-subgroup of (H, P^*) if and only if $P \subseteq B$ and B is a subgroup of $(H,\cdot)$.*

3. Quotient n-ary P-H_v-groups

Throughout in this section $P \subseteq H$, $P \cap Z(H) \neq \phi$, (H, P^*) is an n-ary P-H_v-group over the group $(H,\cdot)$ and (L, P^*) is an n-ary P-H_v-subgroup of H, so by Theorem 2.6, $P \subseteq L$ and $(L,\cdot)$ is a subgroup of $(H,\cdot)$. Also, for n-ary hyperoperation f, we denote the $f(x_1^i, \underbrace{y, \cdots, y}_{n-i})$ by $f(x_1^i, y_.)$, for $x_1, \cdots, x_i,\ y \in H$ and $1 \leq i < n$.

Definition 3.1. An n-ary P-H_v-subgroup (N, P^*) of (H, P^*) is called an *n-ary P-H_v-normal subgroup of H* if for $a_2, \cdots, a_n \in H$ and $2 \leq i, j \leq n$,

$$P^*(a_2^i, N, a_{i+1}^n) = P^*(a_2^j, N, a_{j+1}^n).$$

Theorem 3.2. *Let $N \subseteq H$ and $P \subseteq Z(H) \cap N$. Then (N, P^*) is an n-ary P-H_v normal subgroup of (H, P^*) if and only if $(N,\cdot)$ is a normal subgroup of $(H,\cdot)$.*

Proof. Suppose (N, P^*) is an n-ary P-H_v-normal subgroup of (H, P^*) then by Theorem 2.6, N is a subgroup of H. If $a \in H$ then

$$\begin{aligned} aN &= aPN, \text{ since } P \subseteq N \text{ and } (N,\cdot) \text{ is a group} \\ &= PaN, \text{ since } P \subseteq Z(H) \\ &= P^*(a, N, e.) \end{aligned}$$

$$
\begin{aligned}
&= P^*(N, a, e.), \text{ since } (N, P^*) \text{ is an } n\text{-ary } H_v\text{-normal subgroup} \\
&= PNa \\
&= Na, \text{ since } P \subseteq N \text{ and } (N, \cdot) \text{ is a group.}
\end{aligned}
$$

Therefore $(N, \cdot)$ is a normal subgroup of $(H, \cdot)$.

Conversely, let $(N, \cdot)$ be a normal subgroup of $(H, \cdot)$ and $a_2, \cdots, a_n$ in H then for every $2 \leq i, j \leq n$ we have:

$$
\begin{aligned}
P^*(a_2^{i-1}, N, a_i^n) &= Pa_2 \cdots a_{i-1} N a_i \cdots a_n \\
&= Pa_2 \cdots a_{j-1} N a_j \cdots a_n, \text{ since } (N, \cdot) \text{ is a normal subgroup} \\
&= P^*(a_2^{j-1}, N, a_j^n).
\end{aligned}
$$

By Theorem 2.6, the proof is completed. □

The relation θ on H^{n-1} is defined by

$$(u_2^n)\theta(v_2^n) \text{ if and only if } P^*(L, u_2^n) = P^*(L, v_2^n),$$

for sequences u_2^n, v_2^n in H^{n-1}.
It is clear that the relation θ is an equivalence relation on H^{n-1}, which is called *θ relation corresponding to L*. It is denoted the θ class of (u_2^n) in H^{n-1} by $\mathcal{P}(L, u_2^n)$ and the set includes of all θ classes by H^{n-1}/L. Because for every $u_2, \cdots, u_n \in H$ we have

$$
\begin{aligned}
P^*(L, u_2^n) &= PLu_2 \cdots u_n \\
&= P^*(L, u_2 \cdots u_n, \underbrace{e, \cdots, e}_{n-2}) \\
&= Lu_2 \cdots u_n.
\end{aligned}
$$

Hence $\mathcal{P}(L, u_2^n) = \mathcal{P}(L, u_2 \cdots u_n, e.)$, and

$$H^{n-1}/L = \{\mathcal{P}(L, h, e.) \mid h \in H\},$$

where

$$
\begin{aligned}
\mathcal{P}(L, h, e.) &= \{(u_2^n) \in H^{n-1} \mid P^*(L, u_2^n) = P^*(L, h, e.)\} \\
&= \{(u_2^n) \in H^{n-1} \mid Lu_2 \cdots u_n = Lh\}.
\end{aligned}
$$

Also the relation σ on H is defined by $x\sigma y$ if and only if there exists a sequence u_2^n in H such that $x, y \in P^*(L, u_2^n)$, for every $x, y \in H$, which is called the σ relation on H corresponding to L.

Lemma 3.3. *Consider the σ relation corresponding to L on the n-ary P-H_v-group (H, P^*),*

(i) σ relation is an equivalence relation on H,

(ii) if $x \in P^*(L, u_2^n)$, then

$$x/\sigma = P^*(L, u_2^n) = P^*(L, x, e.) = Lx,$$

where x/σ is the σ-equivalence class of x, *(iii) there is a one-one corresponding between H/σ, H^{n-1}/L and H/L, where H/σ is the set of all equivalence classes of σ.*

Proof. (i) It is clear that σ is a reflexive and symmetric relation. Suppose $x\sigma y$ and $y\sigma z$, so there exist sequences u_2^n, v_2^n in H^{n-1} such that

$$\{x,y\} \subseteq P^*(L,u_2^n),\ \{y,z\} \subseteq P^*(L,v_2^n).$$

If we set $u = u_2 \cdots u_n$ and $v = v_2 \cdots v_n$, then

$$\{x,y\} \subseteq P^*(L,u_2^n) = PLu_2 \cdots u_n = Lu \text{ and } \{y,z\} \subseteq Lv.$$

Hence

$$y \in Lu \cap Lv \neq \phi,$$
$$\{x,y,z\} \subseteq P^*(L,u_2^n) = Lu = Lv = P^*(L,v_2^n).$$

Therefore $x\sigma z$.

(ii) Let $x \in P^*(L,u_2^n)$ and $y \in x/\sigma$ then there exist $v_1,\cdots,v_n \in H$ such that $\{x,y\} \subseteq P^*(L,v_2^n)$. So

$$x \in P^*(L,u_2^n) \cap P^*(L,v_2^n).$$

This concludes

$$P^*(L,u_2^n) = P^*(L,v_2^n),\ y \in P^*(L,u_2^n),$$
$$x/\sigma \subseteq P^*(L,u_2^n).$$

Conversely, if $x \in P^*(L,u_2^n)$ then by definition of σ relation we have $P^*(L,u_2^n) \subseteq x/\sigma$. Therefore $x/\sigma = P^*(L,u_2^n)$. Moreover

$$Lx = P^*(L,x,e.) = P^*(L,u_2^n) = x/\sigma.$$

(iii) By (ii) and definition of θ relation, it is clear that there is a one-one corresponding between

$$H^{n-1}/L = \{\mathcal{P}^*(L,x,e.) \mid x \in H\} \text{ and}$$
$$H/L = \{Lx \mid x \in H\} = \{x/\sigma \mid x \in H\} = H/\sigma.$$

□

Theorem 3.4. *Let N be an n-ary P-H_v-normal subgroup of n-ary P-H_v-group (H,P^*). Then $(H^{n-1}/N, e^*/N)$ is an n-ary group where for every elements $\mathcal{P}(N,a_1,e.),\cdots,\mathcal{P}(N,a_n,e.) \in H^{n-1}/N$,*

$$e^*/N(\mathcal{P}(N,a_1,e.),\cdots,\mathcal{P}(N,a_n,e.)) = \mathcal{P}(N,a_1\cdots a_n,e.).$$

Proof. Suppose $\mathcal{P}(N,a_1,e.),\cdots,\mathcal{P}(N,a_n,e.),\mathcal{P}(N,b_1,e.),\cdots,\mathcal{P}(N,b_n,e.))$ are in H^{n-1}/N and $\mathcal{P}(N,a_i,e.) = \mathcal{P}(N,b_i,e.)$, for $i = 1,\cdots,n$. Then

$$\begin{aligned} & P^*(N,a_i,e.) = P^*(N,b_i,e.),\ \text{for } i = 1,\cdots,n \\ \Rightarrow\ & Na_i = Nb_i,\ \text{for } i = 1,\cdots,n \\ \Rightarrow\ & Na_1\cdots a_n = Nb_1\cdots b_n. \end{aligned}$$

Therefore

$$e^*/N(\mathcal{P}(N,a_1,e.),\cdots,\mathcal{P}(N,a_n,e.)) = e^*/N(\mathcal{P}(N,b_1,e.),\cdots,\mathcal{P}(N,b_n,e.))$$

and e^*/N is an n-ary operation.

It is straightforward to see that e^*/N on H^{n-1}/N is associative.

For elements $\mathcal{P}(N, a_1, e.), \cdots, \mathcal{P}(N, a_{i-1}, e.), \mathcal{P}(N, a_{i+1}, e.), \cdots, \mathcal{P}(N, a_n, e.)$, $\mathcal{P}(N, b, e.)$ in H^{n-1}/N, if

$$x = (a_1 \cdots a_{i-1})^{-1} b \, (a_{i+1} \cdots a_n)^{-1}$$

then we have

$e^*/N(\mathcal{P}(N, a_1, e.), \ \cdots, \ \mathcal{P}(N, a_{i-1}, e.), \ \mathcal{P}(N, x, e.), \ \mathcal{P}(N, a_{i+1}, e.), \cdots,$
$\mathcal{P}(N, a_n, e.))$

$$\begin{aligned} &= \mathcal{P}(N, a_1 \cdots a_{i-1} x a_{i+1} \cdots a_n), e.) \\ &= \mathcal{P}(N, a_1 \cdots a_{i-1} (a_1 \cdots a_{i-1})^{-1} b \, (a_{i+1} \cdots a_n)^{-1} a_{i+1} \cdots a_n), e.) \\ &= \mathcal{P}(N, b, e.). \end{aligned}$$

□

For any n-ary P-H_v-normal subgroup (N, P^*) of (H, P^*), N is a normal subgroup of H and $(H^{n-1}/N, e^*/N)$ induces the n-ary group $(H/N, e^*)$ which is the simple n-ary group over the ordinary quotient group $(H/N, \oplus)$.

Example 3.5. Consider $(\mathbb{Z}, +)$, the group of integers. Suppose $P \subseteq 4\mathbb{Z}$, $n = 3$ and $P^* : \mathbb{Z}^3 \to \mathcal{P}^*(\mathbb{Z})$ be the hyperoperation defined by

$$P^*(x, y, z) = P + x + y + z, \text{ for } (x, y, z) \in \mathbb{Z}^3.$$

Then $(4\mathbb{Z}, P^*)$ is a 3-ary P-H_v-subgroup of $(\mathbb{Z}, P^*)$. By Theorem 3.3

$$\mathbb{Z}/\sigma = \{4\mathbb{Z}, \ 4\mathbb{Z}+1, \ 4\mathbb{Z}+2, \ 4\mathbb{Z}+3\},$$

and $\mathbb{Z}^2/4\mathbb{Z}$ have four classes, where

$$\begin{aligned} (u_1, u_2)/\theta &= \{(x, y) \in \mathbb{Z}^2 | \ P^*(4\mathbb{Z}, x, y) = P^*(4\mathbb{Z}, u_1, u_2)\} \\ &= \{(x, y) \in \mathbb{Z}^2 | \ x + y \in 4\mathbb{Z} + (u_1 + u_2)\}. \end{aligned}$$

Thus we can consider the θ classes as the following:

$$\begin{aligned} \mathcal{P}(4\mathbb{Z}, 0, 0) = (0, 0)/\theta &= \{(x, y) \in \mathbb{Z}^2 | \ x + y \in 4\mathbb{Z}\}, \\ \mathcal{P}(4\mathbb{Z}, 1, 0) = (1, 0)/\theta &= \{(x, y) \in \mathbb{Z}^2 | \ x + y + 4\mathbb{Z} + 1\}, \\ \mathcal{P}(4\mathbb{Z}, 2, 0) = (2, 0)/\theta &= \{(x, y) \in \mathbb{Z}^2 | \ x + y \in 4\mathbb{Z} + 2\}, \\ \mathcal{P}(4\mathbb{Z}, 3, 0) = (3, 0)/\theta &= \{(x, y) \in \mathbb{Z}^2 | \ x + y \in 4\mathbb{Z} + 3\}. \end{aligned}$$

By Theorem 3.4, $(\mathbb{Z}^2/4\mathbb{Z}, e^*/4\mathbb{Z})$ is a quotient 3-ary group with above four elements.

Example 3.6. Let $(\mathbb{Z}, +)$ be the group of integers and $n = 4$. By Theorem 3.4, $(\mathbb{Z}^3/2\mathbb{Z}, e^*/2\mathbb{Z})$ is a 4-ary group with two elements $\mathcal{P}(2\mathbb{Z}, 0, 0, 0)$, $\mathcal{P}(2\mathbb{Z}, 1, 0, 0)$.

Definition 3.7. Let (H, P^*), (G, Q^*) be n-ary P-H_v-groups and $f : H \to G$ be a map. Then f is called a homomorphism of n-ary P-H_v-groups if

$$f(P^*(x_1^n)) \cap Q^*(f(x_1), \cdots, f(x_n)) \neq \phi$$

and f is called strong homomorphism if

$$f(P^*(x_1^n)) = Q^*(f(x_1), \cdots, f(x_n)).$$

Finally, if f is a bijection strong homomorphism then f is called an isomophism which is written $(H, P^*) \cong (G, Q^*)$.

Example 3.8. For $h \in Z(H)$, consider the left by h translation L_h in $(H, \cdot)$ which obviously is a one-one and onto mapping. Also

$$\begin{aligned} L_h((h^{n-1}P)^*(x_1^n)) &= h(h^{n-1}Px_1 \cdots x_n) \\ &= h^n P x_1 \cdots x_n \\ &= P(hx_1) \cdots (hx_n) \\ &= P^*(L_h(x_1), \cdots, L_h(x_n)). \end{aligned}$$

So L_h establishes an isomorphism on $(H, (h^{n-1}P)^*)$ and (H, P^*).

Lemma 3.9. *Let (H, P^*), (G, Q^*) be n-ary P-H_v-groups on $(H, \cdot)$ and $(G, \cdot)$ respectively and $f : H \to G$ be a homomorphism of groups. Then*

$$f : (H, P^*) \to (G, Q^*)$$

(i) is a homomorphism of n-ary P-H_v-groups if and only if $f(P) \cap Q \neq \phi$;
(ii) is a strong homomorphism of n-ary P-H_v-groups if and only if $f(P) = Q$.

Proof. (i) If f is a homomorphism of n-ary P-H_v-groups, then

$$\begin{aligned} f(P^*(e.)) \cap Q^*(f(e.)) \neq \phi &\Rightarrow f(Pe \cdots e) \cap Qf(e) \cdots f(e) \neq \phi \\ &\Rightarrow f(P) \cap Q \neq \phi. \end{aligned}$$

Conversely; if $q \in f(P) \cap Q$ then for every $x_1, \cdots, x_n \in P$ we have

$$\begin{aligned} qf(x_1) \cdots f(x_n) &\in f(P)f(x_1 \cdots x_n) \cap Qf(x_1) \cdots f(x_n) \\ &= f(P^*(x_1^n)) \cap Q^*(f(x_1), \cdots, f(x_n)). \end{aligned}$$

(ii) Suppose f is a strong homomorphism then

$$f(P^*(e.)) = Q^*(f(e.)) \text{ and } f(P) = Q.$$

Conversely, if $f(P) = Q$, for $x_1, \cdots, x_n \in H$, we have

$$\begin{aligned} f(P^*(x_1^n)) &= f(Px_1 \cdots x_n) \\ &= f(P)f(x_1) \cdots f(x_n) \\ &= Qf(x_1) \cdots f(x_n) \\ &= Q^*(f(x_1), \cdots, f(x_n)). \end{aligned}$$

□

Corollary 3.10. *Let $(H, \cdot)$ and $(G, \cdot)$ be two groups, and $f \in Hom(H, G)$ be onto. Then, f is a strong homomorphism from the n-ary P-H_v-groups (H, P^*) to $(G, f(P)^*)$.*

Proof. $x \in P \cap Z(H)$ implies $f(x) \in f(P) \cap Z(G)$. By (*ii*) of Lemma 2.4 (H, P^*) and $(G, f(P)^*)$ are n-ary P-H_v-groups. By (*ii*) of Lemma 3.9 the proof is completed. □

We observe that every $f \in End(H)$ induces a strong homomorphism on the n-ary P-H_v-group $(H, P^*), (H, f(P)^*)$. Similarly, every $f \in Aut(H)$ induces an isomorphism on $(H, P^*), (H, f(P)^*)$. In the special case $f(P) = P$, we have respectively or an endomorphism or an automorphism in (H, P^*). As an example of the last case we give the following:
any element x of the centralizer of P, $x \in C_H(P)$, induces an inner automorphism $h \mapsto x^{-1}hx$, which induces an automorphism on (H, P^*). Because $f(P) = x^{-1}Px = P$ and so $P^* = f(P)^*$.

Definition 3.11. We define

$$P_c^*(x_1^n) = x_n{}^{-1} \cdots x_1{}^{-1}P,$$

which is called n-ary P_c-hyperoperation. The (H, P_c^*) becomes an n-ary P-H_v-group which is called n-ary P_c-H_v-group.

Consider the surjection ${}^{-1} : H \to H,\ x \mapsto x^{-1}$. Then we have

$${}^{-1}(P^*(x_1^n)) = (Px_1 \cdots x_n)^{-1} = x_n^{-1} \cdots x_1^{-1}P^{-1} = (P^{-1})_c^*(x_1^n),$$

for $x_1, \cdots, x_n \in H$. Therefore, we have the isomorphism

$$(H, P^*) \cong (H, (P^{-1})_c^*).$$

The element $w \in H$ is called a left unit in (H, P^*) if

$$x \in P^*(w, x, e.),\ \text{for all } x \in H,$$

and is called a right unit if

$$x \in P^*(x, w, e.),\ \text{for all } x \in H.$$

But if w is a left unit in (H, P^*) then for every $x \in H$ we can find $p_x \in P$ such that $p_x w = e$. Then $w = p_x^{-1}$ and for every $y \in H$ we have

$$y \in P^*(p_x{}^{-1}, y, e.) = P^*(w, y, e.).$$

Conversely; for every $x \in H$ we have $x \in PP^{-1}x = P^*(P^{-1}, x, e.)$. Therefore all left units of (H, P^*) are elements of the set P^{-1}.

Let $x \in H$ and $p_0 \in P$. Then an element $x' \in H$ is a left inverse of x with respect to the unit p_0^{-1} if

$$p_0{}^{-1} \in P^*(x', x, e.).$$

So there must exists an element $p_x \in P$ such that $p_0{}^{-1} = p_x x' x$ and thus

$$x' \in P^{-1}p_0{}^{-1}x^{-1}.$$

Conversely; for every element $p_1{}^{-1}p_0{}^{-1}x^{-1} \in P^{-1}p_0^{-1}x^{-1}$ we have

$$p_0{}^{-1} = p_0{}^{-1}x^{-1}x \in Pp_1{}^{-1}p_0{}^{-1}x^{-1}x = P^*(p_1{}^{-1}p_0{}^{-1}x^{-1}, x, e.).$$

Therefore the left inverse elements of x with respect to the unit p_0^{-1} are exactly the elements of $P^{-1}p_0{}^{-1}x^{-1}$.

Definition 3.12. An n-ary H_v-group (G, f) with neutral element is called reversible in itself when any relation $x \in f(x_1^n)$ implies that there exist inverses $x_1{}^{-1}, \cdots, x_n{}^{-1}$ such that

$$x_i \in f(x_{i-1}{}^{-1}, \cdots, x_1{}^{-1}, x, x_n{}^{-1}, \cdots, x_{i+1}{}^{-1}),$$

for any $1 \leq i \leq n$.

Theorem 3.13. *The n-ary P-H_v-group (H, P^*) is reversible in itself.*

Proof. It is straightforward. □

Lemma 3.14. *Let $f : H \to G$ be a homomorphism, $P \subseteq K = kerf$ and $e \in Q \subseteq G$. Then f induces the homomorpkism $f : (H, P^*) \to (G, Q^*)$ of n-ary P-H_v-groups with (K, P^*) kernel which is an n-ary P-H_v-normal subgroup of (H, P^*).*

Proof. $P \subseteq K$ implies $f(P) \subseteq f(K) = e \in Q$ and $f(P) \cap Q \neq \phi$. By (i) of Lemma 3.9, f is a homomorphism of n-ary P-H_v-groups.

If $a_1, \cdots, a_n \in H$ and $1 \leq i, j \leq n$, then

$$\begin{aligned} P^*(a_2^i, K, a_{i+1}^n) &= Pa_2 \cdots a_i K a_{i+1} \cdots a_n \\ &= Pa_2 \cdots a_j K a_{j+1} \cdots a_n, \text{ since } K \text{ is normal in } H \\ &= P^*(a_2^j, K, a_{j+1}^n). \end{aligned}$$

Therefore, by (ii) of Lemma 2.4, (K, P^*) is an n-ary P-H_v-normal subgroup of H. □

Lemma 3.15. *Let $f : H \to G$ be a homomorphism of groups and $P \subseteq K = kerf$. Then for $k \in \mathbb{N}$ and $x_1, \cdots, x_{k(n-1)+1} \in H$ we have*

$$f(P^*_{(k)}(x_1, \cdots, x_{k(n-1)+1})) = f(x_1) \cdots f(x_{k(n-1)+1}).$$

Proof. If $k = 2$ and $x_1, \cdots, x_{2n-1} \in H$ then for $2 \leq i \leq n-1$

$$P^*(x_1^{i-1}, P^*(x_i^{n+i-1}), x_{n+i}^{2n-1}) = Px_1 \cdots x_{i-1} P x_i \cdots x_{n-i+1} \cdots x_{2n-1}.$$

So

$$\begin{aligned} &f(P^*(x_1^{i-1}, P^*(x_i^{n+i-1}), x_{n+i}^{2n-1})) = f(P)f(x_1) \cdots f(x_{i-1}) f(P) f(x_i) \cdots f(x_{2n-1}) \\ &\quad = f(x_1) \cdots f(x_{2n-1}), \text{ since } P \subseteq K. \end{aligned}$$

Therefore we can write

$$f(P^*_{(2)}(x_1^{2n-1})) = f(x_1) \cdots f(x_{2n-1}).$$

By using induction on k it is easy to prove that for every $k \in \mathbb{N}$

$$f(P^*_{(k)}(x_1, \cdots, x_{k(n-1)+1})) = f(x_1) \cdots f(x_{k(n-1)+1}).$$

□

Theorem 3.16. *Let $f : H \to G$ be an epimorphism of groups and $K = kerf$.* Then

$$(H^{n-1}/K, e^*/K) \cong (G, e^*).$$

Proof. We define

$$\varphi : H^{n-1}/K \to G,\ \varphi(\mathcal{P}(K,h,e.)) = f(h).$$

If $\mathcal{P}(K,h,e.),\ \mathcal{P}(K,t,e.) \in H^{n-1}/K$, then

$$\begin{array}{ll} & \mathcal{P}(K,h,e.) = \mathcal{P}(K,t,e.) \\ \Leftrightarrow & P^*(K,h,e.) = P^*(K,t,e.) \\ \Leftrightarrow & PKh = PKt \\ \Leftrightarrow & Kh = Kt,\ \text{since } P \subseteq K \\ \Leftrightarrow & f(Kh) = f(Kt),\ \text{since } K = kerf \\ \Leftrightarrow & f(K)f(h) = f(K)f(t),\ \text{since } f \text{ is a homomorphism} \\ \Leftrightarrow & f(h) = f(t),\ \text{since } K = kerf \\ \Leftrightarrow & \varphi(\mathcal{P}(K,h,e.)) = \varphi(\mathcal{P}(K,t,e.)). \end{array}$$

Therefore φ is a one-one function.

Because f is onto, for $g \in G$ there exists $h \in H$ such that $f(h) = g$, then

$$\varphi(\mathcal{P}(K,h,e.)) = f(h) = g,$$

so φ is onto.

For $\mathcal{P}(K,h_1,e.),\cdots,\mathcal{P}(K,h_n,e.) \in H^{n-1}/K$ we have

$$\begin{array}{ll} & \varphi(e^*/K(\mathcal{P}(K,h_1,e.),\cdots,\mathcal{P}(K,h_n,e.))) \\ = & \varphi(\mathcal{P}(K,h_1\cdots h_n),e.)), \\ = & f(h_1\cdots h_n) \\ = & f(h_1)\cdots f(h_n),\ \text{by Lemma 3.9} \\ = & e^*(f(h_1),\cdots,f(h_n)) \\ = & e^*(\varphi(\mathcal{P}(K,h_1,e.)),\cdots,\varphi(\mathcal{P}(K,h_n,e.))). \end{array}$$

Therefore φ is an isomorphism of n-ary groups. □

Theorem 3.17. *If N be a normal subgroup of $(H,.)$, then*

$$(H^{n-1}/N, e^*/N) \cong (H/N, e^*).$$

Proof. We consider the natural epimorphism

$$f : H \longrightarrow H/N.$$

Because $N = kerf$, by Theorem 3.16, $(H^{n-1}/N, e^*/N) \cong (H/N, e^*)$. □

Example 3.18. Suppose $k,n \in I\!N_0$ and $f : \mathbb{Z} \to \mathbb{Z}_k,\ f(z) = [z]$. Then, by Theorem 3.16, $(\mathbb{Z}_k, e^*)$ and $(\mathbb{Z}^{n-1}/k\mathbb{Z}, e^*/k\mathbb{Z})$ are isomorph as two n-ary groups where

$$\mathbb{Z}^{n-1}/k\mathbb{Z} = \{\mathcal{P}(k\mathbb{Z},t,0.) \mid t \in \mathbb{Z}\},\ \text{and}$$

$$\mathcal{P}(k\mathbb{Z},t,0.) = \{(u_2,\cdots,u_n) \in \mathbb{Z}^{n-1} \mid u_2+\cdots+u_n \in k\mathbb{Z}+t\}.$$

Theorem 3.19. *Let N, K be normal subgroups of H and $N \subseteq K$. Then*

$$(H^{n-1}/N)^{n-1}/(K^{n-1}/N) \cong H^{n-1}/K.$$

Proof. Suppose $\mathcal{P}(N,h,e.) \in H^{n-1}/N$ and define

$$\begin{aligned}\varphi : (H^{n-1}/N)^{n-1}/(K^{n-1}/N) &\rightarrow H^{n-1}/K,\\ \varphi(\mathcal{P}/N(K^{n-1}/N, \mathcal{P}(N,h,e.),N.) &= \mathcal{P}(K,h,e.).\end{aligned}$$

If the elements $\mathcal{P}/N(K^{n-1}/N, \mathcal{P}(N,h,e.),N.), \mathcal{P}/N(K^{n-1}/N, \mathcal{P}(N,h',e.),N.)$ are in $(H^{n-1}/N)^{n-1}/(K^{n-1}/N)$, then

$$\begin{aligned}&\mathcal{P}/N(K^{n-1}/N, \mathcal{P}(N,h,e.),N.) = \mathcal{P}/N(K^{n-1}/N, \mathcal{P}(N,h',e.),N.)\\ \Leftrightarrow\ & K/N(Nh) = K/N(Nh'), \text{ by } (iii) \text{ of Lemma 3.3}\\ \Leftrightarrow\ & Nh(Nh')^{-1} \in K/N\\ \Leftrightarrow\ & h(h')^{-1} \in K\\ \Leftrightarrow\ & Kh = Kh'\\ \Leftrightarrow\ & \mathcal{P}(K,h,e.) = \mathcal{P}(K,h',e.)\\ \Leftrightarrow\ & \varphi(\mathcal{P}/N(K^{n-1}/N, \mathcal{P}(N,h,e.),N.)) = \varphi(\mathcal{P}/N(K^{n-1}/N, \mathcal{P}(N,h',e.),N.)).\end{aligned}$$

Therefore, φ is a one-one function.
Now, for $\mathcal{P}/N(K^{n-1}/N, \mathcal{P}(N,h_1,e.),N.), \cdots, \mathcal{P}/N(K^{n-1}/N, \mathcal{P}(N,h_n,e.),N.)$ in $(H^{n-1}/N)^{n-1}/(K^{n-1}/N)$ we have

$$\begin{aligned}&\varphi(e^*/(K^{n-1}/N)(\mathcal{P}/N(K^{n-1}/N, \mathcal{P}(N,h_1,e.),N.), \cdots,\\ &\mathcal{P}/N(K^{n-1}/N, \mathcal{P}(N,h_n,e.),N.)))\\ &= \varphi(\mathcal{P}/N(K^{n-1}/N, e^*/N(\mathcal{P}(N,h_1,e.), \cdots, \mathcal{P}(N,h_n,e.)),N.)\\ &= \varphi(\mathcal{P}/N(K^{n-1}/N, \mathcal{P}(N,h_1\cdots h_n,e.),N.))\\ &= \mathcal{P}(K,h_1\cdots h_n,e.)\\ &= e^*/K(\mathcal{P}(K,h_1,e.), \cdots, \mathcal{P}(K,h_n,e.))\\ &= e^*/K(\varphi(\mathcal{P}/N(K^{n-1}/N, \mathcal{P}(N,h_1,e.),N.)), \cdots,\\ &\varphi(\mathcal{P}/N(K^{n-1}/N, \mathcal{P}(N,h_n,e.),N.))).\end{aligned}$$

It is clear φ is onto. So φ is an isomorphism. □

Theorem 3.20. *Let $L_1, \cdots, L_n$ be subgroups of H such that L_1 is normal and $\ell_i\ell_j = \ell_j\ell_i$, for $\ell_i \in L_i$, $\ell_j \in L_j$, $1 \leq i, j \leq n$. Then*

$$(e^*(L_1^n))^{n-1}/L_1 \cong (e^*(e, L_2^n))^{n-1}/e^*(e, L_2^n) \cap L_1.$$

Proof. We set $L = L_2 \cdots L_n$. Then L is a subgroup of $(H,\cdot)$. Similarly, $e^*(L_1^n) = L_1L$ and $L \cap L_1$ are subgroups of $(H,\cdot)$ and $L \cap L_1$ is normal in L. If

$$T \in (e^*(L_1^n))^{n-1}/L_1,\ U \in L^{n-1}/L \cap L_1,$$

then for some $\ell, \ell' \in L, \ell_1 \in L_1$, $T = \mathcal{P}(L_1, \ell_1\ell, e.)$ and $U = \mathcal{P}(L \cap L_1, \ell', e.)$. We Define

$$\varphi : (e^*(L_1^n))^{n-1}/L_1 \rightarrow (e^*(e, L_2^n))^{n-1}/e^*(e, L_2^n) \cap L_1$$

$$\varphi(T) = \varphi(\mathcal{P}(L_1, \ell_1\ell, e.)) = \mathcal{P}(L \cap L_1, \ell, e.).$$

For $T_1 = \mathcal{P}(L_1, \ell_1\ell, e.)$, $T_2 = \mathcal{P}(L_1, \ell_1'\ell', e.) \in L/L_1$ we have

$$\begin{aligned}
T_1 = T_2 \quad &\Leftrightarrow \quad L_1\ell_1\ell = L_1\ell_1'\ell', \text{ by } (iii) \text{ of Lemma 3.3}\\
&\Leftrightarrow \quad L_1\ell = L_1\ell'\\
&\Leftrightarrow \quad \ell(\ell')^{-1} \in L_1 \cap L\\
&\Leftrightarrow \quad (L_1 \cap L)\ell = (L_1 \cap L)\ell'\\
&\Leftrightarrow \quad \mathcal{P}(L_1 \cap L, \ell, e.) = \mathcal{P}(L_1 \cap L, \ell', e.), \text{ by } (iii) \text{ of Lemma 3.3}\\
&\Leftrightarrow \quad \varphi(\mathcal{P}(L_1, \ell_1\ell, e.)) = \varphi(\mathcal{P}(L_1, \ell_1'\ell', e.)).
\end{aligned}$$

Thus, φ is a one-one function. It is clear that φ is onto.
Suppose, $\mathcal{P}(L_1, \ell_{11}\ell_1, e.), \cdots, \mathcal{P}(L_1, \ell_{1n}\ell_n, e.) \in (e^*(L_1^n))^{n-1}/L_1$ then,

$$\begin{aligned}
&\varphi(e^*/L_1(\mathcal{P}(L_1, \ell_{11}\ell_1, e.), \cdots, \mathcal{P}(L_1, \ell_{1n}\ell_n, e.)))\\
&= \varphi(\mathcal{P}(L_1, \ell_{11}\ell_1 \cdots \ell_{1n}\ell_n, e.))\\
&= \varphi(\mathcal{P}(L_1, (\ell_{11} \cdots \ell_{1n})(\ell_1 \cdots \ell_n), e.), \text{ since } L_1 \text{ is normal}\\
&= \mathcal{P}(L_1 \cap L, \ell_1 \cdots \ell_n, e.)\\
&= (e^*/(L_1 \cap L))(\mathcal{P}(L_1 \cap L, \ell_1, e.), \cdots, \mathcal{P}(L_1 \cap L, \ell_n, e.))\\
&= (e^*/(L_1 \cap L))(\varphi(\mathcal{P}(L_1, \ell_{11}\ell_1, e.)), \cdots, \varphi(\mathcal{P}(L_1, \ell_{1n}\ell_n, e.))).
\end{aligned}$$

Therefore, φ is an isomorphism. □

Definition 3.21. Let (H, f), (G, g) be n-ary and m-ary H_v-groups respectively such that $n \leq m$ and G has the e_2 neutral element. Then the map $\varphi : H \to G$ is called an (n, m)-homomorphism if

$$\varphi(f(x_1^n)) = g((\varphi(x_1), \cdots, \varphi(x_n)), e_2.).$$

Theorem 3.22. *Consider the $(H, \cdot)$ group with normal subgroup N. Then for $n > 2$*

(i) $I_H : (H, \cdot) \to (H, P^*)$ is a $(2, n)$-isomomorphism;

(ii) the map $\varphi : H/N \to H^{n-1}/N$, $\varphi(Nh) = \mathcal{P}(N, h, e.)$ is an isomorphism of quotient binary and n-ary groups. Indeed φ is a $(2, n)$-isomomorphism;

(iii) $\overline{n}_2 : (H, P^*) \to (H^{n-1}/N, \ e^*/N)$, $\overline{n}_2(x) = \mathcal{P}(N, x, e.)$ is a strong epimorphism of n-ary P-H_v-groups which is called the n-ary natural epimorphism;

(iv) the below diagram is commutative:

$$\begin{array}{ccc}
(H, \cdot) & \xrightarrow{\overline{n}} & (H/N, \cdot)\\
\downarrow I_H & & \downarrow \varphi\\
(H, e^*) & \xrightarrow{\overline{n}_2} & (H^{n-1}/N, e^*/N)
\end{array},$$

where $(H, \cdot) \xrightarrow{\overline{n}} (H/N, \cdot)$ is the natural epimorphism.

Proof. (i)

$$\begin{aligned} I_H(xy) &= xy \\ &= xy\underbrace{e\cdots e}_{n-2} \\ &= e^*(x, y, e.) \\ &= e^*(I_H(x), I_H(y), e.). \end{aligned}$$

(ii) Suppose $Nh_1,\ Nh_2 \in H/N$. Then

$$\begin{aligned} &\Leftrightarrow\ Nh_1 = Nh_2 \\ &\Leftrightarrow\ \mathcal{P}(N, h_1, e.) = \mathcal{P}(N, h_2, e.) \\ &\Leftrightarrow\ \varphi(Nh_1) = \varphi(Nh_2). \end{aligned}$$

Therefore φ is a one-one function. For $Nh_1, Nh_2 \in H/N$ we have

$$\begin{aligned} \varphi(Nh_1.Nh_2) &= \varphi(Nh_1h_2) \\ &= \mathcal{P}(N, h_1h_2, e.) \\ &= e^*/N(\mathcal{P}(N, h_1, e.), \mathcal{P}(N, h_2, e.), \mathcal{P}(N, e.).). \end{aligned}$$

So φ is a $(2, n)$-homomorphism. Also φ is onto.
(iii)

$$\begin{aligned} \overline{n}_2(e^*(x_1^n)) &= \overline{n}_2(x_1\cdots x_n) \\ &= \mathcal{P}(N, x_1\cdots x_n, e.), \\ &= e^*/N(\mathcal{P}(N, x_1, e.), \cdots, \mathcal{P}(N, x_n, e.)) \\ &= e^*/N(\overline{n}_2(x_1), \cdots, \overline{n}_2(x_n)). \end{aligned}$$

(iv) It is straightforward. □

Corollary 3.23. *If $k, m, n \in I\!N_0$, then $\mathbb{Z}^{n-1}/kZ$ and $\mathbb{Z}^{m-1}/k\mathbb{Z}$ as n-ary and m-ary groups respectively are isomorph.*

Proof. By (ii) of Theorem 3.22, it is clear. □

Theorem 3.24. *Let $\{e\} \to H \xrightarrow{f} K \xrightarrow{g} L \to \{e\}$ be a short exact sequence of groups, $n_1 \leq n_2 \leq n_3$ and $e \in P \subsetneq H,\ Q \subseteq K,\ R \subseteq L$ such that $f(P) = Q,\ g(Q) = R$. Then the sequence*

$$\{e\} \to (H, P^*) \to (K, Q^*) \to (L, R^*) \to \{e\} \qquad (*)$$

is a short exact sequence of (n_1, n_2, n_3)-ary P-H_v-groups.

Proof. Because $e \in P$,

$$\begin{aligned} e = f(e) &\in f(P) = Q \text{ and} \\ e = g(e) &\in g(Q) = R. \end{aligned}$$

By (ii) of Lemma 2.4, $(H, P^*), (K, Q^*), (L, R^*)$ are n-ary P-H_v-groups. By (ii) of Lemma 3.9, all maps in $(*)$ are homomorphisms of n-ary P-H_v-groups. □

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