J. Appl. Math. & Informatics Vol. 26(2008), No. 5 - 6, pp. 851 - 860
Website: http://www.kcam.biz

EXISTENCE OF SPANNING 4-SUBGRAPHS OF AN INFINITE STRONG TRIANGULATION

HWAN-OK JUNG

ABSTRACT. A countable locally finite triangulation is a strong triangulation if a representation of the graph contains no vertex- or edge-accumulation points. In this paper we exhibit the structure of an infinite strong triangulation and prove the existence of connected spanning subgraph with maximum degree 4 in such a graph

AMS Mathematics Subject Classification : 05C38, 05C75
Key words and phrases : strong triangulations, spanning subgraphs, infinite graphs

1. Introduction

All graphs considered here are supposed to be simple and undirected. For terminology not defined here the reader is referred to [2]. Let H be a subgraph of a graph. For a vertex v of H, the *neighborhood* of v respective to H, denoted by $N_H(v)$, is $\{u \in V(H) \mid uv \in E(H)\}$. The *degree* $d_H(v)$ of a vertex v respective to H is the number of $N_H(v)$. H is a *k-subgraph* if $d_H(v) \leq k$ for all $v \in V(H)$. A u, v*-path* in a graph G is a path connecting vertices u and v of G, and in this case u and v are the *endvertices* of the path. A k*-walk* in a graph is a walk that visits every vertex at least once and at most k times. Clearly a 1-walk is just a hamiltonian cycle.

Let G be a connected plane graph. Given a cycle C in G, let $\widehat{C}$ denote the subgraph of G consisted of only the vertices and edges of C but also those lying

Received January 9, 2008. Revised June 7, 2008. Accepted June 10, 2008.

in the interior of C. A cycle C is a *k-cycle* if C contains exactly k vertices. C is called a *facial cycle* if it is a boundary of a face of G, i.e., $\widehat{C} = C$. The *outer cycle* of a finite 2-connected plane graph G is the boundary of the unbounded face of G, i.e., $\widehat{C} = G$. A planar graph G is a *circuit graph* (due to D. Barnette [1]) if it is isomorphic to $\widehat{C}$, where C is a cycle in a 3-connected plane graph. A 2-connected plane graph G is *triangular*, if every inner facial cycle of G is a 3-cycle. Obviously every triangular graph is a circuit graph.

A *vertex-accumulation point* (= *VAP*) of an infinite plane graph G, embedded in $\mathbb{R}^2$, is a point P of $\mathbb{R}^2$ such that the set $\{Q \in \mathbb{R}^2 \mid |Q - P| < \epsilon\}$ contains infinitely many vertices, for all real numbers $\epsilon > 0$. If an embedding of G contains no vertex-accumulation points, it is called a *VAP-free* graph. An *edge-accumulation point free* (=*EAP-free*) graph can also be similarly defined. A *triangulation* is a countable locally finite plane graph, each of which edges is contained in two non-separating triangles. If the graph is VAP- and EAP-free, then it is called a *strong triangulation* (following C. Thomassen [7]).

The search for connected spanning subgraphs of low degree in graphs is a natural generalization of the search for hamiltonian paths or spanning subtrees (see [6] or [10], and compare [4]). It is well known that every finite triangulation of the sphere with no separating 3-cycles is hamiltonian [9]. On the other hand, a classical result of Barnette [1] says that every 3-connected planar graph has a spanning tree with maximum degree at most 3. The notion of the k-walk in a family of walks or paths was introduced as a generalization of various results on the path problems in graphs. It is not difficult to see that the existence of a k-walk in a graph implies the existence of a spanning tree with maximum degree $k + 1$. Gao & Richter [3] proved that every 3-connected planar graph has a 2-walk, and subsequently Thomassen [8] showed that every finite triangulation of an orientable surface of genus g with no noncontractible cycle of length 2^{3g+4} contains a spanning tree of maximum degree 4. As a corollary Gao & Richter gave an interesting result about the existence of a spanning subgraph; namely they showed that every circuit graph with outer cycle C contains a connected spanning 4-subgraph H with $C \subseteq H$ and $d_H(u) = d_H(v) = 2$ for distinct $u, v \in V(C)$.

The aim of this paper is to exhibit of the structure of an infinite strong triangulation and to prove the existence of connected spanning 4-subgraphs in such a graph, which is a slight extension of the theorem of Gao & Richter [3] to infinite graphs. Namely we show the following theorem as a main result.

Theorem. *Every infinite strong triangulation G contains a connected spanning subgraph with $d_G(v) \leq 4$ for all $v \in V(G)$.*

It may be mentioned that Gao & Richter [3] showed that in finite case their result implies the theorem of Barnette [1]. The author also attempts to obtain a similar result from the main theorem in this paper; i.e., using our result, can we prove the existence of spanning 3-tree in infinite strong triangulation? But our attempt unfortunately was unsuccessful.

2. Preliminaries

Let G be a finite connected plane graph. Clearly there exist subgraphs $Q_0, Q_1, \ldots, Q_m$ of G with

$$G = \bigcup_{k=0}^{m} Q_k \quad \text{and} \quad \left[\bigcup_{i=0}^{k-1} Q_i\right] \bigcap Q_k = \{z_k\},\ k = 1, \ldots, m$$

such that Q_k $(k = 1, \ldots, m)$ is either 2-connected or isomorphic to K_2. In the decomposition, the subgraphs Q_k is called *blocks* and the vertices z_k *articulations* of G. In particular, the block is *trivial* if Q_k is isomorphic to K_2, and it is *nontrivial* otherwise. If a block contains at most one articulation, it is called an *endblock* of G. G has a *linear decomposition* (or G is *linear*) if it contains at most 2 endblocks in its decomposition. For a block Q, we may denote ∂Q the outer cycle of Q if Q is nontrivial; but if Q is trivial, we define $\partial Q = Q$.

Gao & Richter [3] proved the following lemma.

Lemma 2.1. *Let G be a circuit graph with outer cycle C and let $u, v \in V(C)$. Then there exists a partition $V_1, \ldots, V_m$ of $V(G) \setminus V(C)$ and there exist vertices $v_1, \ldots, v_m \in V(C) \setminus \{u, v\}$ such that:*

(1) *the subgraph H_i induced by $V_i \cup \{v_i\}$ is a connected graph with linear decomposition, and*
(2) *v_i is contained in an endblock Q_i of H_i with $v_i \in V(Q_i) \setminus \{z_i\}$, where z_i is the articulation of Q_i, for all $i \in \{1, \ldots, m\}$.*

We are now ready to state a modification form of Lemma 2.1 concerning triangular graphs, which is the essential tool in Section 3. But, since it can be shown by similar arguments in the proof of the lemma (see [3]), we omit to prove it in this paper.

Theorem 2.2. *Let G be a 3-connected triangular graph with outer cycle C, and let $u, v, w \in V(C)$ be distinct vertices. Then there exists a spanning subgraph $\widetilde{G}$ of $G - w$ such that:*

(1) *Every nontrivial component of $\widetilde{G}$ has a linear decomposition,*
(2) *u and v lie on a component of $\widetilde{G}$ in common, and each of the remaining components contains exactly one of $V(C) \setminus \{u, v, w\}$.*

In succession, we investigate the structure of an infinite strong triangulation. First, let us recall some concept from [5]. Let C and C' be two disjoint cycles in an infinite strong triangulation G, where C lies in the interior of C'. A (C, C')-*ring* is a subgraph of G, which consists of not only C and C' but also the vertices and edges lying between C and C'. For a (C, C')-ring $\mathcal{R}$, a (C, C')-*bridge* (or simply *bridge*) of $\mathcal{R}$ is either an edge of $\mathcal{R}$ joining C and C' (such a bridge is called *trivial*), or it is a connected component of $\mathcal{R} - (C \cup C')$ together with all edges of $\mathcal{R}$ joining this component to $C \cup C'$. A (C, C')-ring $\mathcal{R}$ is *tight* if it satisfies the following properties:

[T1] C and C' are induced cycles.
[T2] $|V(B) \cap V(C')| \leq 2$, for every bridge B of $\mathcal{R}$.
[T3] If $V(B) \cap V(C') = \{z, z'\}$, $z \neq z'$, for a bridge B, it must hold $zz' \in E(G)$.

Lemma 2.3. *Let C be an induced cycle of an infinite strong triangulation G. Then there exists a cycle C' such that the (C, C')-ring is tight.*

Proof. First, we construct a cycle C' in G satisfying the hypothesis of this Lemma. Let $F := \{J \mid J \text{ is a facial cycle in } G \text{ such that } V(J) \cap V(C) \neq \emptyset\}$ and let E be the set of all vertices of the cycles in F. Then we can see that $|E| < \infty$, since E contains only finite cycles and F is also finite. Furthermore, set $H := G[E]$, i.e. H is the induced subgraph of G containing all elements of E, and let C' be its outer cycle of H. We will now show the (C, C')-ring $\mathcal{R}$ is tight.

As an induced subgraph H of G, C' is an induced cycle. The assertion [T3] is also obvious from the assumption. To show that C and C' are disjoint, suppose to contrary that there exists a vertex x with $x \in V(C) \cap V(C')$. Let y be a vertex on C' adjacent to x. Then, from the fact that all facial cycles in G are triangles, we can find a facial cycle $J = \{x, y, z\}$ such that $yz \notin E(C')$. But since the cycle must be contained in F (since $V(J) \cap V(C) \neq \emptyset$), it follows that $y, z \in E$. Hence we have $yz \in E(H)$, which contradicts our construction of C'.

It remains to be shown that $|V(B) \cap V(C')| \leq 2$ for every bridge B of $\mathcal{R}$. Suppose there exists a bridge B such that $V(B) \cap V(C') = \{y_1, \cdots, y_r\}$, $r \geq 3$. Since B is not a chord and $V(B) \backslash V(C \cup C') \neq \emptyset$, it follows that there exists a y_1, y_r-path P in $B - (C \cup \{y_2, \cdots, y_{r-1}\})$. Thus the facial cycle in $\mathcal{R}$ containing the edge $y_k y_{k+1}$ $(k = 1, \cdots, r-1)$ is not contained in F, and therefore it holds that $y_k \notin E$, $k = 2, \cdots, r-1$, which also contradicts our construction of C'. □

Lemma 2.4. *For any cycle C of an infinite strong triangulation, the subgraph $\widehat{C}$ is a triangular graph.*

Proof. Since every strong triangulation VAP-free, it follows that $\widehat{C}$ is a finite subgraph, and hence it is triangular. □

Now we prove the main result of this section.

Proposition 2.5. *Let G be a strong triangulation and let C_0 be a facial cycle of G. Then there exists a sequence of induced cycles $\{C_0, C_1, C_2, \cdots,\}$ which holds the following properties:*

(1) *The (C_j, C_{j+1})-ring is tight for all $j \in \mathbb{N}$.*

(2) $V(G) = V\Big(\bigcup_{j=0}^{\infty} \widehat{C}_j\Big).$

Proof. It is clear that C_0 is an induced cycle of G. For $j \in \mathbb{N}$ the existence of C_j, related to C_{j-1}, satisfying the condition (1) follows from Lemma 2.3. It remains only to show that the resulting cycles $\{C_0, C_1, C_2, \cdots\}$ hold the condition (2).

Let $x \in V(G)$ be an arbitrary vertex. Since C_j lies in the interior of C_{j+1} ($j \in \mathbb{N}$), it follows that $x \in V(\widehat{C}_{n_x})$, where n_x is a metric distance between x and C_0. Because of $V(\widehat{C}_{n_x}) \subset V\Big(\bigcup_{j=0}^{\infty} \widehat{C}_j\Big)$, we have $V(G) \subseteq V\Big(\bigcup_{j=0}^{\infty} \widehat{C}_j\Big)$. Since it holds clearly that $V(G) \supseteq V\Big(\bigcup_{j=0}^{\infty} \widehat{C}_j\Big)$, we can conclude $V(G) = V\Big(\bigcup_{j=0}^{\infty} \widehat{C}_j\Big)$.

□

Let C be an induced cycle in an infinite strong triangulation G. According to Lemma 2.3 we can construct a cycle C' in G such that (C, C')-ring $\mathcal{R}$ is tight. We denote F be the set of all trivial bridges of $\mathcal{R}$ and let $H := (C \cup C') \cup F$. Then we have exactly $|F|$ facial cycles in G, up to the interior of C and the exterior of C'. For a facial cycle J of G the induced subgraph $\hat{J}$ of G is called a *cell of* $\mathcal{R}$. If $J = \hat{J}$, then the cell J is *empty*. Clearly, in the interior of a cell lies at most one bridge of $\mathcal{R}$ since G is maximal planar.

Now let L be an empty or nonempty cell of $\mathcal{R}$. Because of the conditions (2) and (3) in definition of tightness, L must be one of following two types:

(i) $|V(L) \cap V(C')| = 1,$

(ii) $|V(L) \cap V(C')| = 2.$

In the former case we say that L is *of type 1* and in the latter case L is *of type 2.*

3. 4-subgraphs in circuit and triangular graphs

We begin with the lemma presented by Gao & Richter [3], which is essential tool for the proofs of the successional results.

Lemma 3.1. *Let G be a circuit graph with outer cycle C and let $u, v \in V(C)$ with $u \neq v$. Then there exists a connected spanning 3-subgraph H of G such *that:*

(1) H *contains all edges of* C,
(2) $d_H(u) = d_H(v) = 2$.

Proof. See [3] □

Proposition 3.2. *Let* G *be a connected plane graph with linear decomposition, whose blocks are circuit graphs. Let* Q *and* Q' $(Q \neq Q')$ *be the endblocks of* G *with the articulations* z *and* z', *respectively. Then, for arbitrary given vertices* $v \in V(\partial Q) \setminus \{z\}$ *and* $v' \in V(\partial Q') \setminus \{z'\}$, *there exists a spanning 4-subgraph* H *of* G *such that:*

(1) $E(\partial Q) \subseteq E(H)$ *for each block* Q *of* G,
(2) $d_H(v) = d_H(v') = 2$.

Proof. By assumption, G contains at least 2 blocks; let

$$Q = Q_1, Q_2, \dots, Q_n = Q'$$

be the blocks of G with with the articulations $z_j \in Q_{j-1} \cap Q_j$ $(j = 2, \dots, n)$, and set $z_0 = v$ and $z_{n+1} = v'$. By Lemma 3.1, for each $j \in \{1, \dots, n\}$, there exists a spanning 4-subgraph H_j of Q_j with $E(\partial Q_j) \subseteq E(H_j)$ and $d_{H_j}(z_j) \leq 2$ and $d_{H_j}(z_{j+1}) \leq 2$. Note here that, if Q_j is a trivial block, $d_{H_j}(z_j) = d_{H_j}(z_{j+1}) = 1$. Then $H = \bigcup_{j=1}^{n} H_j$ is a spanning 4-subgraph of G satisfying the assertions in the proposition. □

Combining Proposition 3.2 with Theorem 2.2, we can obtain the following 2 results, which play a prominent role in the proof of our main theorem.

Theorem 3.3. *Let* G *be a 3-connected triangular graph with outer cycle* C. *Further let* $u, v, w \in V(C)$ *be pairwise distinct. Then there exists spanning 4-subgraph* H *of* $G - w$ *with* $|V(C)| - 2$ *components such that:*

(1) u *and* v *lie on a component of* H *in common, and each of the remaining components contains exactly one of* $V(C) \setminus \{u, v, w\}$,
(2) $d_H(u) = d_H(v) = 1$ *and* $d_H(x) \leq 2$ *for all* $x \in V(C) \setminus \{u, v, w\}$.

Proof. Let $\widetilde{G}$ be the spanning subgraph of $G - w$ obtained from Theorem 2.2 which satisfies the conditions (1) and (2) in the theorem. Since every triangular graph is a circuit graph, for each component of $\widetilde{G}$, we use Proposition 3.2 to obtain a spanning 4-subgraph H' of $\widetilde{G}$ (or G). Now consider the degree of u and v. First note that $d_H(u) \leq 1$ and $d_H(v) \leq 1$, since u and v are contained in a component (say K) of H'. If $d_H(u) = d_H(v) = 1$, we set $H := H'$. On the other hand, if $d_H(u) = 2$, we delete one of the two edges of H' incident to u as follows: (The case $d_H(v) = 2$ can be similarly examined.)

First note that in this case u is contained in a nontrivial endblock (say Q) of K. Let x and y be the vertices of Q adjacent to u; in particular assume without loss of generality that x is not an articulation of K. Since Q is triangular, x and y must be adjacent in Q, which follows that $K - uy$ has a linear decomposition with trivial endblock ux. □

Corollary 3.4. *Let G and C as in Theorem 3.3 be given, and let $v, w \in V(C)$ be distinct vertices. Then there exists spanning 4-subgraph of $G - \{v, w\}$ with $|V(C)| - 2$ components such that:*

(1) *Each component of H contains exactly one vertex of $V(C) \setminus \{v, w\}$,*
(2) *$d_H(x) \leq 2$ for all $x \in V(C) \setminus \{v, w\}$.*

Proof. If we denote H' the spanning 4-subgraph of $G - w$ obtained from Theorem 3.3, then $H = H' - v$ clearly is a spanning 4-subgraph of $G - \{v, w\}$ as desired. □

4. Proof of the main theorem

Let G be an infinite strong triangulation and let C be an arbitrary induced cycle in G. Then, by Lemma 2.3, there exists a cycle C' in G such that the (C, C')-ring, say $\mathcal{R}$, is tight. We first choose a cell L_0 of type 1 or 2 with $|V(L_0) \cap V(C)| \geq 2$. Note that, since $|V(C)| \geq 3$, such a cell must exist.

Now we construct a subgraph H_L of each cell L of $\mathcal{R}$ covering most vertices (except for one or two vertices) of L, which satisfies certain degree conditions for the vertices of the outer cycle and the remaining vertices of L.

(a) The cell L_0

Case 1: L_0 is of type 1: Set

$$V(L_0) \cap V(C) = \{x_1, \ldots, x_r\}, r \geq 2, \quad \text{and} \quad V(L_0) \cap V(C') = \{z\}.$$

By Theorem 3.3 we have a spanning 4-subgraph H_{L_0} of $L_0 - x_r$ with $r - 1$ components such that:

(1) z and x_{r-1} are contained in a component of H_{L_0} in common, and each of the remaining components contains exactly one of $\{x_1, \ldots, x_{r-2}\}$,
(2) $d_{H_{L_0}}(z) = 1$, $d_{H_{L_0}}(x_{r-1}) = 1$ and $d_{H_{L_0}}(x_i) \leq 2$ for all $i = 1, \ldots, r - 2$.

Case 2: L_0 is of type 2: Set

$$V(L_0) \cap V(C) = \{x_1, \ldots, x_r\}, r \geq 1, \quad \text{and} \quad V(L_0) \cap V(C') = \{z, z'\}.$$

Note that, from selection of L_0, $|V(L_0) \cap V(C)| \geq 2$; i.e., $r \geq 2$. By Theorem 3.3 we can find a spanning 4-subgraph H_{L_0} of $L_0 - x_r$ with $r-1$ components such that:

(1) z and x_{r-1} are contained in a component of H_{L_0} in common, and each of the remaining components contains exactly one of $\{x_1, \ldots, x_{r-2}, z'\}$,
(2) $d_{H_{L_0}}(z) = 1$, $d_{H_{L_0}}(x_{r-1}) = 1$, $d_{H_{L_0}}(z') \leq 2$ and $d_{H_{L_0}}(x_i) \leq 2$ for all $i = 1, \ldots, r-2$.

(b) $L(\neq L_0)$ **is a cell of** R

Case 1: L is of type 1:

Let $x_1, \ldots, x_r$ and z as Case 1 in (a) be given. By Corollary 3.4 there exists a spanning 4-subgraph H_L of $L - \{x_r, z\}$ with $r-1$ components such that:

(1) Each component of H_L contains exactly one of $\{x_1, \ldots, x_{r-1}\}$,
(2) $d_{H_L}(x_i) \leq 2$ for all $i = 1, \ldots, r-1$.

Case 2: L is of type 2:

Let $x_1, \ldots, x_r$ and z, z' as Case 2 in (a) be given. By Theorem 3.3 we can obtain a spanning 4-subgraph H_L of $L - x_r$ with r components such that:

(1) z and z' are contained in a component of H_L in common, and each of the remaining components contains exactly one of $\{x_1, \ldots, x_{r-1}\}$,
(2) $d_{H_L}(z) = d_{H_L}(z') = 1$ and $d_{H_L}(x_i) \leq 2$ for all $i = 1, \ldots, r-1$.

We are now in position to construct a connected spanning 4-subgraph of $\widehat{C'}$ from a given 4-subgraph of $\widehat{C}$. To do this, assume that, for the cycle C in G, a connected spanning 4-subgraph of $\widehat{C}$ with $d_H(x) \leq 2$ for all $x \in V(C)$ is already constructed. As a 'connection cell' between C and C', we select a cell L_0 of the (C, C')-ring $\mathcal{R}$ with $|V(L_0) \cap V(C)| \geq 2$ as above.

We first set H_{L_0} the spanning 4-subgraph of $L_0 - x_r$ satisfying the conditions in Case (a), and then we denote H_L the spanning 4-subgraph of $L - x_r$ satisfying the conditions in Case (b) for each remaining cells L of $\mathcal{R}$. In order to make a success of the degree condition for the constructed subgraph, we have to delete an edge from the graph. To do this, let us denote $\tilde{L}$ a cell of type 2 of $\mathcal{R}$ with $\tilde{L} \neq L_0$ which contains the vertex z, where z is the vertex of $V(L_0) \cap V(C')$ defined in Case (a). Notice that the number of such cells is one (in the case that L_0 is of type 2) or two (in the case L_0 is of type 1). Finally, by denoting $\tilde{z}$ the vertex of $H_{\tilde{L}}$ adjacent to z, we set

$$H' = H \cup \left[\bigcup_{L \text{ is a cell of } \mathcal{R}} H_L \right] - \{z\tilde{z}\}$$

Then we can show that the constructed graph H' is a spanning 4-subgraph of $\widehat{C'}$ with $d_{H'}(z) \leq 2$ for all $z \in V(C')$. To see this, consider the degree of a vertex on C. By construction we can verify that every vertex on C has degree at most 2. Since $d_H(x) \leq 2$ by assumption, we can conclude $d_{H'}(x) \leq 2+2=4$ for all $x \in V(C)$. Next, consider the vertices on C'. Since every vertex of C' in each cell of type 2 containing this vertex has the degree 1 and since such vertex is contained exactly 2 cells of type 2, it follows that $d_{H'}(z) = 2$ for all $z \in V(C')$. It is not hard to verify the remaining assertions.

We are now prepared to prove the main theorem. Let G be an infinite strong triangulation and let C_0 be a facial cycle of G. By Proposition 2.5, we get a sequence of induced cycles $\{C_0, C_1, C_2, \cdots,\}$ such that the (C_j, C_{j+1})-ring is tight for all $j \in \mathbb{N}$ and $V(G) = V\Big(\bigcup_{j=0}^{\infty} \widehat{C}_j\Big)$.

Obviously $H_0 := C_0$ is a spanning 4-subgraph of $\widehat{C}_0$. Now assume that, for $j \geq 1$, a spanning 4-subgraph H_j of $\widehat{C}_j$ with $d_{H_j}(x) \leq 2$ for all $x \in V(H_j)$ is already constructed. Then, by the claim above and the fact that the (C_j, C_{j+1})-ring is tight, we can obtain a spanning 4-subgraph H_{j+1} of $\widehat{C}_{j+1}$ with the corresponding properties. Therefore we have a sequence of 4-subgraphs $\{H_0, H_1, H_2, \cdots,\}$ in G with $H_j \subseteq H_{j+1}$ and $V(H_j) = V(\widehat{C}_j)$ for all $j \in \mathbb{N}$. By setting $H = V\Big(\bigcup_{j=0}^{\infty} H_j\Big)$, we obtain a spanning 4-subgraph of G, and thus our proof is complete.

References

1. D.W. Barnette, *Trees in polyhedral graphs*, Canad. J. Math. 18 (1966), 731–736.
2. R. Diestel, *Graph Theory* (3nd Ed.), Springer-Verlag, Heidelberg, 2005. 83–93.
3. Z. Gao and R. B. Richter, *2-walks in circuit graphs*, J. Comb. Th. (B) 62 (1994), 259–267.
4. J. Hou, G. Liu, Y. Xin and M. Lan, *Some results on total colorings of planar graphs*, J. Appl. Math. & Informatics 26 (2008), 481–491.
5. H. O. Jung, *Structural properties for certain classes of infinite planar graphs*, J. Appl. Math. & computing 13 (2004), 105–115.
6. C. K. Lin, H. M. Huang, J. J. M. Tan and L. H. Hsu, *On spanning connected graphs*, Disc. Math. 308 (2008), 1330–1333.
7. C. Thomassen, *Straight line representations of infinite planar graphs*, J. London Math. Soc. 16 (1977), 411–423.
8. C. Thomassen, *Trees in triangulations*, J. Comb. Th. (B) 60 (1994), 56–62.
9. H. Whitney, *A theorem on graphs*, Ann. of Math. 32 (1931), 378–390.
10. X. Yu, *Disjoint paths, planarizing cycles, and spanning walks*, Trans. Amer. Math. Soc. 349 (1997), 1333–1358.

Hwan-Ok Jung received his BS degree from Seoul National University, Korea, and MS and Ph. D. from Universität Hamburg, Germany, under the direction of Prof. Dr. R. Halin. Since

1989 he has been a full professor of Hanshin University. His research interests: spanning cycles, planar graphs, hamiltonian paths, spanning subgraphs and factors in infinite graphs.

Department of Mathematics, Hanshin University, Osan, 447-791 Korea
e-mail: jungok@hs.ac.kr