J. Appl. Math. & Informatics Vol. 26(2008), No. 5 - 6, pp. 925 - 932
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WEAK AND CONCRETE FILTERS OF WFI-ALGEBRAS

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ABSTRACT. The notion of weak filters in WFI-algebras is introduced. Relations between weak filters and concrete filters are established. A condition for a filter to be closed is given. Also, a condition for a filter to be a weak filter is provided. Characterizations of weak and concrete filters are discussed. A condition for a subalgebra to be a concrete filter is given.

AMS Mathematics Subject Classification: 03G25, 03C05, 08A05
Key words and Phrases : (medial, simulative) WFI-algebra, subalgebra, strong unit, (weak, concrete) filter.

1. Introduction

In 1990, W. M. Wu [6] introduced the notion of fuzzy implication algebras (FI-algebra, for short), and investigated several properties. In [5], Z. Li and C. Zheng introduced the notion of distributive (resp. regular, commutative) FI-algebras, and investigated the relations between such FI-algebras and MV-algebras. In [1], the present author discussed several aspects of WFI-algebras. He introduced the notion of associative (resp. normal, medial) WFI-algebras, and investigated several properties. He gave conditions for a WFI-algebra to be associative/medial, and provided characterizations of associative/medial WFI-algebras, and showed that every associative WFI-algebra is a group in which every element is an involution. He also verified that the class of all medial WFI-algebras is a variety. Y. B. Jun and S. Z. Song [4] introduced the notions of simulative and/or mutant WFI-algebras and investigated some properties. They established characterizations of a simulative WFI-algebra, and gave a relation between an associative WFI-algebra and a simulative WFI-algebra. They also found some types for a simulative WFI-algebra to be mutant. Jun et al. [3]

Received November 2, 2007. Accepted January 17, 2008.
This work was supported by the fund of Research Promotion Program (RPP-2007-042), Gyeongsang National University.

introduced the concept of ideals of WFI-algebras, and gave relations between a filter and an ideal. Moreover, they provided characterizations of an ideal, and established an extension property for an ideal. In [2], the present author discussed perfect and concrete filters. In this paper, we introduce the notion of weak filters in WFI-algebras. We establish relations between weak filters and concrete filters. We give a condition for a filter to be closed. We also provide a condition for a filter to be a weak filter. We discuss characterizations of weak and concrete filters, and we give a condition for a subalgebra to be a concrete filter.

2. Preliminaries

Let $K(\tau)$ be the class of all algebras of type $\tau = (2, 0)$. By a *WFI-algebra* we mean a system $\mathfrak{X} := (X, \ominus, 1) \in K(\tau)$ in which the following axioms hold:

(a1) $(x \in X)\ (x \ominus x = 1)$,
(a2) $(x, y \in X)\ (x \ominus y = y \ominus x = 1 \Rightarrow x = y)$,
(a3) $(x, y, z \in X)\ (x \ominus (y \ominus z) = y \ominus (x \ominus z))$,
(a4) $(x, y, z \in X)\ ((x \ominus y) \ominus \big((y \ominus z) \ominus (x \ominus z)\big) = 1)$.

We call the special element 1 the *unit*. For the convenience of notation, we shall write $[x, y_1, y_2, \cdots, y_n]$ for

$$(\cdots((x \ominus y_1) \ominus y_2) \ominus \cdots) \ominus y_n.$$

We define $[x, y]^0 = x$, and for $n > 0$, $[x, y]^n = [x, y, y, \cdots, y]$, where y occurs n-times. We use the notation $x^n \ominus y$ instead of

$$x \ominus (\cdots (x \ominus (x \ominus y)) \cdots)$$

in which x occurs n-times.

Proposition 2.1. [1] *In a WFI-algebra $\mathfrak{X}$, the following are true:*

(b1) $x \ominus [x, y]^2 = 1$,
(b2) $1 \ominus x = 1 \Rightarrow x = 1$,
(b3) $1 \ominus x = x$,
(b4) $x \ominus y = 1 \Rightarrow (y \ominus z) \ominus (x \ominus z) = 1,\ (z \ominus x) \ominus (z \ominus y) = 1$,
(b5) $(x \ominus y) \ominus 1 = (x \ominus 1) \ominus (y \ominus 1)$,
(b6) $[x, y]^3 = x \ominus y$.

A nonempty subset S of a WFI-algebra $\mathfrak{X}$ is called a *subalgebra* of $\mathfrak{X}$ if $x \ominus y \in S$ whenever $x, y \in S$. A nonempty subset F of a WFI-algebra $\mathfrak{X}$ is called a *filter* of $\mathfrak{X}$ if it satisfies:

(c1) $1 \in F$,
(c2) $(\forall x \in F)\ (\forall y \in X)\ (x \ominus y \in F \Rightarrow y \in F)$.

A filter F of a WFI-algebra $\mathfrak{X}$ is said to be *closed* [1] if F is also a subalgebra of $\mathfrak{X}$.

Proposition 2.2. [1] *Let F be a filter of a WFI-algebra $\mathfrak{X}$. Then F is closed if and only if $x \ominus 1 \in F$ for all $x \in F$.*

Proposition 2.3. [1] *In a finite WFI-algebra, every filter is closed.*

We now define a relation "$\preceq$" on $\mathfrak{X}$ by $x \preceq y$ if and only if $x \ominus y = 1$. It is easy to verify that a WFI-algebra is a partially ordered set with respect to $\preceq$. A WFI-algebra $\mathfrak{X}$ is said to be *associative* [1] if it satisfies $[x, y, z] = x \ominus (y \ominus z)$ for all $x, y, z \in X$. A WFI-algebra $\mathfrak{X}$ is said to be *medial* [1] if it satisfies

$$(x \ominus y) \ominus (a \ominus b) = (x \ominus a) \ominus (y \ominus b)$$

for all $x, y, a, b \in X$. For a WFI-algebra $\mathfrak{X}$, the set

$$\mathcal{S}(\mathfrak{X}) := \{x \in X \mid x \preceq 1\}$$

is called the *simulative part* of $\mathfrak{X}$. A WFI-algebra $\mathfrak{X}$ is said to be *simulative* [4] if it satisfies

(S) $x \preceq 1 \Rightarrow x = 1$.

Note that the condition (S) is equivalent to $\mathcal{S}(\mathfrak{X}) = \{1\}$.

Proposition 2.4. [4] *The simulative part $\mathcal{S}(\mathfrak{X})$ of a WFI-algebra $\mathfrak{X}$ is a filter of $\mathfrak{X}$.*

3. Weak and concrete filters

In what follows, let $\mathfrak{X}$ denote a WFI-algebra unless otherwise specified.

Definition 3.1. The unit 1 in $\mathfrak{X}$ is said to be *strong* if it satisfies:

$$(\forall x \in X)\,(x^2 \ominus 1 = 1). \tag{3.1}$$

Proposition 3.2. *If the unit 1 is strong, then every filter of $\mathfrak{X}$ is closed.*

Proof. Let F be a filter of $\mathfrak{X}$. For any $x \in F$, we have $x^2 \ominus 1 = 1 \in F$. It follows from (c2) that $x \ominus 1 \in F$. Hence F is a closed filter of $\mathfrak{X}$ by Proposition 2.2. □

Proposition 3.3. *Every filter F of $\mathfrak{X}$ that contains the simulative part of $\mathfrak{X}$ satisfies the following implication.*

$$(\forall x \in X)\,(\forall y \in F)\,(x \preceq y \Rightarrow x \in F). \tag{3.2}$$

Proof. Let F be a filter of $\mathfrak{X}$ such that $\mathcal{S}(\mathfrak{X}) \subseteq F$. Let $x \in X$ and $y \in F$ satisfy $x \preceq y$. Then $x \ominus y = 1$ and so

$$y \ominus x \preceq (x \ominus y) \ominus (y \ominus y) = (x \ominus y) \ominus 1 = 1 \ominus 1 = 1.$$

Hence $y \ominus x \in \mathcal{S}(\mathfrak{X}) \subseteq F$. Since $y \in F$, it follows from (c2) that $x \in F$. □

Proposition 3.4. *Let F be a filter of $\mathfrak{X}$ that satisfies the following implication.*

$$(\forall x \in X)\,(x \preceq 1 \Rightarrow x \in F). \tag{3.3}$$

Then F contains the simulative part of $\mathfrak{X}$.

Proof. Straightforward. □

Corollary 3.5. *Let F be a filter of $\mathfrak{X}$. Then the following are equivalent.*

(i) *F contains the simulative part of $\mathfrak{X}$.*

(ii) $(\forall x \in X)$ $(\forall y \in F)$ $(x \preceq y \Rightarrow x \in F)$.

Definition 3.6. A filter F of $\mathfrak{X}$ is said to be *weak* if it satisfies:

$$(\forall x \in F)\,(\forall y \in X \setminus F)\,(y \ominus x \in X \setminus F). \tag{3.4}$$

Example 3.7. Let $X = \{1, a, b, c, d, e, x, y, z\}$ be a set with the following Cayley table:

$\ominus$	1	a	b	c	d	e	x	y	z
1	1	a	b	c	d	e	x	y	z
a	1	1	b	b	d	e	x	x	z
b	1	a	1	a	d	d	x	y	z
c	1	1	1	1	d	d	x	x	z
d	d	d	e	e	1	b	z	z	x
e	d	d	d	d	1	1	z	z	x
x	x	y	x	y	z	z	1	a	d
y	x	x	x	x	z	z	1	1	d
z	z	z	z	z	x	x	d	d	1

Then $\mathfrak{X} := (X, \ominus, 1)$ is a WFI-algebra. Note that $F_1 := \{1, a, b, c, x, y\}$ and $F_2 := \{1, a, b, c, d, e\}$ are weak filters of $\mathfrak{X}$, but the set $F_3 := \{1, a, d\}$ is not a weak filter of $\mathfrak{X}$.

Theorem 3.8. *Every weak filter of $\mathfrak{X}$ contains the simulative part $\mathcal{S}(\mathfrak{X})$ of $\mathfrak{X}$.*

Proof. Let F be a weak filter of $\mathfrak{X}$. Let $x \in \mathcal{S}(\mathfrak{X})$. If $x \notin F$, then $x \ominus 1 \notin F$ by (3.4). But $x \in \mathcal{S}(\mathfrak{X})$ implies $x \ominus 1 = 1 \in F$, a contradiction. Hence $x \in F$, and so $\mathcal{S}(\mathfrak{X}) \subseteq F$. □

Corollary 3.9. *Every weak filter F of $\mathfrak{X}$ satisfies* (3.2).

Theorem 3.10. *Let F be a closed filter of $\mathfrak{X}$. If F contains the simulative part $\mathcal{S}(\mathfrak{X})$ of $\mathfrak{X}$, then F is weak.*

Proof. Let F be a closed filter of $\mathfrak{X}$ that contains the simulative part $\mathcal{S}(\mathfrak{X})$ of $\mathfrak{X}$. Let $x \in F$ and $y \in X \setminus F$. If $y \ominus x \in F$, then $[y,x]^2 \in F$ since F is closed. Since $y \ominus [y,x]^2 = 1$, it follows from (b4) and (a1) that

$$1 = ([y,x]^2 \ominus y) \ominus (y \ominus y) = ([y,x]^2 \ominus y) \ominus 1.$$

Hence $[y,x]^2 \ominus y \in \mathcal{S}(\mathfrak{X}) \subseteq F$, and so $y \in F$. This is a contradiction, and thus $y \ominus x \in X \setminus F$. Therefore F is weak. □

The following is a characterization of a weak filter.

Theorem 3.11. *A filter F of $\mathfrak{X}$ is weak if and only if it satisfies:*

$$(\forall x, y \in X)\,(x \ominus y \in F \ \& \ y \in F \ \Rightarrow \ x \in F). \tag{3.5}$$

Proof. Assume that F is a weak filter of $\mathfrak{X}$. Let $x, y \in X$ be such that $x \ominus y \in F$ and $y \in F$. If $x \notin F$, then $x \ominus y \notin F$ by (3.4). This is a contradiction. Hence $x \in F$. Conversely let F be a filter of $\mathfrak{X}$ in which the condition (3.5) is valid. Let $x \in F$ and $y \in X \setminus F$. Suppose $y \ominus x \in F$. Then $y \in F$ by (3.5), a contradiction. Hence $y \ominus x \in X \setminus F$, and F is weak. □

Definition 3.12. [2] A nonempty subset F of $\mathfrak{X}$ is called a *concrete filter* of $\mathfrak{X}$ if it satisfies (c1) and

$$(\forall x, z \in X)\,(\forall y \in F)\,((x \ominus y) \ominus (x \ominus z) \in F \ \Rightarrow \ z \in F). \tag{3.6}$$

Note that every concrete filter of $\mathfrak{X}$ is a filter of $\mathfrak{X}$, but the converse is not true in general (see [2]).

Lemma 3.13. *The simulative part $\mathcal{S}(\mathfrak{X})$ of $\mathfrak{X}$ is a concrete filter of $\mathfrak{X}$.*

Proof. Let $x, y, z \in X$ be such that $y \in \mathcal{S}(\mathfrak{X})$ and $(x \ominus y) \ominus (x \ominus z) \in \mathcal{S}(\mathfrak{X})$. Using (a3) and (a4), we have

$$(y \ominus z) \ominus ((x \ominus y) \ominus (x \ominus z)) = (x \ominus y) \ominus ((y \ominus z) \ominus (x \ominus z)) = 1.$$

Putting $w = (x \ominus y) \ominus (x \ominus z)$ and using (a1) and (b4), we obtain

$$(w \ominus (y \ominus z)) \ominus 1 = (w \ominus (y \ominus z)) \ominus (w \ominus w) = 1,$$

i.e., $((x \ominus y) \ominus (x \ominus z)) \ominus (y \ominus z) \preceq 1$. Hence $((x \ominus y) \ominus (x \ominus z)) \ominus (y \ominus z) \in \mathcal{S}(\mathfrak{X})$. Since $\mathcal{S}(\mathfrak{X})$ is a filter of $\mathfrak{X}$ (see Proposition 2.4), it follows from (c2) that $z \in \mathcal{S}(\mathfrak{X})$. Therefore $\mathcal{S}(\mathfrak{X})$ is a concrete filter of $\mathfrak{X}$. □

Theorem 3.14. *Let F be a filter of $\mathfrak{X}$. Then F is concrete if and only if $\mathcal{S}(\mathfrak{X}) \subseteq F$.*

Proof. Assume that F is a concrete filter of $\mathfrak{X}$. Let $x \in \mathcal{S}(\mathfrak{X})$. Then $x \ominus 1 = 1$, and so

$$(x \ominus 1) \ominus (x \ominus x) = 1 \ominus 1 = 1 \in F.$$

It follows from (3.6) that $x \in F$. Hence $\mathcal{S}(\mathfrak{X}) \subseteq F$.

Conversely, suppose $\mathcal{S}(\mathfrak{X}) \subseteq F$. Let $x, y, z \in X$ be such that $y \in F$ and $(x \ominus y) \ominus (x \ominus z) \in F$. By the similar argument as the proof of Lemma 3.13, we have

$$((x \ominus y) \ominus (x \ominus z)) \ominus (y \ominus z) \in \mathcal{S}(\mathfrak{X}) \subseteq F.$$

Since F is a filter of $\mathfrak{X}$, it follows from (c2) that $z \in F$. Hence F is a concrete filter of $\mathfrak{X}$. □

Combining Lemma 3.13 and Theorem 3.14, we know that the simulative part $\mathcal{S}(\mathfrak{X})$ of $\mathfrak{X}$ is the least concrete filter of $\mathfrak{X}$.

Lemma 3.15. [2] *A filter F of $\mathfrak{X}$ is concrete if and only if it satisfies the following implication.*

$$(\forall x \in X)\,([x, 1]^2 \in F \Rightarrow x \in F). \tag{3.7}$$

Theorem 3.16. *Every weak filter is a concrete filter.*

Proof. Let F be a weak filter of $\mathfrak{X}$. Let $x \in X$ be such that $[x, 1]^2 \in F$. Since $x \ominus [x, 1]^2 = 1 \in F$, it follows from Theorem 3.11 that $x \in F$. Hence F is a concrete filter of $\mathfrak{X}$ by Lemma 3.15. □

Let $\mathbb{Z}$ be the set of integers. Then $\mathfrak{Z} := (\mathbb{Z}; \preceq, 0)$ is a simulative WFI-algebra, where $x \preceq y = y - x$ for all $x, y \in \mathbb{Z}$. Note that $F := \{0, 1, 2, 3, \cdots\}$ is a concrete filter of $\mathfrak{Z}$ which is not a weak filter of $\mathfrak{Z}$. This shows that the converse of Theorem 3.16 is not true in general.

Denote by $r_c(\mathfrak{X})$ the set

$$r_c(\mathfrak{X}) := \bigcap_{\alpha \in X} r_c(\alpha),$$

where $r_c(\alpha) := \{x \in X \mid x \preceq \alpha \Rightarrow x = \alpha\}$. The *doubly simulative part* of $\mathfrak{X}$ is defined to be the set (see [3])

$$\mathcal{DS}(\mathfrak{X}) := \{x \in X \mid [x, 1]^2 = x\}.$$

Note that $\mathcal{DS}(\mathfrak{X})$ is a subalgebra of $\mathfrak{X}$ (see [3]).

Lemma 3.17. *We have the following assertions:*

(i) $r_c(\mathfrak{X}) = \mathcal{DS}(\mathfrak{X})$.

(ii) *$\mathfrak{X}$ is simulative if and only if $X = r_c(\mathfrak{X})$.*

Proof. (i) Let $x \in r_c(\mathfrak{X})$. Since $x \preceq [x,1]^2$ by (b1), we have $x = [x,1]^2$. Therefore $x \in \mathcal{DS}(\mathfrak{X})$. Conversely, let $x \in \mathcal{DS}(\mathfrak{X})$ and $y \in X$ be such that $x \preceq y$. Then

$$\begin{aligned} y \ominus x &= y \ominus [x,1]^2 = (x \ominus 1) \ominus (y \ominus 1) \\ &= (x \ominus 1) \ominus (y \ominus (x \ominus y)) \\ &= (x \ominus 1) \ominus (x \ominus 1) = 1, \end{aligned}$$

and so $x = y$, i.e., $x \in r_c(\mathfrak{X})$.

(ii) If $\mathfrak{X}$ is simulative, then $X = \mathcal{DS}(\mathfrak{X}) = r_c(\mathfrak{X})$ by [4, Theorem 3.4] and (i). The converse is clear. □

Note that a subalgebra of $\mathfrak{X}$ may not be a concrete filter of $\mathfrak{X}$. In the following theorem, we give a condition for a subalgebra to be a concrete filter.

Theorem 3.18. *Every subalgebra of $\mathfrak{X}$ is a concrete filter of $\mathfrak{X}$ if and only if the following assertion is valid:*

$$(\forall x \in X)\,(x \neq 1 \Rightarrow x \in r_c(\mathfrak{X}) \setminus \{1\}). \tag{3.8}$$

Proof. Let F be a subalgebra of $\mathfrak{X}$. Obviously, $1 \in F$. Assume that the assertion (3.8) is valid. Then $X = r_c(\mathfrak{X})$, and so $\mathfrak{X}$ is simulative by Lemma 3.17(ii). Hence $\mathfrak{X}$ is medial by [4, Theorem 3.10]. Thus

$$(x \ominus 1) \ominus (x \ominus y) = (x \ominus x) \ominus (1 \ominus y) = 1 \ominus y = y \tag{3.9}$$

and

$$(x \ominus y) \ominus (x \ominus z) = (x \ominus x) \ominus (y \ominus z) = 1 \ominus (y \ominus z) = y \ominus z \tag{3.10}$$

for all $x, y, z \in X$. Now let $x, y, z \in X$ be such that $(x \ominus y) \ominus (x \ominus z) \in F$ and $y \in F$. Using (3.9) and (3.10), we get

$$z = (y \ominus 1) \ominus (y \ominus z) = (y \ominus 1) \ominus ((x \ominus y) \ominus (x \ominus z)) \in F$$

since F is a subalgebra of $\mathfrak{X}$. Hence F is a concrete filter of $\mathfrak{X}$. Conversely, suppose that every subalgebra of $\mathfrak{X}$ is a concrete filter of $\mathfrak{X}$. Then $r_c(\mathfrak{X})$ is a subalgebra of $\mathfrak{X}$, and so a concrete filter of $\mathfrak{X}$. It follows from Theorem 3.14 that $\mathcal{S}(\mathfrak{X}) \subseteq F$. Since $\mathcal{DS}(\mathfrak{X}) \cap \mathcal{S}(\mathfrak{X}) = \{1\}$, we get $\mathcal{S}(\mathfrak{X}) = \{1\}$ and $X = r_c(\mathfrak{X})$. This shows that the assertion (3.8) is valid. □

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