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ON SOME NEW NONLINEAR DELAY AND WEAKLY SINGULAR INTEGRAL INEQUALITIES

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ABSTRACT. This paper deals with some new nonlinear delay and weakly singular integral inequalities of Gronwall-Bellman type. These results generalize the inequalities discussed by Xiang and Kuang [19]. Several other inequalities proved by Medveď [15] and Ou-Iang [17] follow as special cases of this paper. This work can be used in the analysis of various problems in the theory of certain classes of differential equations, integral equations and evolution equations. A modification of the Ou-Iang type inequality with delay is also treated in this paper.

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1. Introduction

Integral inequalities play a vital role in the study of many differential and integral equations and their applications. In recent years, considerable attention has been given to various generalizations, extensions and refinements of integral inequalities by many authors. In particular, the Gronwall-Bellman inequality and new nonlinear delay and weakly singular integral inequalities of Gronwall-Bellman type have received a special attention by many authors including Cheung and Ma [4-5], Lipovan [8, 9], Ma and his collaborators [10-14], Medveď [15-16], Tartar [18], Xiang and Kuang [19] for the investigation of differential, integral and evolution equations.

In order to investigate the optimal control problem for a class of delay systems, Xiang and Kuang [19] have considered two useful linear integral inequalities with delay and singularity as follows:

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$$\left.\begin{aligned} &\|x(t)\| \le a + b\int_0^t \|x(s)\|ds + c\int_0^t \|x_s\|_C ds, && 0 \le t \le T, \\ &x(t) = \psi(t), && -\tau \le t \le 0, \end{aligned}\right\} \tag{1.1}$$

and

$$\left.\begin{aligned} &\|x(t)\| \le a + b\int_0^t (t-s)^{\beta-1}\|x(s)\|ds + c\int_0^t \|x_s\|_C ds, && 0 < t < T, \\ &x(t) = \psi(t), && -\tau \le t \le 0, \end{aligned}\right\} \tag{1.2}$$

The main purpose of this paper is to generalize inequalities (1.1) and (1.2) to more general nonlinear integral inequalities that can be used as an effective tool compared to other available inequalities in the literature. In other words, some new nonlinear delay and weakly singular integral inequalities of Gronwall-type are proved in this paper. These results are extensions of delay and weakly singular inequalities discussed by Xiang and Kuang [19]. In addition, several other inequalities proved by Medveď [15] and Ou-Iang [17] follow as special cases of results in this paper.

2. Main result

Throughout this paper, let X and Y be two Banach spaces, $\mathcal{L}(X,Y)$ denote the space of bounded linear operators from X to Y. Particularly, $\mathcal{L}(X) = \mathcal{L}(X,X)$ whose norm is denoted by $\|\cdot\|_{\mathcal{L}}$. Suppose $r > 0$, $T > 0$ and $I = [0,T]$. Let $C([-r,a],X)$ be the Banach space of continuous functions from $[-r,a]$ to X with the usual supremum norm. If $a = 0$, this space is simply denoted by C with its norm denoted by $\|\cdot\|_C$. Obviously, for any $x \in C([-r,T],X)$ and $t \in I$, we define $x_t(\theta) = x(t+\theta)$ for $r \le \theta \le 0$ so that $x_t \in C$.

Theorem 2.1. *Let $\psi \in C$, $a(t)$, $b(t)$, $c(t)$ and $w(t) \in C(I,R_+)$. Let $a(t)$ and $w(t)$ be nondecreasing with $w(t) > 0$ for $t > 0$. Let $x \in C([-r,T],X)$ and satisfy the following inequality:*

$$\left.\begin{aligned} &\|x(t)\| \le a(t) + \int_0^t b(s)w(\|x(s)\|)ds + \int_0^t c(s)\|x_s\|_C ds, && 0 \le t \le T, \\ &x(t) = \psi(t), && -\tau \le t \le 0, \end{aligned}\right\} \tag{2.1}$$

then

$$\|x(t)\| \le G^{-1}\left\{G\left[\exp\left(\int_0^t c(s)ds\right)\left(a(t) + \|\psi\|_C\int_0^t c(s)ds\right)\right] + \exp\left(\int_0^t c(s)ds\right)\int_0^t b(s)ds\right\}, \quad 0 \le t \le T_1, \tag{2.2}$$

where

$$G(v) = \int_{v_0}^v \frac{d\xi}{w(\xi)}, \qquad v \ge v_0 > 0, \tag{2.3}$$

G^{-1} is the inverse of function G, and $0 < T_1 \le T$ is chosen so that the quantity in the curly brackets of (2.2) is in the range of G.

Proof. We define a nonnegative and nondecreasing function $u(t)$ by the right-hand side of (2.1)

$$u(t) = a(t) + \int_0^t b(s)w(\|x(s)\|)ds + \int_0^t c(s)\|x_s\|_C ds, \tag{2.4}$$

then

$$\|x(t)\| \le u(t), \qquad 0 \le t \le T. \tag{2.5}$$

Note that for $0 \le t \le T$,

$$\|x_t\| = \sup_{-r\le\theta\le 0} \|x(t+\theta)\| \le \sup_{-r\le\tau\le 0} \|x(\tau)\| + \sup_{0\le\tau\le t} \|x(\tau)\| \le \|\psi\|_C + u(t). \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4) gives

$$u(t) \le a(t) + \int_0^t b(s)w(\|u(s)\|)ds + \int_0^t c(s)(\|\psi\|_C + u(s))ds. \tag{2.7}$$

Setting

$$\overline{u}(t) = a(t) + \int_0^t c(s)\|\psi\|_C ds + \int_0^t b(s)w(u(s))ds,$$

then inequality (2.7) can be written as

$$u(t) \le \overline{u}(t) + \int_0^t c(s)u(s)ds. \tag{2.8}$$

Since $\overline{u}(t)$ is nondecreasing, by the Bellman inequality (see Mitrinovic et al. [2]), it follows from (2.8) that

$$u(t) \le \left(\exp\int_0^t c(s)ds\right)\left(a(t) + \int_0^t c(s)\|\psi\|_C ds + \int_0^t b(s)w(u(s))ds\right) \tag{2.9}$$

We fix any positive number $\widetilde{T}$ $(\le T_1)$, so that we obtain from (2.9) that

$$u(t) \le \left(\exp\int_0^{\widetilde{T}} c(s)ds\right)\left(a(\widetilde{T}) + \int_0^{\widetilde{T}} c(s)\|\psi\|_C ds + \int_0^t b(s)w(u(s))ds\right), \qquad 0 \le t \le \widetilde{T}.$$

Let $\varepsilon > 0$ be an arbitrary small constant, we define a positive function $v(t)$ by

$$v(t) \tag{2.10}$$
$$= \left(\exp \int_0^{\widetilde{T}} c(s)ds\right)\left(a(\widetilde{T}) + \varepsilon + \int_0^{\widetilde{T}} c(s)\|\psi\|_C ds + \int_0^t b(s)w(u(s))ds\right),$$
$$0 \le t \le \widetilde{T}.$$

Then

$$u(t) \le v(t), \qquad 0 \le t \le \widetilde{T}. \tag{2.11}$$

By differentiation, we derive from (2.10) and (2.11) that

$$\frac{dv}{dt} \le \left(\exp \int_0^{\widetilde{T}} c(s)ds\right) b(t)w(v(t)),$$

i.e.,

$$dG(v(t)) = \frac{dv}{w(v(t))} \le \left(\exp \int_0^{\widetilde{T}} c(s)ds\right) b(s).$$

Integrating both sides of the last inequality yields

$$G(v(t)) \le G\left[\left(\exp \int_0^{\widetilde{T}} c(s)ds\right)\left(a(\widetilde{T}) + \varepsilon \int_0^{\widetilde{T}} c(s)\|\psi\|_C ds\right)\right] + \int_0^t \left(\exp \int_0^{\widetilde{T}} c(s)ds\right) b(s)ds.$$

Taking $t = \widetilde{T}$ in the last inequality and then letting $\varepsilon \to 0$, we obtain

$$G(v(\widetilde{T})) \le G\left[\left(\exp \int_0^{\widetilde{T}} c(s)ds\right)\left(a(\widetilde{T}) + \int_0^{\widetilde{T}} c(s)\|\psi\|_C ds\right)\right] + \int_0^{\widetilde{T}} \left(\exp \int_0^{\widetilde{T}} c(s)ds\right) b(s)ds.$$

Since $\widetilde{T} \in (0, T_1]$ is arbitrary, it follows from (2.11) and the last inequality that

$$u(t) \le v(t), \quad 0 < t \le T_1, \tag{2.12}$$

and

$$G(v(t)) \le G\left[\left(\exp \int_0^t c(s)ds\right)\left(a(t) + \int_0^t c(s)\|\psi\|_C ds\right)\right] + \left(\exp \int_0^t c(s)ds\right)\int_0^t b(s)ds, \qquad 0 < t \le T_1,$$

or,

$$v(t) \le G^{-1}\left\{G\left[\left(\exp\int_0^t c(s)ds\right)\left(a(t)+\int_0^t c(s)\|\psi\|_C ds\right)\right] + \left(\exp\int_0^t c(s)ds\right)\int_0^t b(s)ds\right\}, \quad 0<t\le T_1. \quad (2.13)$$

Using (2.5) and (2.12)-(2.13), we obtain

$$\|x(t)\| \le G^{-1}\left\{G\left[\left(\exp\int_0^t c(s)ds\right)\left(a(t)+\int_0^t c(s)\|\psi\|_C ds\right)\right] + \left(\exp\int_0^t c(s)ds\right)\int_0^t b(s)ds\right\}, \quad 0<t\le T_1. (2.14)$$

In view of (2.1), inequality (2.14) also holds when $t=0$. □

Remark 1. (i) Under the hypotheses of Theorem 2.1, if $w \in H_\phi$ (see Dannan [6]), i.e., there is a nonnegative continuous function ϕ such that $w(uv) \le \phi(u)w(v)$ for u, $v \ge 0$, we can get a sharper estimation to $\|x(t)\|$ as follows:

$$\|x(t)\| \le \left(\exp\int_0^t c(s)ds\right) G^{-1}\left\{G\left(a(t)+\int_0^t c(s)\|\psi\|_C ds\right) + \int_0^t b(s)\exp\left(-\int_0^s c(\tau)d\tau\right)\phi\left(\exp\int_0^s c(\tau)d\tau\right)ds\right\}, \quad 0\le t\le T_1', \quad (2.15)$$

where G and G^{-1} are defined as in Theorem 2.1, and $0 < T_1' \le T$ is chosen so that the quantity of the curly brackets in (2.15) in the range of G.

(ii) When $w(u) \equiv u$, $a(t) \equiv a$, $b(t) \equiv b$ and $c(t) \equiv c$, by (2.15), we can get the result of Lemma 1.1 of Xiang and Kuang [19].

Theorem 2.2. *Let $a(t)$, $b(t)$, $c(t)$ and $\psi(t)$ be as in Theorem 2.1. Let $\omega \in C^1(R_+, R_+)$ with ω' nondecreasing and $\omega'(u) > 0$ for $u > 0$. If $x \in C([-r,T], X)$ satisfies the following inequality*

$$\left.\begin{aligned} &\omega\|x(t)\| \le a(t) + \int_0^t b(s)\,\omega'(\|x(s)\|)\,ds \\ &\qquad\qquad \times\left[\omega(\|x(s)\|)\,ds + \int_0^t c(s)\,\|x_s\|_C\right]ds, \\ &\qquad\qquad\qquad 0\le t\le T, \\ &x(t) = \psi(t), \qquad -\tau\le t\le 0, \end{aligned}\right\} \quad (2.16)$$

then

$$\|x(t)\| \le G^{-1}\left\{G\left[\exp\left(\int_0^t c(s)ds\right)\left(\omega^{-1}(a(t)) + \|\psi\|_C\int_0^t c(s)ds\right)\right] + \exp\left(\int_0^t c(s)ds\right)\int_0^t b(s)ds\right\}, \quad 0\le t\le T_2, \quad (2.17)$$

where G and G^{-1} are defined as in Theorem 2.1, ω^{-1} is the inverse function of ω and $0 < T_2 \le T$ is chosen so that the quantity in the curly brackets of (2.17) is in the range of G.

Proof. Let $\varepsilon > 0$ be an arbitrary small real number. Fixing any positive number $\widetilde{T}$ ($\le T_2$), we define a positive nondecreasing function $u(t)$ by

$$\omega(u(t)) = a(\widetilde{T}) + \varepsilon + \int_0^t \omega'(\|x(s)\|)\,[b(s)w(\|x(s)\|) + c(s)\|x_s\|_C]\,ds, \qquad 0 \le t \le \widetilde{T}\,. \tag{2.18}$$

Then

$$\|x(t)\| \;\le\; u(t), \qquad 0 \le t \le \widetilde{T}\,. \tag{2.19}$$

By differentiation, we derive from (2.18) and (2.6) that

$$\begin{aligned}\omega'(u(t))\frac{du}{dt} &= \omega'(\|x(t)\|)\,[b(t)w(\|x(t)\|) + c(t)\|x_t\|_C] \\ &\le \omega'(\|u(t)\|)\,[b(t)w(\|u(t)\|) + c(t)u(t) + c(t)\|\psi\|_C]\,,\end{aligned}$$

i.e.,

$$\frac{du}{dt} \;\le\; b(t)w(\|u(t)\|) + c(t)u(t) + c(t)\|\psi\|_C$$

since $u(t) > 0$ for $0 \le t \le \widetilde{T}$, ω' is nondecreasing with $\omega > 0$ for $u > 0$ and (2.19) holds.

Integrating the both sides of the last inequality from 0 to t, we obtain

$$u(t) \;\le\; \omega^{-1}(a(\widetilde{T} + \varepsilon)) + \int_0^t [b(s)w(\|u(s)\|) + c(s)u(s) + c(s)\|\psi\|_C]\,ds, \qquad 0 \le t \le \widetilde{T}\,. \tag{2.20}$$

Using procedure similar to (2.7) and (2.14) as stated in the proof of Theorem 2.1, we can derive from (2.19) and (2.20) that

$$\|x(t)\| \le G^{-1}\left\{G\left[\exp\left(\int_0^t c(s)ds\right)\left(\omega^{-1}(a(t)) + \|\psi\|_C\int_0^t c(s)ds\right)\right] + \exp\left(\int_0^t c(s)ds\right)\int_0^t b(s)ds\right\}, \qquad 0 \le t \le T_2. \tag{2.21}$$

By (2.16), (2.21) also holds when $t = 0$. □

Remark 2. By choosing suitable function ω in Theorem 2.2, we can obtain some useful and interesting inequalities. For instance, let $\omega = u^p$ ($p > 1$ is a constant) in Theorem 2.2,inequality (2.16) is a new class of Ou-Iang [17] delay inequality which is different from the delay inequalities discussed in [4, 8-11] and [13].

We next use a modification of Medveď's [15] method to study a class of new delay integral inequalities with weakly singular and delay which are nonlinear extension of (1.2). For convenience, we cite one definition and two Lemmas from Ma and Yang [12] as follows:

Definition. One ordered nonnegative real numbers $[x, y, z]$ belongs to the first class distribution and denoted by $[x, y, z] \in H_1$, if $x \in (0, 1]$, $y \in (\frac{1}{2}, 1)$ and $z \geq \frac{3}{2} - y$; and it is called the second class distribution and denoted by $[x, y, z] \in H_2$, if $x \in (0, 1]$, $y \in (0, \frac{1}{2}]$ and $z > (1 - 2y^2)/(1 - y^2)$.

Lemma 2.1. *Let α, β, γ and p be positive constants, then*

$$\int_0^t (t^\alpha - s^\alpha)^{p(\beta-1)} s^{p(\gamma-1)} ds = \frac{t^\theta}{\alpha} B\left[\frac{p(\gamma - 1) + 1}{\alpha}, p(\beta - 1)\right], \quad t \geq 0,$$

where $B[\xi, \eta]$ is the Beta function defined by

$$B[\xi, \eta] = \int_0^1 s^{\xi-1}(1 - s)^{\eta-1} ds \ (\Re\xi > 0, \Re\eta > 0)$$

and $\theta = p[\alpha(\beta - 1) + \gamma - 1] + 1$.

Lemma 2.2. *If positive real numbers α, β, γ, p_1 and p_2 satisfy*
(*i*) $[\alpha, \beta, \gamma] \in H_1, p_1 = 1/\beta$; *or*
(*ii*) $[\alpha, \beta, \gamma] \in H_2, p_2 = (1 + 4\beta)/(1 + 3\beta)$, *then we have*

$$B\left[\frac{p_i(\gamma - 1) + 1}{\alpha}, p_i(\beta - 1) + 1\right] \in (0, +\infty)$$

and $\theta_i = p[\alpha(\beta - 1) + \gamma - 1] + 1 \geq 0$ for $i = 1, 2$.

Theorem 2.3. *Suppose that functions $a(t)$, $b(t)$, $c(t)$, $\psi(t)$ and $w(t)$ are defined as in Theorem 2.1. If $x \in C([-r, T], X)$ satisfies the following inequality:*

$$\left.\begin{aligned} \|x(t)\| \leq a(t) &+ \int_0^t (t^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} b(s) w(\|x(s)\|) ds \\ &+ \int_0^t (t^\alpha - s^\alpha)^{\beta-1} s^{\gamma-1} c(s) \|x_s\|_C ds, \quad 0 \leq t \leq T, \\ x(t) = \psi(t), &\qquad -r \leq t \leq 0. \end{aligned}\right\} \quad (2.22)$$

Then, we have

(*i*) *if $[\alpha, \beta, \gamma] \in H_1$,*

$$u(t) \leq \left\{G_1^{-1}\left[G_1(A_1(t) \exp C_1(t)) + B_1(t) \exp C_1(t)\right]\right\}^{1-\beta}, \ t \in [0, T_3], \quad (2.23)$$

where

$$\begin{aligned} A_1(t) &= 3^{\frac{\beta}{1-\beta}}\, a^{\frac{1}{1-\beta}}(t) + \|\psi\|_C^{\frac{1}{1-\beta}} C_1(t), \\ B_1(t) &= 3^{\frac{\beta}{1-\beta}}\, m_1^{\frac{\beta}{1-\beta}}(t) \int_0^t b^{\frac{1}{1-\beta}}(s)ds, \\ C_1(t) &= 6^{\frac{\beta}{1-\beta}}\, m_1^{\frac{\beta}{1-\beta}}(t) \int_0^t c^{\frac{1}{1-\beta}}(s)ds, \\ G_1(t) &= \int_{v_0}^{v} \frac{d\xi}{w^{\frac{1}{1-\beta}}(\xi^{1-\beta})}, v \geq v_0 > 0, \\ m_1(t) &= \frac{1}{\alpha} B\left[\frac{\beta+\gamma-1}{\alpha\beta}, \frac{2\beta-1}{\beta}\right] t^{[\alpha(\beta-1)+\beta+\gamma-1]/\beta}, \end{aligned}$$

where G_1^{-1} is the inverse of G, and $0 < T_3 \leq T$ is chosen so that the quantity in the square brackets of (2.23) is in the range of G_1;

(ii) if $[\alpha, \beta, \gamma] \in H_2$, then

$$u(t) \leq \left\{G_2^{-1}\left[G_2(A_2(t)\exp C_2(t)) + B_1(t)\exp C_2(t)\right]\right\}^{\frac{\beta}{1+4\beta}}, \quad t \in [0, T_4], \tag{2.24}$$

where

$$\begin{aligned} A_2(t) &= 3^{\frac{1+3\beta}{\beta}}\, a^{\frac{1+4\beta}{\beta}}(t) + \|\psi\|_C^{\frac{1+4\beta}{\beta}} C_2(t), \\ B_2(t) &= 3^{\frac{1+3\beta}{\beta}}\, m_2^{\frac{1+\beta}{\beta}}(t) \int_0^t b^{\frac{1+4\beta}{\beta}}(s)ds, \\ C_2(t) &= 6^{\frac{1+\beta}{\beta}}\, m_2^{\frac{1+3\beta}{\beta}}(t) \int_0^t c^{\frac{1+4\beta}{\beta}}(s)ds, \\ G_2(t) &= \int_{v_0}^{v} \frac{d\xi}{w^{\frac{1+4\beta}{\beta}} \xi^{\frac{\beta}{1+4\beta}}}, v \geq v_0 > 0, \\ m_2(t) &= \frac{1}{\alpha} B\left[\frac{\gamma(1+4\beta)-\beta}{\alpha(1+3\beta)}, \frac{4\beta^2}{1+3\beta}\right] t^{[\alpha(1+4\beta)(\beta-1)+\gamma(1+4\beta-\beta]/(1+3\beta)}, \end{aligned}$$

and G_2^{-1} is the inverse of G, and $0 < T_4 \leq T$ is chosen so that the quantity in the square brackets of (2.24) is in the range of G_2.

Proof. To prove (i) and (ii) we introduce the following indices: if $[\alpha, \beta, \gamma] \in H_1$, let $p_1 = \frac{1}{\beta}$, $q_1 = \frac{1}{1-\beta}$; if $[\alpha, \beta, \gamma] \in H_2$, let $p_2 = \frac{1+4\beta}{1+3\beta}$, $q_2 = \frac{1+4\beta}{\beta}$, where $p_i^{-1}\, q_i^{-1} = 1 \quad (i = 1, 2)$.

Using Hölder's inequality with indices p_i and q_i, we obtain from (2.22) that

$$\|x(t)\| \leq a(t) + \left[\int_0^t (t^\alpha - s^\alpha)^{p_i(\beta-1)} s^{p_i(\gamma-1)} ds\right]^{1/p_i} \times \left[\left(\int_0^t b^{q_i}(s) w^{q_i}(\|x(s)\| ds\right)^{1/q_i} + \left(\int_0^t c^{q_i}(s) \|x_s\|_C^{q_i} ds\right)^{1/q_i}\right], 0 \leq t \leq T.$$

By Lemmas 2.1 and 2.2, the last inequality can be rewritten as

$$\|x(t)\| \leq a(t) + \left\{ \frac{1}{\alpha} B \left[\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1 \right] t_i^{\theta} \right\}^{1/p_i} \times \left[\left(\int_0^t b^{q_i}(s) w^{q_i}(\|x(s)\| ds \right)^{1/q_i} + \left(\int_0^t c^{q_i}(s) \|x_s\|_C^{q_i} ds \right)^{1/q_i} \right], \ 0 \leq t \leq T, \tag{2.25}$$

where the value of Beta function B is positive and $\theta_i = p[\alpha(\beta-1)+\gamma-1]+1$ is nonnegative.

Setting

$$m_i(t) = \frac{1}{\alpha} B \left[\frac{p_i(\gamma-1)+1}{\alpha}, p_i(\beta-1)+1 \right] t^{\theta_i}, \tag{2.26}$$

and using the following well known consequence of the Jensen inequality,

$$(K_1 + K_2 + \cdots + K_n)^r \leq n^{r-1}(K_1^r + K_2^r + \cdots + K_n^r), \tag{2.27}$$

where n is a positive integer, $r \geq 1$, $K_1, K_2, \cdots, K_n$ are nonnegative constants, we obtain from (2.25) that

$$\|x(t)\|^{q_i} \leq 3^{q_i-1} \left[a^{q_i}(t) + m_i^{q_i/p_i}(t) \left(\int_0^t b^{q_i}(s) w^{q_i}(\|x(s)\|) ds + \int_0^t c^{q_i}(s) \|x_s\|_C^{q_i} ds \right) \right]. \tag{2.28}$$

By (2.6) and using (2.27) again, we obtain

$$\|x_s\|_C^{q_i} \leq 2^{q_i-1} \left(\|\psi\|_C^{q_i} + \|x(s)\|^{q_i} \right). \tag{2.29}$$

Substituting (2.29) into (2.28) gives

$$\|x(t)\|^{q_i} \leq \left[A_i(t) + 3^{q_i-1} m_i^{q_i/p_i}(t) \int_0^t b^{q_i}(s) w^{q_i}(\|x(s)\|) \right] + 6^{q_i-1} m_i^{q_i/p_i}(t) \int_0^t c^{q_i}(s) \|x_s\|_C^{q_i} ds, \tag{2.30}$$

where

$$A_i(t) = 3^{q_i-1} \left(a^{q_i}(t) + 2^{q_i-1} \|\psi\|_C^{q_i} m_i^{q_i/p_i}(t) \int_0^t c^{q_i}(s) ds \right).$$

Since the function in the square brackets of (2.30) and $6^{q_i-1} m_i^{q_i/p_i}(t)$ are nondecreasing, using a generalized Bellman inequality in Dannan [6] to (2.30), we obtain

$$\|x(t)\|^{q_i} \leq \left[A_i(t) + 3^{q_i-1} m_i^{q_i/p_i}(t) \int_0^t b^{q_i}(s) w^{q_i}(\|x(s)\|) \right] \exp C_i(t), \tag{2.31}$$

where $C_i(t) = 6^{q_i-1} m_i^{q_i/p_i}(t) \int_0^t c^{q_i}(s) ds$.

Let $\varepsilon > 0$ be an arbitrary small constant and fix any positive number $\bar{t}_i \in (0, T_{i+2}]$. Since $A_i(t), C_i(t)$ and $m_i(t)$ are nonnegative and nondecreasing, it follows from (2.31) that

$$\|x(t)\| \leq Z_i^{1/q_i}(t), \qquad t \in [0, \bar{t}_i], \tag{2.32}$$

where

$$Z_i(t) = \left[A_i(\bar{t}_i) + \varepsilon + 3^{q_i-1} m_i^{q_i/p_i}(\bar{t}_i) \int_0^t b^{q_i}(s) w^{q_i}(\|x(s)\|)\right] \exp C_i(\bar{t}_i). \tag{2.33}$$

By differentiation and use of (2.32), we derive from (2.33) that

$$\frac{dZ_i(t)}{dt} \leq \left(3^{q_i-1} m_i^{q_i/p_i}(\bar{t}_i) \exp C_i(\bar{t}_i)\right) b^{q_i}(t) w^{q_i}(Z_i^{1/q_i}(t)),$$

or,

$$\frac{dZ_i(t)}{w^{q_i}(Z_i^{1/q_i}(t))} \leq \left(3^{q_i-1} m_i^{q_i/p_i}(\bar{t}_i) \exp C_i(\bar{t}_i)\right) b^{q_i}(t).$$

Integrating both sides of the last relation from 0 to t, by the definition of G_i, and in view of $Z_i(0) = (A_i(\bar{t}_i) + \varepsilon) \exp C_i(\bar{t}_i)$ from (2.33), we have

$$\begin{aligned} G_i(Z_i(t)) \leq\ & G_i[(A_i(\bar{t}_i) + \varepsilon) \exp C_i(\bar{t}_i)] \\ & + 3^{q_i-1} m_i^{q_i/p_i}(\bar{t}_i)(\exp C_i(\bar{t}_i)) \int_0^t b^{q_i}(s) ds, \qquad t \in [0, \bar{t}_i]. \end{aligned}$$

Taking $t = \bar{t}_i$ in the last inequality and then letting $\varepsilon \to 0$, we obtain

$$G_i(Z_i(t)) \leq G_i[A_i(\bar{t}_i) \exp C_i(\bar{t}_i)] + 3^{q_i-1} m_i^{q_i/p_i}(\bar{t}_i)(\exp C_i(\bar{t}_i)) \int_0^{\bar{t}_i} b^{q_i}(s) ds.$$

Since $\bar{t}_i - \in (0, T_i i + 2]$ is arbitrary, from the last relation and (2.32) we obtain

$$G_i(Z_i(t)) \leq G_i[A_i(t) \exp C_i(t)] + 3^{q_i-1} m_i^{q_i/p_i}(t)(\exp C_i(t)) \int_0^t b^{q_i}(s) ds, \quad t \in (0, T_{i+2}] \tag{2.34}$$

or

$$Z_i(t) \leq G^{-1}\left[G_i[A_i(t) \exp C_i(t)] + 3^{q_i-1} m_i^{q_i/p_i}(t)(\exp C_i(t)) \int_0^t b^{q_i}(s) ds\right], \quad t \in (0, T_{i+2}], \tag{2.35}$$

and

$$\|x(t)\| \leq Z_i^{1/q_i}(t), \qquad t \in (0, T_{i+2}]. \tag{2.36}$$

Hence, by (2.35) and (2.36), we have

$$\|x(t)\| \leq G^{-1}\left[G_i[A_i(t) \exp C_i(t)] + 3^{q_i-1} m_i^{q_i/p_i}(t)(\exp C_i(t)) \int_0^t b^{q_i}(s) ds\right] \quad t \in (0, T_{i+2}]. \tag{2.37}$$

In view of inequality (2.22), (2.37) also holds when $t = 0$.

Finally, considering two situations for $i = 1, 2$ and using parameters α, β and γ to denote p_i, q_i and θ_i in (2.37), we can get the desired inequalities (2.23) and (2.24), respectively. □

Remark 3. In Theorem 2.3, letting $a(t) \equiv a$, $b(t) \equiv b$, $c(t) \equiv c$(a, b and c are constants), $\alpha = \gamma = 1$ and $w(u) \equiv u$, then it follows result of Lemma 1.2 of Xiang and Kuang [19].

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