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SPECTRAL ANALYSIS OF TIME SERIES IN JOINT SEGMENTS OF OBSERVATIONS

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ABSTRACT. Spectral analysis of a strictly stationary r-vector valued time series is considered under the assumption that some of the observations are missed due to some random failure. Statistical properties and asymptotic moments are derived. Asymptotic normality is discussed.

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1. Introduction

The spectral analysis of time series is one of the oldest and most widely used analysis techniques in the physical sciences. The basic idea behind spectral analysis is to decompose the variance (covariance) of a time series into a number of components each one of which can be associated with a particular frequency. Brillinger (1969), considered the case where, $X(t)(t = 0, \pm 1, ...)$ is a zero mean r-vector valued strictly stationary time series satisfying a particular assumptions about the near independence of widely separated values. For the observations $X(t)(t = 0, 1, ..., T - 1)$, he constructed asymptotically unbiased and consistent estimates of the matrix of spectral measures, the matrix of covariances and the matrix of the average of spectral densities. The statistics were based on the matrix of second order periodograms, see also, Dahlhaus(1985), Ghazal (1999) and Ghazal and Farag [(1998), (2001)].

When observing a time series at equal spaced intervals of time, it might be happen that the device being used to observe the series will miss an observation because of some random failure. The extension to estimate the spectral density in the case where some observations are missed utilizes an idea introduced by Jones (1962), who examined the case where a block of observations is periodically unobtainable. Parzen (1962) developed the theory of amplitude-modulated stationary processes, and applied this theory to missing data problems (Parzen

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(1963)), considering in detail the case where observations are missed in some periodic way. The amplitude-modulated series is constructed by replacing missing observations in the original series by their mean value. Scheinok (1965) considered the case where an observation is made or not according to the out come of a Bernoulli trial. Bloomfield (1970) considered the case where a more general random mechanism is involved. Ghazal and Elhassanein (2004) construct an estimate of the spectral density matrix of a strictly stationary r-vector valued time series with randomly missing observations on non-crossed intervals observations using periodogram, see also, Ghazal and Elhassanein (2006). In this paper we will discuss the spectral analysis of a strictly stationary r-vector valued time series with randomly missing observations in joint segments of observation.

In the paper $W_r(\gamma, \sum)$ will denote an $r \times r$ symmetric matrix-valued Wishart variate with covariance matrix $\sum$ and γ degree of freedom. Let $W_r^c(\gamma, \sum)$ denote an $r \times r$ Hermitian matrix-valued complex Wishart variate with covariance matrix $\sum$ and γdegree of freedom. Let $N_r(\mu_Z, \sum_{ZZ})$ denote the multivariate normal distribution with mean μ_Z and covariance matrix $\sum_{ZZ}$ where Z is an r-vector valued random variable having real-valued components. Let $N_r^c(\mu_Z, \sum_{ZZ})$, the complex multivariate normal distribution with mean μ_Z and covariance matrix $\sum_{ZZ}$ where Z is of complex-valued components.

2. Observed series

Let $X(t)(t = 0, \pm 1, ...)$ be a zero mean r-vector valued strictly stationary time series with

$$E\{X(t+u)\overline{X}'(t)\} = C_{XX}(u) \tag{1}$$

and

$$\sum_{u=-\infty}^{\infty} |C_{XX}(u)| \prec \infty, \tag{2}$$

where $|C_{XX}(u)|$denotes the matrix of absolute values, the bar denotes the complex conjugate and $'$ denotes the matrix transpose. We may then define $f_{XX}(\lambda)$ the $r \times r$ matrix of second order spectral densities by

$$f_{XX}(\lambda) = (2\pi)^{-1} \sum_{u=-\infty}^{\infty} C_{XX}(u) \exp(-i\lambda u), \tag{3}$$

see Brillinger (2001) P.24.

Using the assumed stationary, we then set down

Assumption I. $X(t)$ is a strictly stationary series all of whose moments exist. For each $j = 1, 2, ..., k-1$ and any k-tuple $a_1, a_2, ..., a_k$ we have

$$\sum_{u_1 \cdots u_{k-1}} |u_j C_{a_1,...,a_k}(u_1, ..., u_{k-1})| < \infty, k = 2, 3, ... \tag{4}$$

where

$$C_{a_1,\ldots,a_k}(u_1,u_2,\ldots,u_{k-1}) = cum\{X_{a_1}(t+u_1), X_{a_2}(t+u_2), \ldots, X_{a_k}(t)\}, \quad (5)$$

$(a_1, a_2, \ldots, a_k = 1, 2, \ldots, r\,; u_1, u_2, \ldots, u_{k-1}, t = 0, \pm 1, \ldots; k = 2, \ldots)$.

Because cumulants are measures of the joint dependance of random variables, (4) is seen to be a form of mixing or asymptotic independence requirement for values of $X(t)$ well separated in time. If $X(t)$satisfies Assumption I we may define its cumulant spectral densities by

$$f_{a_1,\ldots,a_k}(\lambda_1,\ldots,\lambda_{k-1}) \quad (6)$$
$$= (2\pi)^{-k+1} \sum_{u_1\ldots u_{k-1}} C_{a_1,\ldots,a_k}(u_1,\ldots,u_{k-1}) \exp\Big(-i\sum_{j=1}^{k-1}\lambda_j u_j\Big),$$

$(-\infty < \lambda < \infty, a_1, a_2, \ldots, a_k = 1, 2, \ldots, r; k = 2, \ldots)$. If $k = 2$ the cross-spectra $f_{a_1a_2}(\lambda)$ are collected together in the matrix $f_{XX}(\lambda)$ of (3).

Assumption II. $\Phi(t)$ is bounded, is of bounded variation and vanishes for $t < 0$, $t > T-1$ that is called data window. Let

$$\Phi^{(T)}_{a_1\ldots a_k}(\lambda) = \sum_t \prod_{j=1}^{k} \Phi_{a_j}(t)\exp(-i\lambda t), \quad (7)$$

for $-\infty < \lambda < \infty$ and $a_1, \ldots, a_k = 1, 2, \ldots, r$.

3. Modified series

Let $H(t) = \{H_a(t)(t = 0, \pm 1, \ldots)\}_{a=1,2,\ldots,r}$ be a process independent of $X(t)$ such that, for every t

$$P\{H_a(t) = 1\} = p, P\{H_a(t) = 0\} = q, \quad (8)$$

note that

$$E\{H_a(t)\} = p. \quad (9)$$

The success of recording an observation not depend on the fail of another and so it is independent. We may then define the modified series

$$Y(t) = H(t)X(t), \quad (10)$$

with components,

$$Y_a(t) = H_a(t)X_a(t), \quad (11)$$

where

$$H(t) = \begin{cases} 1 & \text{if } X(t) \text{ is observied ;} \\ 0 & \text{if } X(t) \text{ is missed.} \end{cases} \quad (12)$$

4. Expanded finite Fourier transform in L-joint segments of observations

Let $X(t)(t=0,1,...,T-1)$ be an observed stretch of data with some randomly missing observations. Let $T=L(N-M)+M$, where L, is the number of joint segments and N, is the length of each segment and M is the length of joint partes, $0 \leq M < N$ then the observations may be represented as

$$X[l(N-M)], X[l(N-M)+1], ..., X[(l+1)(N-M)+M-1], l=0,1,...,L-1.$$

The expanded finite Fourier transform of a given stretch of data, is defined by

$$d_Y^{l(N-M)}(\lambda) = \left(2\pi \sum_{t=l(N-M)}^{(l+1)(N-M)+M-1} \Phi^2[t-l(N-M)]\right)^{-\frac{1}{2}} \tag{13}$$

$$\times \sum_{t=lN}^{(l+1)N-1} \Phi[t-l(N-M)] \exp(-i\lambda t) Y(t),$$

where $-\infty < \lambda < \infty$, and $\Phi(t)$ is the data window satisfies Assumption II.

In this section we will study the statistical properties of the modified series, the expanded finite Fourier transform and the modified periodograms.

Theorem 4.1. *Let $X(t)(t=0,\pm1,...)$ be a strictly stationary r-vector valued time series with mean zero, and satisfy Assumption I. Let $d_a^{l(N-M)}(\lambda)$ be defined as (13), and $\Phi_a(t)$ satisfy Assumption II, for $a=1,2,...,r$. Then*

$$E\{d_a^{l(N-M)}(\lambda)\} = 0, \tag{14}$$

$$Cov\left\{d_a^{l(N-M)}(\lambda_1), d_b^{l(N-M)}(-\lambda_2)\right\}$$
$$= p^2 R_{ab}(l,l) \Phi_{ab}^{l(N-M)}(\lambda_1+\lambda_2) f_{ab}(\lambda_1) + O(N^{-1}), a \neq b \tag{15}$$

$$Cov\left\{d_a^{l(N-M)}(\lambda_1), d_a^{l(N-M)}(-\lambda_2)\right\} = pqC_{aa}(0)R_{aa}(l,l)\Phi_{aa}^{l(N-M)}(\lambda_1+\lambda_2)$$
$$+ p^2 R_{aa}(l,l)\Phi_{ab}^{l(N-M)}(\lambda_1+\lambda_2) f_{aa}(\lambda_1) + O(N^{-1}), \tag{16}$$

at $\lambda_1 = \lambda_2 = \lambda$

$$Cov\left\{d_a^{l(N-M)}(\lambda), d_b^{l(N-M)}(-\lambda)\right\}$$
$$= p^2 R_{ab}(l,l)\Phi_{ab}^{(N-M)}(0) f_{ab}(\lambda_1) + O(N^{-1}) \tag{17}$$

where $O(N^{-1})$ is uniform in λ as $N \to \infty$.

$$Cum\{d_{a_1}^{l_1(N-M)}(\lambda_1), ..., d_{a_k}^{l_k(N-M)}(\lambda_k)\}$$
$$= (2\pi)^{\frac{k}{2}-1} p^k R_{a_1...a_k}(l_1,...,l_k) \Phi_{a_1...a_k}^{(N)}\Big(\sum_{j=1}^{k} \lambda_j\Big) \tag{18}$$
$$\times \exp(-il_k(N-M)\sum_{j=1}^{k} \lambda_j) f_{a_1 a_2 ... a_k}(\lambda_1, ..., \lambda_k) + O(N^{-\frac{k}{2}})$$

where $O(N^{-\frac{k}{2}})$ is uniform in $\lambda_1, \lambda_2, ..., \lambda_{k-1}$ as $N \to \infty$, $k = 2, ...$ and

$$R_{a_1...a_k}(l_1, ..., l_k) = \left(\sum_{t_1=l_1(N-M)}^{(l_1+1)(N-M)+M-1} \Phi_{a_1}^2[t_1 - l(N-M)] \right)^{-\frac{1}{2}} \cdots \times \left(\sum_{t_2=l_k(N-M)}^{(l_k+1)(N-M)+M-1} \Phi_{a_k}^2[t_1 - l(N-M)] \right)^{-\frac{1}{2}}$$

Proof. The proof of (14) comes directly from (13). Since $E\{Y(t)\} = 0$,

$$\begin{aligned}
&Cov\left\{d_a^{l(N-M)}(\lambda_1), d_b^{l(N-M)}(-\lambda_2)\right\} \\
&= E\left\{d_a^{l(N-M)}(\lambda_1) d_b^{l(N-M)}(\lambda_2)\right\} \\
&= (2\pi)^{-1} R_{ab} E\Big\{ \sum_{t_1=l(N-M)}^{(l+1)(N-M)+M-1} \Phi_a[t_1 - l(N-M)] \exp(-i\lambda_1 t_1) Y_a(t_1). \\
&\sum_{t_2=l(N-M)}^{(l+1)(N-M)+M-1} \Phi_b[t_2 - l(N-M)] \exp(-i\lambda_2 t_2) Y_b(t_2) \Big\} \\
&= (2\pi)^{-1} R_{ab} \Big[\sum_{t=l(N-M)}^{(l+1)(N-M)+M-1} \Phi_a[t - l(N-M)] \Phi_b[t - l(N-M)] \\
&\times \exp(-i(\lambda_1 + \lambda_2)t) E\{Y_a(t) Y_b(t)\} \\
&+ \sum_{t_1=l(N-M)}^{(l+1)(N-M)+M-1} \underset{\neq}{} \sum_{t_2=l(N-M)}^{(l+1)(N-M)+M-1} \Phi_a[t_1 - l(N-M)] \Phi_b[t_2 - l(N-M)] \\
&\times \exp(-i(\lambda_1 t_1 + \lambda_2 t_2) E\{Y_a(t_1) Y_b(t_2)\} \Big].
\end{aligned}$$

Let $a = b$. Then

$$\begin{aligned}
&Cov\left\{d_a^{l(N-M)}(\lambda_1), d_a^{l(N-M)}(-\lambda_2)\right\} \\
&= (2\pi)^{-1} p R_{aa} C_{aa}(0) \Phi_{aa}^{l(N-M)}(\lambda_1 + \lambda_2) \\
&+ (2\pi)^{-1} p^2 R_{aa} \sum_{t_1=l(N-M)}^{(l+1)(N-M)+M-1} \underset{\neq}{} \sum_{t_2=l(N-M)}^{(l+1)(N-M)+M-1} \Phi_a[t_1 - l(N-M)] \\
&\times \Phi_b[t_2 - l(N-M)] \exp(-i(\lambda_1 t_1 + \lambda_2 t_2) C_{aa}(t_1 - t_2).
\end{aligned}$$

let $t_1 - t_2 = u$, $u \neq 0$, $t_2 = t$, and by Assumption II and Lemma (4.1) in Ghazal and Elhassanein(2004), then

$$Cov\left\{d_a^{l(N-M)}(\lambda_1), d_a^{l(N-M)}(-\lambda_2)\right\}$$
$$= (2\pi)^{-1} p R_{aa} C_{aa}(0) \Phi_{aa}^{l(N-M)}(\lambda_1 + \lambda_2)$$
$$+(2\pi)^{-1} p^2 R_{aa} \Phi_{aa}^{l(N-M)}(\lambda_1 + \lambda_2) \sum_{0 \neq u = -(N-M+1)}^{(l+1)(N-M)+M-1} \exp(-i\lambda_1 u) C_{aa}(u)$$
$$+O(N^{-1})$$

by (3) the proof of (16) is completed. We can get the proof of (15) and (17) by the same structure in (16). Since

$$Cum\{d_{a_1}^{l_1(N-M)}(\lambda_1), ..., d_{a_k}^{l_k(N-M)}(\lambda_k)\}$$
$$= Cum\Bigg\{ \left(2\pi \sum_{t_1=l_1(N-M)}^{(l_1+1)(N-M)+M-1} \Phi_{a_1}^2[t_1 - l_1(N-M)]\right)^{-\frac{1}{2}}$$
$$\times \sum_{t_1=l_1(N-M)}^{(l_1+1)(N-M)+M-1} \Phi_{a_1}[t_1 - l_1(N-M)] \exp(-i\lambda t_1) Y_{a_1}(t_1), ...,$$

$$\left(2\pi \sum_{t_k=l_k(N-M)}^{(l_k+1)(N-M)+M-1} \Phi_{a_1}^2[t_k - l_k(N-M)]\right)^{-\frac{1}{2}}$$
$$\times \sum_{t_k=l_k(N-M)}^{(l_k+1)(N-M)+M-1} \Phi_{a_k}[t_k - l_k(N-M)] \exp(-i\lambda t_k) Y_{a_k}(t_k)\Bigg\}$$
$$= (2\pi)^{-\frac{k}{2}} \left(\sum_{t_1=l_1(N-M)}^{(l_1+1)(N-M)+M-1} \Phi_{a_1}^2[t_1 - l_1(N-M)]\right)^{-\frac{1}{2}}$$

$$\times ... \left(\sum_{t_k=l_k(N-M)}^{(l_k+1)(N-M)+M-1} \Phi_{a_1}^2[t_k - l_k(N-M)]\right)^{-\frac{1}{2}}$$
$$\times \sum_{t_1=l_1(N-M)}^{(l_1+1)(N-M)+M-1} \cdots \sum_{t_k=l_k(N-M)}^{(l_k+1)(N-M)+M-1} \Phi_{a_1}[t_1 - l_1(N-M)]$$

$$\times ... \times \Phi_{a_k}[t_k - l_k(N-M)] \exp\left(-i\sum_{j=1}^{k}\lambda_j t_j\right) Cum\{Y_{a_1}(t_1), ..., Y_{a_k}(t_k)\}$$

$$= (2\pi)^{-\frac{k}{2}} R_{a_1...a_k} \sum_{t_1=l_1(N-M)}^{(l_1+1)(N-M)+M-1} \cdots \sum_{t_k=l_k(N-M)}^{(l_k+1)(N-M)+M-1} \Phi_{a_1}[t_1 - l_1(N-M)]$$

$$\times ... \times \Phi_{a_k}[t_k - l_k(N-M)] \exp\left(-i\sum_{j=1}^{k}\lambda_j t_j\right) C_{a_1...a_k}(t_1 - t_k, ..., t_{k-1} - t_k),$$

let $t_j - l_j(N-M) = u_j$, $j = 1, ..., k-1$ and by Lemma (P.4.1) in Brillinger (2001) PP. 420 and (6), we get the proof of (18). □

Corollary 4.1. *Under the conditions of Theorem (4.1), we have*

$$E\{d_a^{(lN)}(\pm\lambda_j)\} = 0,$$

$$N^{-1}Cov\left\{d_a^{(lN)}(\pm\lambda_j), d_b^{(lN)}(\pm\lambda_l)\right\}$$
$$= N^{-1}P^2 R_{ab}\Phi_{ab}^{(lN)}(\pm\lambda_j \mp \lambda_l) f_{ab}(\pm\lambda_j) + O(N^{-2}),$$

that tends to 0 as $N \to \infty$ if $\lambda_j \pm \lambda_l \not\equiv 0(mod 2\pi)$, it tends to $2\pi p^2 R_{ab}\Phi_{ab}^{(T)}(0)$ $f_{ab}(\pm\lambda_j)$ if $\pm\lambda_j \equiv \pm\lambda_l(mod 2\pi)$.

$$N^{-\frac{k}{2}} Cum\left\{d_{a_1}^{l(N-M)}(\pm\lambda_{j_1}), ..., d_{a_k}^{l(N-M)}(\pm\lambda_{j_k})\right\}$$
$$= N^{-\frac{k}{2}}(2\pi)^{\frac{k}{2}-1} p^k R_{a_1...a_k}\Phi_{a1.....a_k}^{l(N-M)}(\pm\lambda_{j_1}(T) \pm ... \pm \lambda_{j_k}(T))$$
$$\times f_{a_1 a_2 ... a_k}(\pm\lambda_{j_1}(T), ..., \pm\lambda_{j_k}(T)) + O(N^{-k}),$$

that tends to 0 as $N \to \infty$ if $k > 2$.

Proof. The proof comes directly by Theorem (4.1). □

Theorem 4.2. *Let $X(t)(t = 0, \pm 1, ...)$ be a zero mean strictly stationary r-vector valued time series satisfy Assumption I and $d_Y^{(T)}(\lambda_j)$ be given by (13). Suppose $2\lambda_j, \lambda_j \pm \lambda_k \not\equiv 0(mod 2\pi)$ for $1 \le j < k \le J$.*

Then $d_Y^{l(N-M)}(\lambda j)$, $\lambda_j \not\equiv 0(mod 2\pi), j = 1, 2, ...J$ are asymptotically independent $N_r^c\left(0, 2\pi p^2 R_{ab}\Phi_{ab}^{l(N-M)}(0) f_{ab}(\lambda_j)\right)$, variates. Also if $\lambda = 0, \pm 2\pi, ..., d_Y^{(T)}(\lambda)$ is asymptotically $N_r\left(0, 2\pi p^2 R_{ab}\Phi_{ab}^{l(N-M)}(0) f_{ab}(\lambda)\right)$ independent of previous variates and if $\lambda = \pm\pi, \pm 3\pi...$ is asymptotically $N_r\left(0, 2\pi p^2 R_{ab}\Phi_{ab}^{l(N-M)}(0) f_{ab}(\lambda)\right)$ independent of previous variates.

Proof. The proof comes directly by Corollary (4.1). □

5. The smoothed spectral density estimate

Using expanded finite Fourier transform (13), we construct the modified periodogram as

$$I_{ab}^{l(N-M)}(\lambda) = (p^2 R_{ab}\Phi_{ab}^{(N)}(0))^{-1} d_a^{l(N-M)}(\lambda)\overline{d_b^{\prime l(N-M)}(\lambda)}, \tag{19}$$

where the bar denotes the complex conjugate. The smoothed spectral density estimate is constructed as

$$f_{ab}^{(T)}(\lambda) = \frac{1}{L}\sum_{l=0}^{L-1} I_{ab}^{l(N-M)}(\lambda), a, b = 1, 2, ..., r \tag{20}$$

Theorem 5.1. *Let $X(t)(t = 0, \pm 1, ...)$ be a strictly stationary r-vector valued time series with mean zero, and satisfy Assumption I.*

Let $I_{YY}^{(T)}(\lambda) = \{I_{ab}^{(T)}(\lambda)\}_{a,b=1,2,...,r}$ be given by (19), and $\Phi_a(t)$ satisfy Assumption II for $a = 1, 2, ..., r$. Then

$$E\{I_{ab}^{l(N-M)}(\lambda)\} = f_{ab}(\lambda) + O(N^{-1}), p \to 1, \tag{21}$$

$$\begin{aligned}
&Cov\left\{I_{a_1b_1}^{l(N-M)}(\lambda_1), I_{a_2b_2}^{l(N-M)}(\lambda_2)\right\} \\
&= \left(R_{a_1b_1}R_{a_2b_2}\Phi_{a_1b_1}^{l(N-M)}(0)\Phi_{a_2b_2}^{l(N-M)}(0)\right)^{-1} \\
&\times\Big[R_{a_1a_2}R_{b_1b_2}\Phi_{a_1a_2}^{l(N-M)}(\lambda_1-\lambda_2)\overline{\Phi_{b_1b_2}^{l(N-M)}(\lambda_1-\lambda_2)}f_{a_1a_2}(\lambda_1)f_{b_1b_2}(-\lambda_1) \\
&+R_{a_1b_2}R_{b_1a_2}\Phi_{a_1b_2}^{l(N-M)}(\lambda_1+\lambda_2)\overline{\Phi_{b_1a_2}^{l(N-M)}(\lambda_1+\lambda_2)}f_{a_1b_2}(\lambda_1)f_{b_1a_2}(-\lambda_1) \\
&+(2\pi)R_{a_1b_1a_2b_2}\Phi_{a_1b_1a_2b_2}^{l(N-M)}(0)f_{a_1b_1a_2b_2}(\lambda_1, -\lambda_1, \lambda_2)\Big] + O(N^{-1})
\end{aligned} \tag{22}$$

$$\begin{aligned}
&Cum\left\{I_{a_1b_1}^{l_1(N-M)}(\lambda_1), ..., I_{a_kb_k}^{l_k(N-M)}(\lambda_k)\right\} \\
&= \left(\prod_{i=1}^{k} R_{a_ib_i}\Phi_{a_ib_i}^{(N)}(0)\right)^{-1} \sum\left\{\prod_{j=1}^{k} R_{a_jb_j}\exp(-il_k(N-M)\sum_{j=1}^{k}(\mu_j+\gamma_j))\right. \\
&\left.\times\Phi_{c_jd_j}^{(N)}(\mu_j+\gamma_j)\right\}\left\{\prod_{j=1}^{k} f_{c_jd_j}(\mu_j)\right\} + O(N^{-1})
\end{aligned} \tag{23}$$

where the summation extends over all partitions $\{(c_1, \mu_1), (d_1, \gamma_1)\}, ..., \{(c_k, \mu_k), (d_k, \gamma_k)\}$, into pairs of the quantities $(a_1, \lambda_1), (b_1, -\lambda_1), ..., (a_k, \lambda_k), (b_k, -\lambda_k)$ excluding the case with $\mu_j = -\gamma_j = \lambda_m$ for some j, m, where $O(N^{-1})$ is uniform in $\lambda_1, ..., \lambda_k$.

Proof. By (5.1), we have

$$E\left\{I_{ab}^{l(N-M)}(\lambda)\right\} = (P^2 R_{ab}\Phi_{ab}^{l(N-M)}(0))^{-1} E\{d_a^{l(N-M)}(\lambda)\overline{d_b^{\prime l(N-M)}(\lambda)}\}$$
$$= Cov\left\{d_a^{l(N-M)}(\lambda), d_b^{l(N-M)}(\lambda)\right\},$$

and then by (15) the proof of (21) is completed. From (19), and by Theorem (2.3.2) in Brillinger (2001) PP.21, we have

$$Cov\{I_{a_1b_1}^{l(N-M)}(\lambda_1), I_{a_2b_2}^{l(N-M)}(\lambda_2)\}$$
$$= Cov\{d_{a_1}^{l(N-M)}(\lambda_1)d_{b_1}^{\prime l(N-M)}(-\lambda_1), d_{a_2}^{l(N-M)}(\lambda_2)d_{b_2}^{\prime l(N-M)}(-\lambda_2)\}$$
$$Cum\{d_{a_1}^{l(N-M)}(\lambda_1), d_{b_1}^{l(N-M)}(-\lambda_1), d_{a_2}^{l(N-M)}(\lambda_2), d_{b_2}^{l(N-M)}(-\lambda_2)\}$$
$$+Cov\{d_{a_1}^{l(N-M)}(\lambda_1), d_{a_2}^{l(N-M)}(\lambda_2)\}Cov\{d_{b_1}^{l(N-M)}(-\lambda_1), d_{b_2}^{l(N-M)}(-\lambda_2)\}$$
$$+Cov\{d_{a_1}^{l(N-M)}(\lambda_1), d_{b_2}^{l(N-M)}(-\lambda_2)\}Cov\{d_{b_1}^{l(N-M)}(-\lambda_1), d_{a_2}^{l(N-M)}(\lambda_2)\}.$$

By Theorem (4.1) the proof of (22) is completed. From (19), we have

$$Cum\{I_{a_1b_1}^{l_1(N-M)}(\lambda_1), ..., I_{a_kb_k}^{l_k(N-M)}(\lambda_k)\}$$
$$= p^{-2k}(\prod_{i=1}^{k} R_{a_ib_i}\Phi_{a_ib_i}^{(N)}(0))^{-1}$$
$$\times Cum\{d_{a_1}^{l_1(N-M)}(\lambda_1)d_{b_1}^{\prime l_1(N-M)}(-\lambda_1), ..., d_{a_k}^{l_k(N-M)}(\lambda_k)d_{b_k}^{\prime l_k(N-M)}(-\lambda_k)\}$$

By Theorem (2.3.2) in Brillinger (2001) PP.21, we get

$$Cum\left\{d_{a_1}^{l_1(N-M)}(\lambda_1)d_{b_1}^{\prime l_1(N-M)}(-\lambda_1), ..., d_{a_k}^{l_k(N-M)}(\lambda_k)d_{b_k}^{\prime l_k(N-M)}(-\lambda_k)\right\}$$
$$= \sum_{\nu} Cum\{d^{l_i(N-M)}(\lambda_i); i \in \nu_1\}...Cum\{d^{l_i(N-M)}(\lambda_i); i \in \nu_s\},$$

where the summation extends over all indecomposable partitions $\nu = [\cup_{j=1}^{s}\nu_j] \in I$, $I = (a_1, ..., a_k\ ;\ b_1, ..., b_k)$, $1 \le s \le k$ of the transformed table

$$\begin{array}{llcll} (a_1, \lambda_1), & (b_1, -\lambda_1) & & \{(c_1, \mu_1), & (d_1, \gamma_1)\} \\ (a_2, \lambda_2), & (b_2, -\lambda_2) & & \{(c_2, \mu_2), & (d_2, \gamma_2)\} \\ . & . & \rightarrow & . & . \\ . & . & & . & . \\ (a_k, \lambda_k), & (b_k, -\lambda_k) & & \{(c_k, \mu_k), & (d_k, \gamma_k)\}. \end{array}$$

Then, by Theorem (4.1), we get the proof of (23). □

Theorem 5.2. *Let $X(t)(t = 0, \pm 1, ...)$ be a strictly stationary r-vector valued time series with mean zero, and satisfy Assumption I. Let $I_{YY}^{l(N-M)}(\lambda) = \{I_{ab}^{l(N-M)}(\lambda)\}_{a,b=1,2,...,r}$ be given by (19),$2\lambda_j, \lambda_j \pm \lambda_k \not\equiv 0(mod 2\pi)$ for $1 \le j < k \le J$ and $\Phi_a(t)$ satisfy Assumption II for $a = 1, 2, ..., r$,.*
Then $I_{YY}^{l(N-M)}(\lambda_j)$, $j = 1, 2, ..J$ are asymptotically independent $W_r^c(1, f_{XX}(\lambda_j))$ variates. Also if $\lambda = \pm\pi, \pm 3\pi...$ then $I_{YY}^{l(N-M)}(\lambda)$ is asymptotically $W_r(1, f_{XX}(\lambda))$ independent of the previous variates.

Proof. The proof comes directly from Theorem (4.2), for more details about Wishart distribution see Anderson (1972). □

Theorem 5.3. *Let $X(t)(t = 0, \pm 1, ...)$ be a strictly stationary r-vector valued time series with mean zero, and satisfy Assumption I. Let $f_{ab}^{(T)}(\lambda)$ be given by (20), $a, b = 1, 2, ..., r,$. Then*

$$E\{f_{ab}^{(T)}(\lambda)\} = f_{ab}(\lambda) + O(N^{-1}), \tag{24}$$

$$\begin{aligned} &Cov\left\{f_{a_1b_1}^{(T)}(\lambda_1), f_{a_2b_2}^{(T)}(\lambda_2)\right\} \\ &= \left(L^2\Phi_{a_1b_1}^{(N)}(0)\Phi_{a_2b_2}^{(N)}(0)\right)^{-1} \sum_{l_1=0}^{L-1}\sum_{l_2=0}^{L-1}(R_{a_1b_1}(l_1,l_1)R_{a_2b_2}(l_2,l_2))^{-1} \\ &\times\Big[R_{a_1a_2}(l_1,l_2)R_{b_1b_2}(l_1,l_2)\Phi_{a_1a_2}^{(N)}(\lambda_1-\lambda_2)\overline{\Phi_{b_1b_2}^{(N)}(\lambda_1-\lambda_2)} \\ &\times\exp(-il_2(N-M)(\lambda_1-\lambda_2))f_{a_1a_2}(\lambda_1)f_{b_1b_2}(-\lambda_1) \\ &+R_{a_1b_2}(l_1,l_2)R_{b_1a_2}(l_1,l_2)\Phi_{a_1b_2}^{(N)}(\lambda_1+\lambda_2)\overline{\Phi_{b_1a_2}^{(N)}(\lambda_1+\lambda_2)} \\ &\times\exp(-il_2(N-M)(\lambda_1+\lambda_2))f_{a_1b_2}(\lambda_1)f_{b_1a_2}(-\lambda_1) \\ &+(2\pi)R_{a_1b_1a_2b_2}(l_1,l_1,l_2,l_2)\Phi_{a_1b_1a_2b_2}^{(N)}(0)f_{a_1b_1a_2b_2}(\lambda_1,-\lambda_1,\lambda_2)\Big] + O(N^{-1}) \end{aligned} \tag{25}$$

Proof. By (20), we have

$$E\left\{f_{ab}^{(T)}(\lambda)\right\} = \frac{1}{L}\sum_{l=0}^{L-1} E\left\{I_{ab}^{l(N-M)}(\lambda)\right\},$$

then by (21) the proof of (24) is completed. From (20), we get

$$\begin{aligned} &Cov\left\{f_{a_1b_1}^{(T)}(\lambda_1), f_{a_2b_2}^{(T)}(\lambda_2)\right\} \\ &= \frac{1}{L^2}\sum_{l_1=0}^{L-1}\sum_{l_2=0}^{L-1} Cov\left\{I_{a_1b_1}^{l(N-M)}(\lambda_1), I_{a_2b_2}^{l(N-M)}(\lambda_2)\right\}. \end{aligned}$$

Which completes the proof of (25). □

Theorem 5.4. *Let $X(t)(t = 0, \pm 1, ...)$ be a strictly stationary r-vector valued time series with mean zero, and satisfy Assumption I. Let $f_{ab}^{l(N-M)}(\lambda)$ be given by (20), $a, b = 1, 2, ..., r$, $2\lambda_j, \lambda_j \pm \lambda_k \not\equiv 0(mod 2\pi)$ for $1 \leq j < k \leq J$, Then $Lf_{ab}^{l(N-M)}(\lambda_j)$, $j = 1, 2, ..J$ are asymptotically independent $W_r^c(L, f_{ab}(\lambda_j))$ variates. Also if $\lambda = \pm\pi, \pm 3\pi..$, then $Lf_{ab}^{l(N-M)}(\lambda)$ is asymptotically $W_r(L, f_{ab}(\lambda))$ independent of the previous variates.*

Proof. The proof comes directly by Theorem (5.2) and Theorem (7.3.2) in Anderson (1972) PP.162.

REFERENCES

1. T. W. Anderson, *An Introduction to Multivariate Statistical Analysis*, Wiley Eastern Limited, New Delhi, 1972.
2. P. Bloomfield, *Spectral Analysis with Randomly Missing Observations*, J. R. Statist. Soc **32** (1970), 336-380.
3. D. R. Brillinger, *Time Series Data Analysis and Theory*, SIAM, 2001.
4. D. R. Brillinger, *Asymptotic properties of Spectral Estimate of Second Order*, Biometrika **56**(2) (1969), 375-390.
5. R. Dahlhaus, *On Spectral Density Estimate Obtained by Averaging Periodograms*, J. Appl. Prop. **22** (1985), 598-610.
6. M. A. Ghazal, *One Spectral Density Estimate on Non-crossed Interval Observation*, The International Journal of Applied Mathematics **1**(8) (1999), 875-882.
7. M. A. Ghazal, and A.Elhassanein, *Periodogram Analysis WithMissing Observations*, J. Appl. Math. & Computing, (22) (2006), 1 - 2, 209 - 222.
8. M. A. Ghazal, and A.Elhassanein, *Spectral Density Estimate of Time Series with Randomly Missing Observations*, The 39th Annual Conference on Statistics, Computer Science and Operation Research Part I (2004), 64-83.
9. M. A. Ghazal and E. A. Farag, *Some Properties of the Continuous Expanded Finite Fourier Transform*, J. Statistica **3** (2001), 577-384.
10. M. A. Ghazal and E. A. Farag, *Asymptotic Distribution of Spectral Density Estimate of Continuous Time Series on Crossed Interval Observations*, The Egyption Statistical ISSR, Cairo Uni., **42**(2) (1998), 197-214.
11. R. H. Jones, *Spectral Analysis with Regularly Missed Observations*, Ann. Math. Statist. **33** (1962), 455-461.
12. E. Parzen, *On Spectral Analysis with Missing Observations and Amplitude Modulation*, Sankhya A **25** (1963), 383-392.
13. E. Parzen, *Spectral Analysis of Asymptotically Stationary Time Series*, Bull. Int. Statist. Inst 33rd Session, Paris 1962.
14. P. A. Scheinok, *Spectral Analysis with Randomly Missed Observations The Binomial Case*, Ann. Math. Statist. **36** (1965), 971-977.

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