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STOCHASTIC SINGLE MACHINE SCHEDULING WITH WEIGHTED QUADRATIC EARLY-TARDY PENALTIES

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ABSTRACT. The problem of scheduling n jobs on a single machine is considered when the machine is subject to stochastic breakdowns. The objective is to minimize the weighted squared deviation of job completion times from a common due date. Two versions of the problem are addressed. In the first one the common due date is a given constant,whereas in the second one the common due date is a decision variable. In each case, a general form of deterministic equivalent of the stochastic scheduling problem is obtained when the counting process $N(t)$ related to the machine uptimes is a Poisson process. It is proved that an optimal schedule must be V-shaped in terms of weighted processing time when the agreeable weight condition is satisfied. Based on the V-shape property, two dynamic programming algorithms are proposed to solve both versions of the problem.

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1. Introduction

Scheduling problems involving quadratic early-tardy penalties have received increasing attention in recent years. A number of results have been published in the literature[6,8]. This model corresponds to the situations where only small deviation of the job completion times from due date are acceptable. The problem is to determine a sequence so that the weighted sum of squared deviation(WSSD) of the job completion times about a common due date is minimized. In general, the WSSD problem involves arbitrary processing times and weights (If jobs have equal weights it is SSD problem). There are two versions of the problem, constrained problem and unconstrained problem. In the constrained problem

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the common due date is given, whereas in the unconstrained problem the common due date is a decision variable. Bagchi et al.[2] proposed the enumerative procedures for both versions of the WSSD problem. For the unconstrained WSSD problem,it is not hard to show that the optimal due date for any given sequence is equal to the mean completion time. Therefore the unconstrained WMSD problem is equivalent to the problem of minimizing weighted completion time variance(WCTV). Cai[3] considered WCTV problem where the jobs satisfy the agreeable weight condition. They showed that an optimal sequence must be V-shaped in terms of weighted processing time and presented two dynamic programming algorithms. Merten and Muller[12] propose two special cases of the constrained WCTV problem,one with equal weights and the other with equal processing times.They indicated that the characters of the optimal sequence were not obvious even in these special cases. The case with equal weights (CTV) have received extensive studies. Schrage[14]examines the optimal sequence for the CTV problem and gives a conjecture about the nature of optimal sequences. Kubiak[10] proved that the CTV problem is NP-hard. Gupta et al.[9] propose a heuristic procedure based on genetic algorithm. De et al.[7] and Kubiak [11]present pseudopolynomial dynamic programming algorithms for the CTV problem. Cai[4] extended the CTV problem to the case where the processing times are random variables. A sufficient condition is derived under which optimal sequences are V-shaped.

Most studies on the WSSD(WCTV) problem mentioned above are concerned with the case where the machine is continuously available for processing. However, there are many situations where the machine may be subject to unpredictable breakdowns.

Birge et al[1] considered the single machine problem with breakdowns and regular objective function. They provide a description of the stochastic breakdowns problem. Mittenthal and Raghavachari[13] studied the SSD problem with deterministic processing times and machine subject to stochastic breakdowns. When the counting process related to the machine uptimes is a generalized Poisson process, several properties of optimal sequences,such as V-shaped property, are derived for both versions of the problem. Cai and Tu [5]extended the model of Mittenthal and Raghavachari to the case where the processing times are random variables. A sufficient condition is derived under which optimal sequences are V-shaped.

The present work is motivated by the contributions of Mittenthal and Raghavachari[13] and Cai[3]. In[13] Mittenthal and Raghavachari considered the SSD problem, in which the machine is subject to stochastic breakdowns. Cai[3] studied the WCTV problem where the jobs satisfy the agreeable weight condition and machine without breakdowns. In this paper, we consider WSSD problem in which machine subject to stochastic breakdowns. We first develop the deterministic equivalent of the two versions of the stochastic problem. We then show that an optimal schedule must be V-shaped in terms of weighted processing time when the jobs have agreeable processing times. Based on the V-shaped property, two dynamic programming algorithms with pseudopolynomial time complexity are proposed.

2. Problem description

There are given a single machine and a set of n independent jobs immediately available for processing at time 0. Job J_j requires a positive integer processing time p_j and is assigned a positive integer weight w_j. We denote $\sum_{i=1}^{n} p_j$ and $\sum_{i=1}^{n} w_j$ by P and W, respectively.

The breakdowns of the machine is denoted by a sequence of nonnegative vectors $\{U_i, D_i\}_{i=1}^{\infty}$ with U_i representing the duration of the ith machine uptime and D_i the duration of the ith machine downtime. Assume that the uptimes are independent of the downtimes and that the downtimes are independent and have the same mean $\mu = E[D_i] < \infty$, and same $\sigma = E[D_i^2] < \infty$, for all $i \geq 1$.

For the random sequence $\{U_i\}_{i=1}^{\infty}$,we define a counting process$\{N(t), t \geq 0\}$ by

$$N(t) = \sup\{k \geq 0, S_k \leq t\},$$

where $S_n = \sum_{i=1}^{n} U_i, n \geq 1$, and $S_0 \equiv 0$.

In this paper, we are concerned with a $preempt - resume$ model,i.e., if a breakdown occurs during the processing of a job, the work done on the job prior to the breakdown is not lost, and the processing of the disrupted job can be continued from where it was interrupted as soon as the machine becomes operable.

We also assume that the start time of the first job is zero and jobs are continuously processed.

For a given sequence $\pi = [J_1, J_2, \ldots, J_n]$, the actual completion time of job J_j,denoted by R_j, is a random variable that can be expressed as

$$R_j = C_j + \sum_{i=1}^{N(C_j)} D_i,$$

where $C_j = \sum_{i=1}^{j} p_i$.

We will consider the following two problems:

2.1. Constrained problem. Given a common due date d,determine the job sequence π to minimize

$$F(\pi) = E\left[\sum_{j=1}^{n} w_j (R_j - d)^2\right].$$

2.2. Unconstrained problem. Determine the job sequence π and the common due date d, to minimize

$$F(\pi) = E\left[\sum_{j=1}^{n} w_j (R_j - d)^2\right].$$

We consider the case where the $N(t)$ is a Poisson process. If $N(t)$ is a Poisson process, then $E[N(t)] = at$, $Var[N(t)] = at$ for some $a > 0$.

We now establish the deterministic equivalent of the objective function.

From [13], $E[(R_j - d_j)^2] = [(1 + \mu a)C_j - d]^2 + a\sigma C_j$, therefore, we have

Theorem 1. *If $N(t)$ is a Poisson process, the objective of the constrained problem is equivalent to*

$$F(\pi) = \sum_{j=1}^{n} w_j(\alpha C_j - d)^2 + \beta \sum_{j=1}^{n} w_j C_j, \tag{2.1}$$

where $\alpha = 1 + \mu a$ and $\beta = a\sigma$.

For the unconstrained problem, the common due date d is a decision variable. We first determine the optimal common due date d.

Taking derivative of $F(\pi)$ with respect to d,we get

$$\frac{\partial F(\pi)}{\partial d} = \sum_{j=1}^{n} w_j(-2)(\alpha C_j - d).$$

Let

$$\frac{\partial F(\pi)}{\partial d} = 0,$$

then

$$d^* = \alpha \bar{C},$$

where $\bar{C} = \frac{1}{W} \sum_{j=1}^{n} w_j C_j$ is the mean weighted completion time of sequence π. Note that $\bar{C}$ is sequence dependent.

Theorem 2. *If $N(t)$ is a Poisson process, the objective of the unconstrained problem is equivalent to*

$$F(\pi) = \alpha^2 \sum_{j=1}^{n} w_j(C_j - \bar{C})^2 + \beta \sum_{j=1}^{n} w_j C_j, \tag{2.2}$$

where $\alpha = 1 + \mu a$ and $\beta = a\sigma$.

3. V-shape property

Definition 1. The jobs are said satisfy the agreeable weight condition if $p_i < p_j$ implies $w_i \geq w_j$ for all $1 \leq i, j \leq n$.

A sequence is said to be V-shaped in terms of (weighted) processing times if the jobs before and after the job with the smallest (weighted) processing time are,respectively, in non-increasing and non-decreasing order of (weighted) processing times. For the equal weight problem, Mittenthal and Raghavachari[13]show that the optimal sequence possesses a V-shaped structure in terms of processing times. We now show that, provided that some condition is satisfied, an optimal sequence for a problem with job-dependent weights also possesses a V-shaped structure in terms of weighted processing times.

Theorem 3. *For the constrained problem, the optimal sequence that minimizes $F(\pi)$ must be V-shaped in terms of weighted processing time p_j/w_j,if the agreeable weight condition is satisfied.*

Proof. From (1),

$$\begin{aligned} F(\pi) &= \sum_{j=1}^{n} w_j(\alpha C_j - d)^2 + \beta \sum_{j=1}^{n} w_j C_j \\ &= \alpha^2 \sum_{j=1}^{n} w_j C_j^2 + (\beta - 2\alpha d)\sum_{j=1}^{n} w_j C_j + d^2 \sum_{j=1}^{n} w_j, \end{aligned}$$

since $d^2 \sum_{j=1}^{n} w_j$ is a constant independent of sequence, so the objective is equivalent to

$$F_1(\pi) = \alpha^2 \sum_{j=1}^{n} w_j C_j^2 + (\beta - 2\alpha d)\sum_{j=1}^{n} w_j C_j.$$

(1) $(\beta - 2\alpha d) \geq 0$.

If $(\beta - 2\alpha d) \geq 0$, then the optimal sequence can be obtained by sequencing jobs in nondecreasing order of weighted processing time(WSPT rule).

Suppose an optimal sequence π is not WSPT, then there are two consecutive jobs J_j and J_k such that $p_j/w_j > p_k/w_k$. It implies that $p_j > p_k$ and $w_j \leq w_k$. Let π_1 be the sequence obtained by interchange J_j and J_k. We denote the completion time of job J_i under π_1 by $C_i(\pi_1)$.

It is obvious that

$$C_j(\pi_1) = C_j + p_k, C_k(\pi_1) = C_k - p_j.$$

Suppose under π the starting time of J_j is t, then

$$\begin{aligned} & F_1(\pi_1) - F_1(\pi) \\ = \; & [\alpha^2(2t + p_j + p_k) + (\beta - 2\alpha d)](w_j p_k - w_k p_j) + \alpha^2 p_j p_k (w_j - w_k) \\ < \; & 0. \end{aligned}$$

This contradicts the optimality of π. Hence the optimal sequence must satisfy WSPT rule, and WSPT sequence is a spacial case of the V-shaped.

(2) $(\beta - 2\alpha d) < 0$.

If $(\beta - 2\alpha d) < 0$, the objective is equivalent to

$$F_2(\pi) = \sum_{j=1}^{n} w_j (C_j - \bar{d})^2, \tag{3.1}$$

where $\bar{d} = \frac{2\alpha d - \beta}{2\alpha^2}$.

Suppose an optimal sequence π is not V-shaped, then there are three consecutive jobs J_i, J_j and J_k such that $p_j/w_j > p_i/w_i$ and $p_j/w_j > p_k/w_k$. It implies that $p_j > p_i$ and $p_j > p_k$, $w_j \leq w_i$ and $w_j \leq w_k$. Let π_1 be the sequence obtained by interchange J_i and J_j, and π_2 be the sequence obtained by interchange J_j and J_k. We denote the completion time of job J_i in under π_k by $C_i(\pi_k)(k = 1, 2)$.

It is obvious that

$$C_i(\pi_1) = C_i + p_j, C_j(\pi_1) = C_j - p_i, C_k(\pi_1) = C_k;$$

$$C_i(\pi_2) = C_i, C_j(\pi_2) = C_j + p_k, C_k(\pi_2) = C_k - p_j.$$

By direct calculation, we get

$$\begin{aligned} F_2(\pi_1) - F_2(\pi) &= (2C_i - 2\bar{d} + p_j)(w_i p_j - w_j p_i) - p_i w_j (p_j - p_i). \\ F_2(\pi_2) - F_2(\pi) &= (2C_j - 2\bar{d} + p_k)(w_j p_k - w_k p_j) - p_j w_k (p_k - p_j) \\ &= (2C_i - 2\bar{d} + p_j)(w_j p_k - w_k p_j) \\ &+ p_k(w_j p_k - w_k p_j) + p_j p_k (w_j - w_k). \end{aligned}$$

If $(2C_i - 2\bar{d} + p_j) < 0$, then $F_2(\pi_1) - F_2(\pi) < 0$. Otherwise, if $(2C_i - 2\bar{d} + p_j) \geq 0$, then $F_2(\pi_2) - F_2(\pi) < 0$. It is implies that either $F_2(\pi_1) < F_2(\pi)$ or $F_2(\pi_2) < F_2(\pi)$, this contradicts the optimality of π and proves the theorem. □

Similar to that of the constrained case, the unconstrained problem also possesses a V-shaped structure.

Theorem 4. *For the unconstrained problem, the optimal sequence that minimizes $F(\pi)$ must be V-shaped in terms of weighted processing time p_j/w_j, if the agreeable weight condition is satisfied.*

Proof. The proof is similar to that of the constrained case. Suppose an optimal sequence π is not V-shaped, then there are three consecutive jobs J_i, J_j and J_k such that $p_j/w_j > p_i/w_i$ and $p_j/w_j > p_k/w_k$. It is implies that $p_j > p_i$ and $p_j > p_k$, $w_j \leq w_i$ and $w_j \leq w_k$. Let π_1 be the sequence obtained by interchange J_i and J_j, and π_2 be the sequence obtained by interchange J_j and J_k. We denote the completion time of job J_i in under π_k by $C_i(\pi_k)(k = 1, 2)$.

It is obvious that

$$C_i(\pi_1) = C_i + p_j, C_j(\pi_1) = C_j - p_i, C_k(\pi_1) = C_k;$$

$$C_i(\pi_1) = C_i, C_j(\pi_2) = C_j + p_k, C_k(\pi_2) = C_k - p_j.$$

$$\begin{aligned} \bar{C}(\pi_1) &= \bar{C} - \triangle_1, \triangle_1 = \frac{1}{W}(w_j p_i - w_i p_j), \\ \bar{C}(\pi_2) &= \bar{C} - \triangle_2, \triangle_2 = \frac{1}{W}(w_k p_j - w_j p_k). \end{aligned}$$

By direct calculation, we get

$$\begin{aligned} F(\pi_1) - F(\pi) &= \alpha^2[-W\triangle_1^2 - (w_j p_i^2 - w_i p_j^2) \\ &- 2(C_i - p_i - \bar{C} + \frac{\beta}{2\alpha^2})(w_j p_i - w_i p_j) - 2p_i p_j (w_j - w_i)] \\ &= \alpha^2[-W\triangle_1^2 - (w_j p_i^2 - w_i p_j^2) \\ &- 2(C_j - p_j - \bar{C} + \frac{\beta}{2\alpha^2})(w_j p_i - w_i p_j) \\ &- 2p_i p_j (w_j - w_i) + 2p_i(w_j p_i - w_i p_j)]. \end{aligned}$$

$$
\begin{aligned}
F(\pi_2) - F(\pi) &= \alpha^2[-W\triangle_2^2 - (w_k p_j^2 - w_j p_k^2) \\
&- 2(C_j - p_j - \bar{C} + \frac{\beta}{2\alpha^2})(w_k p_j - w_j p_k) - 2p_j p_k (w_k - w_j)].
\end{aligned}
$$

Noting $(w_j p_i - w_i p_j) < 0$ and $(w_k p_j - w_j p_k) > 0$, so we have: if $(C_j - \bar{C} + \frac{\beta}{2\alpha^2}) \leq \frac{p_j}{2}$ then

$$
\begin{aligned}
F(\pi_1) - F(\pi) &\leq \alpha^2[-W\triangle_1^2 - (w_j p_i^2 - w_i p_j^2) - 2(-\frac{1}{2}p_j)(w_j p_i - w_i p_j) \\
&- 2p_i p_j (w_j - w_i) + 2p_i(w_j p_i - w_i p_j] \\
&= \alpha^2[-W\triangle_1^2 - p_i w_j (p_j - p_i)] < 0.
\end{aligned}
$$

If $(C_j - \bar{C} + \frac{\beta}{2\alpha^2}) \geq \frac{p_j}{2}$ then

$$
\begin{aligned}
F(\pi_2) - F(\pi) &\leq \alpha^2[-W\triangle_2^2 - (w_k p_j^2 - w_j p_k^2) \\
&- 2(-\frac{1}{2}p_j)(w_k p_j - w_j p_k) - 2p_j p_k (w_k - w_j)] \\
&= \alpha^2[-W\triangle_2^2 - p_k(w_k p_j - w_j p_k) - p_j p_k (w_k - w_j)] < 0.
\end{aligned}
$$

It implies that either $F_2(\pi_1) < F_2(\pi)$ or $F_2(\pi_2) < F_2(\pi)$, this contradicts the optimality of π and proves the theorem. □

The structure of optimal sequences of the general case is still not clear if no condition is imposed on the problem parameters. The V-shape property, however, does not hold universally even there are not machine breakdowns. Examples are as follows.

Example 1. Consider constrained problem: $n = 3, p_1 = 5, p_2 = 2, p_3 = 4$; $w_1 = 10, w_2 = 3, w_3 = 4; d = 7.4$.

The optimal schedule is $\pi = [J_1, J_3, J_2]$ (the corresponding objective value is 106.72), which is not V-shaped.

Example 2 ([3]). Consider unconstrained problem: $n = 3, p_1 = 10000, p_2 = 2, p_3 = 9; w_1 = 20000, w_2 = 3, w_3 = 9$.

The optimal schedule is $\pi = [J_1, J_3, J_2]$ (the corresponding objective value is 0.0545), which is not V-shaped.

4. Algorithms

In [3], based on the V-shape property of the optimal sequence, Cai proposed a dynamic programming algorithm of the WCTV problem. Along the lines of Cai, we will propose two dynamic programming algorithms to solve both versions of the WSSD problem.

We first consider constrained problem. From the proof of Theorem 3, if $(\beta - 2\alpha d) \geq 0$, then the optimal sequence can be obtained by sequencing jobs in nondecreasing order of weighted processing time. In the following, we assume that $(\beta - 2\alpha d) < 0$. Let Π be the set of V-shaped sequences . Given a d,we want to find $\pi^* \in \Pi$ that minimizes

$$F_2(\pi) = \sum_{j=1}^{n} w_j (C_j - \bar{d})^2.$$

From the V-shape property, job J_2 must be either immediately prior to job J_1 or immediately after job J_1; job J_3 must be either immediately prior to jobs J_1 and J_2 or immediately after jobs J_1 and J_2 and so on. General, for any given set $\mathcal{N}_i = \{J_1, J_2, \ldots, J_i\} (i = 1, 2, \ldots, n)$, job J_i must be either the first or the last in an optimal sequence for the jobs in $\mathcal{N}_i$. Suppose π_i be the optimal sequence for the jobs in $\mathcal{N}_i$. It is a subsequence of the optimal sequence π^*

Let t_i be the starting time of jobs in $\mathcal{N}_i$. It is clear that for any $i \in \mathcal{N}_i$,the possible values of that t_i may take are contained in the set $\mathcal{T}_i = \{0, 1, 2, \ldots, P - P_i\}$ and $t_n = 0$, where $P_i = \sum_{J_j \in \mathcal{N}_i} p_j$. Let

$$f_i(t_i) = \sum_{J_j \in \mathcal{N}_i} w_j (C_j - \bar{d})^2. \tag{4.1}$$

$f_i(t_i)$ be the contribution of the jobs in $\mathcal{N}_i$ to the objective function (3) subject to starting the processing at time t_i.

Given t_i, if job J_i is first job in π_i, the completion time of job J_i will be $t_i + p_i$, then $f_i(t_i) = w_i(t_i + p_i - \bar{d})^2 + f_{i-1}(t_i + p_i)$. Otherwise, if job J_i is last job in π_i,the completion time of job J_i will be $t_i + P_i$, then $f_i(t_i) = w_i(t_i + P_i - \bar{d})^2 + f_{i-1}(t_i)$. Let

$$f_i^a(t_i) = w_i(t_i + p_i - \bar{d})^2 + f_{i-1}(t_i + p_i), \tag{4.2}$$

$$f_i^b(t_i) = w_i(t_i + P_i - \bar{d})^2 + f_{i-1}(t_i). \tag{4.3}$$

By the principle of optimality of dynamic programming, an optimal sequence must sequence the jobs such that

$$f_i(t_i) = \min\{f_i^a(t_i), f_i^b(t_i)\}, for\ i = 1, 2, \ldots, n, \text{and } t_i \in \mathcal{T}_i \tag{4.4}$$

subject to $f_0(t_i) = 0, f_i(t_i) = \infty, \forall t_i \notin \mathcal{T}_i$.

Based on above results, $f_i(t_i), i = 1, 2, \ldots, n$ and $t_i \in \mathcal{T}_i$ can be computed according to the recurrence relation (5)-(7). Since we assume that the machine start processing its first job at time zero($t_n = 0$), the overall minimal objective function of (3) is equal to $f_n(0)$.

In summary, we propose the following algorithm.

Algorithm 1.

1. For $i = 1, 2, \ldots, n$, calculate $f_i(t_i)$ for all $t_i \in \mathcal{T}$ according to (5)-(7), and let
$$m_i(t_i) = \begin{cases} 1, & \text{if } f_i(t_i) = f_i^a(t_i), \\ 2, & \text{if } f_i(t_i) = f_i^b(t_i), \end{cases}$$
2. Let $t_n = 0, F(\pi^*) = f_n(0)$.
3. Construct the sequence π^* that achieves $F(\pi^*)$ by a backward tracking procedure.

3.1. Let $\mathcal{J}^1_{n+1}$ and $\mathcal{J}^2_{n+1}$ be empty.

3.2. For $i = n, n-1, \ldots, 2$, do

$$\mathcal{J}^1_i = \begin{cases} \{J_i, \mathcal{J}^1_{i+1}\}, & \text{if } m_i(t_i) = 1, \\ \mathcal{J}^1_{i+1}, & \text{if } m_i(t_i) = 2, \end{cases}$$

$$\mathcal{J}^2_i = \begin{cases} \mathcal{J}^2_{i+1}, & \text{if } m_i(t_i) = 1, \\ \{\mathcal{J}^2_{i+1}, J_i\}, & \text{if } m_i(t_i) = 2, \end{cases}$$

$$t_{i-1} = \begin{cases} t_i + p_i, & \text{if } m_i(t_i) = 1, \\ t_i, & \text{if } m_i(t_i) = 2, \end{cases}$$

3.3. Let $\pi^* = \{\mathcal{J}^1_2, J_1, \mathcal{J}^2_2\}$.

Theorem 5. *For the constrained problem, if the agreeable weight condition is satisfied, then*

(1) *The solution π^* generated by Algorithm 1 is optimal.*

(2) *The complexity of Algorithm 1 is $O(n^2P)$.*

Proof. From the principle of optimality of dynamic programming, Algorithm 1 generates the best V-shape sequence.

Step 1 of the algorithm needs $O(n^2P)$ times to enumerate t_i. Step 2 needs $O(nP)$ time and Step 3 needs $O(n)$ time.

We now consider unconstrained problem. We first introduce the following auxiliary problem:

$$\min_{d\in\mathcal{D}} G(d) = \{\min_{\pi\in\Pi} g(\pi, d) = \alpha^2 \sum_{j=1}^{n} w_j (C_j - d)^2 + \beta \sum_{j=1}^{n} w_j C_j\}, \tag{4.5}$$

where $\mathcal{D} = \{1, 1 + 1/W, \ldots, P - 1/W, P\}$.

This problem is to find an optimal $d^* \in \mathcal{D}$ such that $G(d^*)$ is minimal. The function $G(d)$ is defined to be the minimum of $g(\pi, d)$ with respect to π. Since $1 \le C_j \le P$ are integers, it is clear that all possible value that $\bar{C}$ can take are contained in $\mathcal{D}$. □

Theorem 6. $\pi^* \in \Pi$ *is an optimal sequence for unconstrained problem (3) if it minimizes* $g(\pi, d^*), \forall \pi \in \Pi$.

Proof. Let d^* be the optimal solution of problem (8), π^* be the optimal sequence of problem (8) when $d = d^*$ is given.$C^*_j (i = 1, 2, \ldots, n)$ be the completion times corresponding to $\pi^*.\bar{C}^* = \frac{1}{W} \sum_{j=1}^{n} w_j C^*_j$.

Clearly $d = \bar{C}^* \in \mathcal{D}$. So

$$G(d^*) = g(\pi^*, d^*) \le G(\bar{C}^*) = g(\pi^0, \bar{C}^*) \le g(\pi^*, \bar{C}^*),$$

where π^0 is the optimal sequence that minimizes $g(\pi, d)$ with $d = \bar{C}^*$.

From (8), we have

$$\alpha^2 \sum_{j=1}^{n} w_j (C^*_j - d^*)^2 + \beta \sum_{j=1}^{n} w_j C^*_j \le \alpha^2 \sum_{j=1}^{n} w_j (C^*_j - \bar{C}^*)^2 + \beta \sum_{j=1}^{n} w_j C^*_j,$$

which gives $W(\bar{C^*} - d^*)^2 \leq 0$, it means $d^* = \bar{C^*}$.

Hence for any $\pi \in \Pi$ and the corresponding C_j,

$$G(d^*) = g(\pi^*, d^*) \leq G(\bar{C}) = g(\bar{\pi}, \bar{C}) \leq g(\pi, \bar{C}),$$

where $\bar{\pi}$ is the sequence that minimizes $g(\pi, \bar{C})$.

This implies that π^* is an optimal sequence for problem (3). □

Theorem 6 indicates that a solution to the unconstrained problem (3) can be obtained by solving the new problem (8). We now propose a dynamic programming algorithm to solve problem (8).

Given a $d \in \mathcal{D}$, the sequence π will minimize $g(\pi, d)$ if it minimizes $\tilde{g}(\pi, d)$, defined as following

$$\tilde{g}(\pi, d) = g(\pi, d) - W\beta d = \sum_{j=1}^{n}[\alpha^2 w_j(C_j - d)^2 + \beta w_j(C_j - d)].$$

Let

$$f_i(t_i) = \sum_{J_j \in \mathcal{N}_i} [\alpha^2 w_j(C_j - d)^2 + \beta w_j(C_j - d)]. \tag{4.6}$$

$f_i(t_i)$ be the contribution of the jobs in $\mathcal{N}_i$ to the overall objective function $\tilde{g}(\pi, d)$ subject to starting the processing at time t_i.Given t_i, if job J_i is first job in π_i, the completion time of job J_i will be t_i+p_i, then $f_i(t_i) = \alpha^2 w_i(t_i+p_i-d)^2+\beta w_i(t_i+p_i-d)+f_{i-1}(t_i+p_i)$. Otherwise, if job J_i is last job in π_i, the completion time of job J_i will be t_i+P_i,then $f_i(t_i) = \alpha^2 w_i(t_i+P_i-d)^2+\beta w_i(t_i+P_i-d)^2+f_{i-1}(t_i)$. Let

$$f_i^a(t_i) = \alpha^2 w_i(t_i + p_i - d)^2 + \beta w_i(t_i + p_i - d) + f_{i-1}(t_i + p_i), \tag{4.7}$$

$$f_i^b(t_i) = \alpha^2 w_i(t_i + P_i - d)^2 + \beta w_i(t_i + P_i - d)^2 + f_{i-1}(t_i). \tag{4.8}$$

By the principle of optimality of dynamic programming, an optimal must be sequence the jobs such that

$$f_i(t_i) = \min\{f_i^a(t_i), f_i^b(t_i)\}, for\ i = 1, 2, \ldots, n, and\ t_i \in \mathcal{T}_i \tag{4.9}$$

subject to $f_0(t_i) = 0, f_i(t_i) = \infty, \forall t_i \notin \mathcal{T}_i$.

Based on above results, $f_i(t_i), i = 1, 2, \ldots, n$ and $t_i \in \mathcal{T}_i$ can be computed according to the recurrence relation (10)-(12). Since we assume that the machine start processing its first job at time zero($r_n = 0$), the overall minimal objective function of (8) is equal to $f_n(d)$.

In summary, we propose the following algorithm.

Algorithm 2.

1. For $i = 1, 2, \ldots, n$,calculate $f_i(t_i)$ for all $t_i \in \mathcal{T}_i$ according to (10)-(12), and let
$$m_i(t_i) = \begin{cases} 1, & \text{if } f_i(t_i) = f_i^a(t_i), \\ 2, & \text{if } f_i(t_i) = f_i^b(t_i), \end{cases}$$
2. Let $t_n = 0, F(\pi^*) = f_n(0)$.

3. Construct the sequence π^* that achieves $F(\pi^*)$ by a backward tracking procedure.
   3.1. Let $\mathcal{J}_{n+1}^1$ and $\mathcal{J}_{n+1}^2$ be empty,
   3.2. For $i = n, n-1, \ldots, 2$, do

$$\mathcal{J}_i^1 = \begin{cases} \{J_i, \mathcal{J}_{i+1}^1\}, & \text{if } m_i(t_i) = 1, \\ \mathcal{J}_{i+1}^1, & \text{if } m_i(t_i) = 2. \end{cases}$$

$$\mathcal{J}_i^2 = \begin{cases} \mathcal{J}_{i+1}^2, & \text{if } m_i(t_i) = 1, \\ \{\mathcal{J}_{i+1}^2, J_i\}, & \text{if } m_i(t_i) = 2. \end{cases}$$

$$t_{i-1} = \begin{cases} t_i + p_i, & \text{if } m_i(t_i) = 1, \\ t_i, & \text{if } m_i(t_i) = 2. \end{cases}$$

   3.3. Let $\pi^* = \{\mathcal{J}_2^1, J_1, \mathcal{J}_2^2\}$.

Theorem 7. *For the unconstrained problem, if the agreeable weight condition is satisfied, then*
(1) The solution π^* generated by Algorithm 1 is optimal.
(2) The complexity of Algorithm 1 is $O(nWP)$.

Proof. From the principle of optimality of dynamic programming, Algorithm 1 generates the best V-shape sequence.

Step 1 of the algorithm needs $O(nWP)$ times to enumerate t_i. Step 2 needs $O(WP)$ time and Step 3 needs $O(n)$ time. □

5. Conclusions

In this paper, we have examined the single machine scheduling problem in which the machine is subject to stochastic breakdowns and jobs have deterministic processing times. The objective is to minimize the weighted squared deviation of job completion times from a common due date. Two versions of the problem are addressed.In the first one the common due date is a given constant, whereas in the second one the common due date is a decision variable. We first develop the deterministic equivalent of the two versions of the stochastic problem when the counting process $N(t)$ related to the machine uptimes is a Poisson process. We then show that an optimal schedule must be V-shaped in terms of weighted processing time when the jobs satisfy the agreeable weight condition. Based on the V-shaped property, two dynamic programming algorithms with pseudopolynomial time complexity are proposed. Further study includes the investigation of the problems where the jobs have arbitrary due dates,or the counting process related to the machine uptimes is a nonhomogeneous Poisson process.

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