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MIXED TYPE DUALITY FOR CONTROL PROBLEMS WITH GENERALIZED INVEXITY

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ABSTRACT. A mixed type dual to the control problem in order to unify Wolfe and Mond-Weir type dual control problem is presented in various duality results are validated and the generalized invexity assumptions. It is pointed out that our results can be extended to the control problems with free boundary conditions. The duality results for nonlinear programming problems already existing in the literature are deduced as special cases of our results..

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1. Introduction

Optimal control models are very prominent amongst constrained optimization models because of their occurrences in a variety of popular contexts, notably, advertising investment, production and inventory, epidemic, control of a rocket etc. The planning of a river system, where it is required to make the best use of the water, can also be modelled as an optimal control problem. Optimal control models are also potentially applicable to economic planning, and to the world models of the 'Limits to Growth' kind.

Necessary optimality conditions for existence of extremal solution for a variational problem in the presence of inequality and equality constraints were obtained by Valentine [10]. Using Valentine's results, Berkovitz [3] obtained corresponding Fritz John type necessary optimality conditions for a control problem. Mond and Hanson [7] pointed out that if the optimal solution for the problem is normal, then the Fritz John type optimality conditions reduce to Kuhn- Tucker conditions. Using these Karush-Kuhn-Tucker optimality conditions, Mond and Hanson [7] presented Wolfe type dual and established weak, strong and converse duality theorems under convexity conditions. Abraham and Buie [1] studied

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duality for continuous programming and optimal control from a unified point of view. Later Mond and Smart [9] proved that for invex functions, the necessary conditions of Berkovitz [3] together with normality of the constraints, are sufficient for optimality and also derived some duality results under invexity.Recently Husain et al. [6] presented Mond-Weir type dual to the control problem with a view to weaken the invexity assumptions in [9] and derived various duality results and also pointed out that their results have analogues in (static) nonlinear programming.

In this paper, we propose, in the spirit of Bector, Chandra, Abha [2], a mixed type dual to nonlinear programming and establish various various duality theorems under generalized invexity conditions on the functionals that appear in the formulations of the control problems. The formulation of mixed type dual to the control problem combines Wolfe and Mond-Weir type dual control problems. Special cases are discussed to show that our results extend some earlier results in the literature.

2. Control problem and related preliminaries

Let R^n denotes an n-dimensional Euclidean space, $I = [a, b]$ be a real interval and $f : I \times R^n \times R^m \to R$ be a continuously differentiable with respect to each of its arguments. For the function $f(t, x, u)$, where $x : I \to R^n$ is differentiable with its derivative $\dot{x}$ and $u : I \to R^m$ is the smooth function, denote the partial derivatives of f by f_t, f_x and f_u, where

$$f_t = \frac{\partial f}{\partial t},\ f_x = \left(\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n}\right)^T,\ f_u = \left(\frac{\partial f}{\partial u^1}, \dots, \frac{\partial f}{\partial u^m}\right)^T,$$

$$x = (x^1, \dots, x^n)^T \text{ and } u = (u_1, \dots, u_m)^T.$$

For an m-dimensional vector function $g(t, x, u)$, the gradient with respect to x is

$$g_x = \begin{pmatrix} \frac{\partial g^1}{\partial x^1}, \dots, \frac{\partial g^p}{\partial x^n} \\ \vdots \\ \frac{\partial g^1}{\partial x^n}, \dots, \frac{\partial g^p}{\partial x^n} \end{pmatrix},$$

which is an $n \times p$ matrix of first order derivatives. Here $u(t)$ is the control variable and $x(t)$ is the state variable, u is related to x via the state equation $\dot{x} = h(t, x, u)$. Gradients with respect to u are defined analogously.

A control problem is to transfer the state vector from an initial state $x(a) = \alpha$ to a final state $x(b) = \beta$ so as to minimize a functional, subject to constraints on the control and state variables. A control problem can be stated formally as,

Problem (CP) (Primal): $\qquad \underset{x \in X, u \in U}{\text{Minimize}} \int_a^b f(t, x, u) dt$

subject to

$$x(a) = \hat{\alpha}, x(b) = \hat{\beta}, \tag{1}$$

$$h(t, x, u) = \dot{x}, t \in I, \tag{2}$$

$$g(t, x, u) \leq 0, t \in I. \tag{3}$$

(i) f is as before, $g : I \times R^n \times R^m \to R^p$ and $h : I \times R^n \times R^m \to R^n$ are continuously differentiable functions with respect to each of its arguments.

(ii) X is the space of continuously differentiable state functions $x : I \to R^n$ such that $x(a) = \alpha, x(b) = \beta$, equipped with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, and u is the space of piecewise continuous control functions $u : I \to R^m$ has the uniform norm $\|.\|_\infty$, and

(iii) The differential equation (2) for x with the initial conditions expressed as $x(t) = x(a) + \int_a^t h(s, x(s), u(s))ds$, $t \in I$, may be written as $Dx = H(x, u)$, where the map $H : X \times U \to C(I, R^n), C(I, R^n)$ being the space of continuous functions from $I \to R^n$, defined by $H(z, u)(t) = h(t, x(t), u(t))$.

Following Craven [4], the control problem can be expressed as,

(ECP) : $\underset{x \in X, u \in U}{\text{Minimize}}\ F(x, u)$

subject to

$Dx = H(x, u)$,

$-G(x, u) \in S$.

Where G is function from $X \times U$ into $C(I, R^p)$ given by $G(x, u)(t) = g(t, x(t), u(t))$ from $x \in H, u \in U$, and $t \in I$; S is the convex cone of functions in $C(I, R^p)$ whose components are non-negative; thus S has interior points.

Necessary optimality conditions for existence of external solution for a variational problem subject to both equality and inequality constraints was given by Valentine [10]. Invoking Valentine's [10] results, Berkovitz [3] obtained corresponding necessary optimality conditions for the above control problem (CP). Here we mention the Fritz John optimality conditions derived by Craven [4] in the form of the following proposition which will be required in the sequel.

Proposition 1 (Necessary optimality conditions). *If $(\bar{x}, \bar{u}) \in X \times U$ an optimal solution of (CP) and the Fréchet derivatives $Q' = (D - H_x(x, u), -H_u(x, u))$ is surjective, then there exist Lagrange multipliers $\lambda_0 \in R$, and piecewise smooth functions $\lambda : I \to R^p$ and $\mu : I \to R^n$ satisfying, for all $t \in I$,*

$$\lambda_0 f_x(t, \bar{x}, \bar{u}) + \lambda(t)^T g_x(t, \bar{x}, \bar{u}) + \mu(t)^T h_x(t, \bar{x}, \bar{u}) + \dot{\mu}(t) = 0,$$

$$\lambda_0 f_u(t, \bar{x}, \bar{u}) + \lambda(t)^T g_u(t, \bar{x}, \bar{u}) + \mu(t)^T h_u(t, \bar{x}, \bar{u}) = 0,$$

$$\lambda(t)^T g(t, \bar{x}, \bar{u}) = 0,$$

$$(\lambda_0, \lambda(t)) \geq 0,$$

$$(\lambda_0, \lambda(t), \mu(t)) \neq 0.$$

The above conditions will become Karush-Kuhn-Tucker conditions if $\lambda_0 > 0$. Therefore, if we assume that the optimal solutions $(\bar{x}, \bar{u})$ is normal, then without any loss of generality, we can set $\lambda_0 = 1$. Thus from the above we have the

Karush-Kuhn- Tucker type optimality conditions

$$f_x(t,\bar{x},\bar{u}) + \lambda(t)^T g_x(t,\bar{x},\bar{u}) + \mu(t)^T h_x(t,\bar{x},\bar{u}) + \dot{\mu}(t) = 0, \ t \in I, \tag{4}$$
$$f_u(t,\bar{x},\bar{u}) + \lambda(t)^T g_u(t,\bar{x},\bar{u}) + \mu(t)^T h_u(t,\bar{x},\bar{u}) = 0, \ t \in I, \tag{5}$$
$$\lambda(t)^T g(t,\bar{x},\bar{u}) = 0, \ t \in I, \tag{6}$$
$$\lambda(t) \geq 0, \ t \in I. \tag{7}$$

Using these optimality conditions, Mond and Hanson [7] constructed following Wolfe type dual.

Problem (CD_0) (Dual):

$$\text{Maximize} \int_a^b \{f(t,x,u) + \lambda(t)^T g(t,x,u) + \mu(t)^T (h(t,x,u) - \dot{x})\}dt$$

subject to

$$f_x(t,x,u) + \lambda(t)^T g_x(t,\bar{x},\bar{u}) + \mu(t)^T h_x(t,x,u) + \dot{\mu}(t) = 0, \ t \in I,$$
$$f_u(t,x,u) + \lambda(t)^T g_u(t,\bar{x},\bar{u}) + \mu(t)^T h_u(t,x,u) = 0, \ t \in I,$$
$$\lambda(t) \geq 0, \ t \in I.$$

In [7], [CP] and (CD_0) are shown to be a dual pair if f, g and h are all convex in x and u. Subsequently, Mond and Smart [9] extended this duality by introducing the following invexity requirement.

Definition 1 ((Invex) [9]). If there exist vector function $\eta(t,x,\bar{x}) \in R^n$ with $\eta = 0$ at t if $x(t) = \bar{x}(t)$, and $\xi(t,u,\bar{u}) \in R^m$ such that for scalar function $\Phi(t,x,\dot{x},u)$, the functional $\Phi(x,\dot{x},u) = \int_a^b \Phi(t,x,\dot{x},u)dt$ satisfies

$$\Phi(x,\dot{x},u) - \phi(\bar{x},\dot{\bar{x}},\bar{u})$$
$$\geq \int_a^b \left[\eta^T h_x(t,\dot{\bar{x}},\bar{x},\bar{u}) + \left(\frac{d\eta}{dt}\right)^T h_{\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u}) + \xi^T h_u(t,\bar{x},\dot{\bar{x}},\bar{u})\right]dt$$

then ϕ is said to be invex in $x, \dot{x}$ and u on I with respect to η and ξ.

In [9] Mond and Smart proved weak, strong and converse duality theorems under the invexity of $\int_a^b f dt$, $\int_a^b \lambda^T g dt$, for $\lambda(t) \in R^p$ with $\lambda(t) \geq 0, t \in I$ and $\int_a^b \mu^T h dt$ for any $\mu(t) \in R^n, t \in I$. Recently Husain et al [6] for relaxing invexity requirements in [9] for duality to hold constructed the following dual in the sprit of Mond and Weir [8]:

Problem (CD) (Dual):

$$\text{Maximize} \int_a^b f(t,x,u)dt$$

subject to

$$x(a) = \alpha, x(b) = \beta, \tag{8}$$

$$f_x(t,x,u) + \lambda(t)^T g_x(t,x,u) + \mu(t)^T h_x(t,x,u) + \dot{\mu}(t) = 0, t \in I, \tag{9}$$

$$f_u(t,x,u) + \lambda(t)^T g_u(t,x,u) + \mu(t)^T h_u(t,x,u) = 0, t \in I, \tag{10}$$

$$\int_a^b (\lambda(t)^T g(t,x,u) + \mu(t)^T (h(t,x,u) - \dot{x}))dt \geq 0, \tag{11}$$

$$\lambda(t) \geq 0, \ t \in I. \tag{12}$$

In the subsequent analysis, we require the following definitions of generalized invexity.

Definition 2 (Pseudoinvex). For a scalar function $\Phi(t,x,\dot{x},u)$, the functional $\phi(x,\dot{x},u) = \int_a^b \phi(t,x,\dot{x},u)dt$ is said to be pseudoinvex in $x,\dot{x}$ and u if there exist vector function $\eta(t,x,\bar{x}) \in R^n$ with $\eta = 0$ at t if $x(t) = \bar{x}(t)$ and $\xi(t,u,\bar{u}) \in R^m$ such that

$$\begin{aligned} &\int_a^b \left(\eta^T \phi_x(t,\bar{x},\dot{\bar{x}},\bar{u}) + \left(\frac{d\eta}{dt}\right)^T \phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u}) + \xi^T \phi_u(t,\bar{x},\dot{\bar{x}},\bar{u})\right)dt \geq 0 \\ \implies \quad &\Phi(x,\dot{x},u) \geq \Phi(\bar{x},\dot{\bar{x}},\bar{u}). \end{aligned}$$

Definition 3 (Strictly Pseudoinvex). The functional Φ is said to be strictly pseudoinvex, if there exist vector functions $\eta(t,x,\bar{x}) \in R^n$ with $\eta = 0$ at t if $x(t) = \bar{x}(t)$ and $\xi(t,u,\bar{u}) \in R^m$ such that

$$\begin{aligned} &\int_a^b \left(\eta^T \phi_x(t,\bar{x},\dot{\bar{x}},\bar{u}) + \left(\frac{d\eta}{dt}\right)^T \phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u}) + \xi^T \phi_u(t,\bar{x},\dot{\bar{x}},\bar{u})\right)dt \geq 0 \\ \implies \quad &\Phi(x,\dot{x},u) > \Phi(\bar{x},\dot{\bar{x}},\bar{u}). \end{aligned}$$

Definition 4 (Quasi-invex). The functional Φ is said to be quasi-invex, if there exist vector functions $\eta(t,x,\bar{x}) \in R^n$ with $\eta = 0$ at t if $x(t) = \bar{x}(t)$ and $\xi(t,u,\bar{u}) \in R^m$ such that

$$\begin{aligned} &\Phi(x,\dot{x},u) \leq \Phi(\bar{x},\dot{\bar{x}},\bar{u}) \\ \implies \quad &\int_a^b \left((\eta^T \phi_x(t,\bar{x},\dot{\bar{x}},\bar{u}) + \left(\frac{d\eta}{dt}\right)^T \phi_{\dot{x}}(t,\bar{x},\dot{\bar{x}},\bar{u}) + \xi^T \phi_u(t,\bar{x},\dot{\bar{x}},\bar{u})\right)dt \leq 0. \end{aligned}$$

3. Mixed type duality

We propose the following mixed type dual (Mix CD) to the control problem (CP) and establish usual duality results:

(Mix CD) :

$$\text{Maximize} \int_a^b \left[f(t,x,u) + \sum_{i \in I_0} \mu^i(t)(h^i(t,x,u) - \dot{x}^j) + \sum_{j \in J_0} \lambda^j(t) g^j(t,x,u) \right] dt$$

subject to

$$x(a) = \alpha, x(b) = \beta \tag{13}$$

$$f_x(t,x,u) + \mu(t)^T h_x(t,x,u) + \lambda(t)^T g_x(t,x,u) + \dot{\mu}(t) = 0, t \in I \tag{14}$$

$$f_u(t,x,u) + \mu(t)^T h_u(t,x,u) + \lambda(t)^T g_u(t,x,u) = 0, t \in I \tag{15}$$

$$\int_a^b \left(\sum_{i \in I_\alpha} \mu^i(t)(h^i(t,x,u) - \dot{x}^i) + \sum_{j \in J_\alpha} \lambda^j(t) g^j(t,x,u) \right) dt \geq 0, \quad \alpha = 1,2,\ldots r \tag{16}$$

$$\lambda(t) \geq 0, t \in I \tag{17}$$

where for $N = \{1,2,\ldots,n\}$ and $K = (1,2,\ldots,k)$,

(i) $I_\alpha \subseteq M, \alpha = 0,1,2,\ldots r$ and $I_\alpha \cap I_\beta = \phi, \alpha \neq \beta$ and $\bigcup_{\alpha=0}^{r} I_\alpha = N$.

(ii) $J_\alpha \subseteq k, \alpha = 0,1,2,\ldots r$ with $J_\alpha \cap J_\beta = \phi, \alpha \neq \beta$ and $\bigcup_{\alpha=0} J_\alpha = K$, and

(iii) $r = \max(r_1, r_2)$, where r_1 is the number of disjoint subsets of M and r_2 is the number of disjoint subsets of K. Then I_α or J_α is empty for $\alpha > \min(r_1, r_2)$.

Theorem 1 (Weak duality). *Let $(\bar{x}, \bar{u})$ be feasible for (CP) and (x,u,λ,μ) be feasible for (Mix CD). If for all feasible $(\bar{x}, \bar{u}, x, u, \lambda, \mu)$, $\int_a^b \left(f + \sum_{i \in I_0} \mu^i(h^i - \dot{x}^i) + \sum_{j \in J_0} \lambda^j g^j \right) dt$ is pseudoinvex and $\int_a^b \left(\sum_{i \in I_\alpha} \mu^i(h^i - \dot{x}^i) + \sum_{j \in J_\alpha} \lambda^j g^j \right) dt$ is quasi-invex with respect to the same η and ξ, then* $\inf(\text{CP}) \geq \sup(\text{MixCD})$.

Proof. Since $(\bar{x}, \bar{u})$ be feasible for (CP) and (x,u,μ,λ) be feasible for (Mix CD), we have

$$\int_a^b \left(\sum_{i \in I_\alpha} \mu^i(t)(h^i(t,x,u) - \dot{x}^i) + \sum_{j \in J_\alpha} \lambda^j(t) g^j(t,x,u) \right) dt$$
$$\leq \int_a^b \left(\sum_{i \in I_\alpha} \mu^i(t)(h^i(t,\bar{x},\bar{u}) - \dot{\bar{x}}^i) + \sum_{j \in J_\alpha} \lambda^j(t) g^j(t,\bar{x},\bar{u}) \right) dt, \ \alpha = 1,2,\ldots,r.$$

By quasi-invexity of $\int_a^b \left(\sum_{i \in I_\alpha} \mu^i(h^i - \dot{x}^i) + \sum_{j \in J_\alpha} \lambda^j g^j \right) dt, \alpha = 1,2,\ldots r$ this inequality yields,

$$\begin{aligned}
0 \geq & \int_a^b \Big[\eta^T\Big(\sum_{i\in I_\alpha}\mu^i(t)h_x^i(t,x,u)+\sum_{j\in J_\alpha}\lambda^j(t)g_x^j(t,x,u)\Big)-\Big(\frac{d\eta}{dt}\Big)\sum_{i\in I_\alpha}\dot{\mu}^i \\
& +\xi^T\Big(\sum_{i\in I_\alpha}\mu^i(t)h_u^i(t,x,u)+\sum_{j\in J_\alpha}\lambda^j(t)g_u^j(t,x,u)\Big)\Big]dt \\
= & \int_a^b \Big[\eta^T\Big(\sum_{i\in I_\alpha}\Big(\mu^i(t)h_x^i(t,x,u)+\dot{\mu}^i(t)\Big)+\sum_{j\in J_\alpha}\lambda^j(t)g_x^j(t,x,u)\Big) \\
& +\xi^T\Big(\sum_{i\in I_\alpha}\mu^i(t)h_u^i(t,x,u)+\sum_{j\in J_\alpha}\lambda^j(t)g^j(t,x,u)\Big)\Big]dt-\eta\sum_{i\in I_\alpha}\dot{\mu}^i\Big|_{t=a}^{t=b} \\
= & \int_a^b \Big[\eta^T\Big(\sum_{i\in I_\alpha}(\mu^i(t)h_x^i(t,x,u)+\dot{\mu}^i(t))+\sum_{j\in J_\alpha}\lambda^j(t)g_x^j(t,x,u)\Big) \\
& +\xi^T\Big(\sum_{i\in I_\alpha}\mu^i(t)h^i(t,x,u)+\sum_{j\in J_\alpha}\lambda^j(t)g^j(t,x,u)\Big)\Big]dt \\
& (\text{using } \eta=0, \text{ at } t=a \text{ and } t=b) \\
= & \int_a^b \Big[\eta^T\Big(\sum_{i\in N\setminus I_0}\Big(\mu^i(t)h_x^i(t,x,u)+\dot{\mu}^i(t)\Big)+\sum_{j\in K\setminus J_0}\lambda^j(t)g^j(t,x,u)\Big) \\
& +\xi^T\Big(\sum_{i\in N\setminus I_0}\mu^i(t)h^i(t,x,u)+\sum_{j\in K\setminus J_0}\lambda^j(t)g^j(t,x,u)\Big)\Big]dt.
\end{aligned}$$

Using (5) and (6), this implies

$$\begin{aligned}
& \int_a^b \Big[\eta^T\Big(\sum_{i\in I_0}\Big(\mu^i(t)h_x^i(t,x,u)+\dot{\mu}^i(t)\Big)+\sum_{j\in J_0}\lambda^j(t)g^j(t,x,u)\Big) \\
& +\xi^T\Big(\sum_{i\in I_0}\mu^i(t)h^i(t,x,u)+\sum_{j\in J_0}\lambda^j(t)g^j(t,x,u)\Big)\Big]dt \geq 0
\end{aligned}$$

This, because of pseudo-invexity of $\int_a^b \Big(f+\sum_{i\in I_0}\mu^i(t)(h^i-\dot{x}^i)+\sum_{j\in J_0}\lambda^j(t)g^j\Big)dt$ yields,

$$\begin{aligned}
& \int_a^b \Big\{f(t,\bar{x},\bar{u})+\sum_{i\in I_0}\mu^i(t)(h^i(t,\bar{x},\bar{u})-\dot{\bar{x}}^i)+\sum_{j\in J_0}\lambda^j(t)g^j(t,\bar{x},\bar{u})\Big\}dt \\
\geq & \int_a^b \{f(t,x,u)+\sum_{i\in I_0}\mu^i(t)(h^i(t,x,u)-\dot{x}^i)+\sum_{j\in J_0}\lambda^j(t)g^j(t,x,u)\}dt. \quad (18)
\end{aligned}$$

Since $\mu(t)^T\Big(h(t,\bar{x},\bar{u})-\dot{\bar{x}}\Big)=0$, and $\lambda(t)^Tg(t,\bar{x},\bar{u})\leq 0$, these respectively imply

$$\sum_{i\in I_0}\mu^i(t)(h^i(t,\bar{x},\bar{u})-\dot{\bar{x}}^i)=0 \quad \text{and} \quad \sum_{j\in J_0}\lambda^j(t)g^j(t,\bar{x},\bar{u})\leq 0, t\in I.$$

Consequently (18) gives

$$\int_a^b f(t,\bar{x},\bar{u})dt$$
$$\geq \int_a^b \left\{ f(t,x,u) + \sum_{i\in I_0} \mu^i(t)(h^i(t,x,u) - \dot{x}^i) + \sum_{j\in J_0} \lambda^j(t) g^j(t,x,u) \right\} dt.$$

That is, inf(CP) $\geq$ sup(Mix CD). □

Theorem 2 (Strong Duality). *If $(\bar{x},\bar{u})$ is an optimal solution of (CP) and is normal, then there exist piecewise smooth $\bar{\mu}: I \to R^n$ and $\bar{\lambda}: I \to R^p$ such that $(\bar{x},\bar{u},\bar{\lambda},\bar{\mu})$ be feasible and the corresponding values of (CP) and (Mix CD) are equal.*

If, also $\int_a^b \left\{ f + \sum_{i\in I_0} \mu^i(h^i - \dot{x}^i) + \sum_{j\in J_0} \lambda^j g^j \right\} dt$ is pseudoinvex and $\int_a^b \left\{ f + \sum_{i\in I_\alpha} \mu^i(h^i - \dot{x}^i) + \sum_{j\in J_\alpha} \lambda^j g^j \right\} dt$ is quasi-invex with respect to the same η and ξ, then $(\bar{x},\bar{u},\bar{\lambda},\bar{\mu})$ is an optimal solution of (Mix CD).

Proof. Since $(\bar{x},\bar{u})$ is an optimal solution to (CP) and is normal then from Proposition 1, there exist piecewise smooth $\bar{\mu}: I \to R^n$ and $\bar{\lambda}: I \to R^p$ such that

$$f_x(t,\bar{x},\bar{u}) + \bar{\mu}(t)^T h_x(t,\bar{x},\bar{u}) + \dot{\mu}(t) + \bar{\lambda}(t)^T g(t,\bar{x},\bar{u}) = 0, t \in I, \quad (19)$$
$$f_u(t,\bar{x},\bar{u}) + \bar{\mu}(t)^T h_u(t,\bar{x},\bar{u}) + \bar{\lambda}(t)^T g(t,\bar{x},\bar{u}) = 0, t \in I, \quad (20)$$
$$\bar{\lambda}(t)^T g(t,\bar{x},\bar{u}) = 0, t \in I, \quad (21)$$
$$\bar{\lambda}(t) \geq 0, t \in I. \quad (22)$$

The relation (21) implies $\sum_{j\in J_0} \lambda^j(t) g^j(t,\bar{x},\bar{u}) = 0$ and $\sum_{j\in J_\alpha} \bar{\lambda}^j(t) g^j(t,\bar{x},\bar{u}) = 0$, $\alpha = 1, 2, \ldots, r$. Also $\bar{\mu}(t)^T(h(t,\bar{x},\bar{u}) - \dot{\bar{x}}) = 0$, implies $\sum_{i\in I_0} \mu^i(t)(h^i(t,\bar{x},\bar{u}) - \dot{\bar{x}}^i) = 0, t \in I$ and $\sum_{i\in I_\alpha} \mu^i(t)(h^i(t,\bar{x},\bar{u}) - \dot{\bar{x}}^i) = 0, t \in$ I. Consequently, $\sum_{i\in I_\alpha} \bar{\mu}^i(t)(h^i(t,\bar{x},\bar{u}) - \dot{\bar{x}}^i) = 0, t \in I$ and $\sum_{j\in J_\alpha} \bar{\lambda}^j(t) g^j(t,\bar{x},\bar{u}) = 0, t \in I$ imply

$$\int_a^b \left[\sum_{i\in I_\alpha} \bar{\mu}^i(t)(h^i(t,\bar{x},\bar{u}) - \dot{\bar{x}}^i) + \sum_{j\in J_\alpha} \lambda^j(t) g^j(t,\bar{x},\bar{u}) \right] dt = 0. \quad (23)$$

From the relations (19), (20), (22) and (23), it implies that $(\bar{x},\bar{u},\bar{\lambda},\bar{\mu})$ is feasible for (Mix CD) and the corresponding objective values of (CP) and (Mix CD) are equal in view of $\sum_{i\in I_0} \bar{\mu}^i(t)(h^i(t,\bar{x},\bar{u}) - \dot{\bar{x}}^i) = 0$ and $\sum_{j\in J_0} \bar{\lambda}^j(t) g^j(t,\bar{x},\bar{u}) = 0, t \in I$.

If $\int_a^b \left(f + \sum_{i\in I_0} \mu^i(h^i - \dot{x}^i) + \sum_{j\in J_0} \lambda^j g^j\right)dt$ is pseudoinvex and $\int_a^b \Big(f + \sum_{i\in I_\alpha} \mu^i(h^i - \dot{x}^i) + \sum_{j\in J_\alpha} \lambda^j g^j\Big)dt$, $\alpha = 1, 2, \ldots r$ is quasi-invex with respect to the same η and ξ, then from Theorem 1, $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu})$ must be an optimal solution of (Mix CD). □

Theorem 3 (Strict Converse duality). *Let $(\bar{x}, \bar{u})$ be an optimal solution of (CP) and normality condition be satisfied at $(\bar{x}, \bar{u})$. Let $(\hat{x}, \hat{u}, \hat{\lambda}, \hat{\mu})$ be an optimal solution of (Mix CD). If $\int_a^b \left(\sum_{i\in I_\alpha} \hat{\mu}^i(h^i - \dot{\hat{x}}^i) + \sum_{j\in J_\alpha} \hat{\lambda}^j(t) g^j\right)dt, \alpha = 1, 2 \ldots r$ is quasi-invex for all feasible $(\hat{x}, \hat{u}, \hat{\lambda}, \hat{\mu})$, and $\int_a^b \Big(f + \sum_{i\in I_0} \hat{\mu}^i(t)(h^i - \dot{\hat{x}}^i) + \sum_{j\in J_0} \hat{\lambda}^j(t) g^j\Big)dt$ is strictly pseudoinvex with respect to the same η and ξ, then $(\hat{x}, \hat{u}) = (\bar{x}, \bar{u})$, i.e., $(\hat{x}, \hat{u})$ is an optimal solution of (CP).*

Proof. We assume that $(\hat{x}, \hat{u}) \neq (\bar{x}, \bar{u})$ and show that this assumption leads to a contradiction. Since $(\bar{x}, \bar{u})$ is an optimal solution of (CP) and is normal, it follows by strong duality (Theorem 2) that there exist piecewise smooth $\mu : I \to R^n$ and such that $(\bar{x}, \bar{u}, \bar{\lambda}, \bar{\mu})$ is an optimal solution of (Mix CD) and

$$\int_a^b f(t, \bar{x}, \bar{u})dt$$
$$= \int_a^b \left[f(t, \bar{x}, \bar{u}) + \sum_{i\in I_0} \hat{\mu}^i(t)\Big(h^i(t, \hat{x}, \hat{u}) - \dot{\hat{x}}^i\Big) + \sum_{j\in J_0} \hat{\lambda}^j(t) g^j(t, \hat{x}, \hat{u})\right]dt \quad (24)$$

Also since $(\bar{x}, \bar{u})$ and $(\hat{x}, \hat{u}, \hat{\lambda}, \hat{\mu})$ are feasible for (CP) and (Mix CD), therefore, for $\alpha = 1, 2, \ldots, r$

$$\int_a^b \left(\sum_{i\in I_\alpha} \hat{\mu}^i(t)(h^i(t, \bar{x}, \bar{u}) - \dot{\bar{x}}^i) + \sum_{j\in J_\alpha} \hat{\lambda}^j(t) g^i(t, \bar{x}, \bar{u})\right)dt$$
$$\leq \int_a^b (\sum_{i\in I_\alpha} \hat{\mu}^i(t)(h^i(t, \hat{x}, \hat{u}) - \dot{\hat{x}}^i) + \sum_{j\in J_\alpha} \hat{\lambda}^j(t) g^j(t, \hat{x}, \hat{u}))dt \quad (25)$$

This, because of quasi-invexity of $\int_a^b \left(\sum_{i\in I_\alpha} \hat{\mu}^i(t)(h^i - \dot{\hat{x}}^i) + \sum_{j\in J_\alpha} \hat{\lambda}^j g^j\right)dt, \alpha = 1, 2, \ldots, r$ is quasi-invex for all feasible $(\bar{x}, \bar{u}, \hat{x}, \hat{u}, \hat{\lambda}, \hat{\mu})$ with respect to η and ξ, therefore, (25) implies that for $\alpha = 1, 2, \ldots, r$,

$$\int_a^b \Bigg[\eta^T\Big(\sum_{i\in I_\alpha}\hat{\mu}^i(t)h_x^i(t,\hat{x},\hat{u}) + \sum_{j\in J_\alpha}\hat{\lambda}^j(t)g_x^j(t,\hat{x},\hat{u})\Big) - \sum_{i\in I_\alpha}\Big(\frac{d\eta}{\mathrm{dt}}\Big)^T\hat{\mu}^i$$
$$+\xi^T\Big(\sum_{i\in I_\alpha}\hat{\mu}^i(t)\Big(h^i(t,\hat{x},\hat{u}) - \dot{\hat{x}}^i\Big) + \sum_{j\in J_\alpha}\hat{\lambda}^j(t)g^j(t,\hat{x},\hat{u})\Big)\Bigg]dt \leq 0,$$

$$\int_a^b \Bigg[\eta^T\Big(\sum_{i\in I_\alpha}\Big(\hat{\mu}^i(t)h^i(t,\hat{x},\hat{u}) + \dot{\hat{\mu}}^i(t)\Big) + \sum_{j\in J_\alpha}\hat{\lambda}^j(t)\Big(g_x^j(t,\hat{x},\hat{u})\Big)\Big)$$
$$+\xi^T\Big(\sum_{i\in I_\alpha}\hat{\mu}^i(t)h_u^i(t,\hat{x},\hat{u}) + \sum_{j\in J_\alpha}\hat{\lambda}^j(t)g_u^j(t,\hat{x},\hat{u})\Big)\Bigg]dt - \eta\sum_{i\in I_\alpha}\hat{\mu}^i(t)\Big|_{t=a}^{t=b}$$

(integration by parts)

$$\int_a^b \Bigg[\eta^T\Big(\sum_{i\in I_\alpha}\Big(\hat{\mu}^i(t)h_x^i(t,\hat{x},\hat{u}) + \dot{\hat{\mu}}^i(t)\Big) + \sum_{j\in J_\alpha}\hat{\lambda}^j(t)g^j(t,\hat{x},\hat{u})\Big)$$
$$+\xi^T\Big(\sum_{i\in I_\alpha}\hat{\mu}^i(t)h_u^i(t,\hat{x},\hat{u}) + \sum_{j\in J_\alpha}\hat{\lambda}^j(t)g^j(t,\hat{x},\hat{u})\Big)\Bigg]dt \leq 0,$$

$(\eta = 0$ at $t = a, t = b)$.

Or

$$\int_a^b \Bigg[\eta^T\Big(\sum_{i\in N\backslash I_0}\Big(\hat{\mu}^i(t)h_x^i(t,\hat{x},\hat{u}) + \dot{\hat{\mu}}^i(t)\Big) + \sum_{j\in K\backslash I_0}\hat{\lambda}^j(t)g_x^j(t,\hat{x},\hat{u})\Big)$$
$$+\xi^T\Big(\sum_{i\in N\backslash I_0}\hat{\mu}^i(t)h_u^i(t,\hat{x},\hat{u}) + \sum_{j\in K\backslash I_0}\hat{\lambda}^j(t)g_u^j(t,\hat{x},\hat{u})\Big)\Bigg]dt \leq 0.$$

Since $(\hat{x}, \hat{u}, \hat{\lambda}, \hat{\mu})$ is feasible for (Mix CD), therefore, by using (14) and (15) in the above inequality, we have

$$\int_a^b \Bigg[\eta^T\Big(f_x(t,\hat{x},\hat{u}) + \sum_{i\in I_0}\Big(\hat{\mu}^i(t)h_x^i(t,\hat{x},\hat{u}) + \dot{\hat{\mu}}^i(t)\Big) + \sum_{j\in J_0}\hat{\lambda}^j(t)g_x^j(t,\hat{x},\hat{u})\Big)$$
$$+\xi^T\Big(f_u(t,\hat{x},\hat{u}) + \sum_{i\in I_0}\hat{\mu}^i(t)h_u^i(t,\hat{x},\hat{u}) + \sum_{j\in J_0}\hat{\lambda}^j(t)g_u^j(t,\hat{x},\hat{u})\Big)\Bigg]dt \geq 0.$$

This, because of strict pseudo-invexity of $\int_a^b\Big(f + \sum_{i\in J_0}\hat{\mu}^i(t)(h^i - \dot{x}^i) + \sum_{j\in J_0}\hat{\lambda}^j(t)g^j\Big)dt$ with respect to η and ξ, yields

$$\int_a^b\Big(f(t,\bar{x},\bar{u}) + \sum_{i\in I_0}\hat{\mu}^i(t)\Big(h^i(t,\bar{x},\bar{u}) - \dot{\bar{x}}\Big) + \sum_{j\in J_0}\hat{\lambda}^j(t)g^j(t,\bar{x},\bar{u})\Big)dt$$
$$> \int_a^b\Big[\Big(f(t,\hat{x},\hat{u}) + \sum_{i\in I_0}\Big(\hat{\mu}^i(t)(h^i(t,\hat{x},\hat{u}) - \dot{\hat{x}})\Big) + \sum_{j\in J_0}\hat{\lambda}^j(t)g^j(t,\hat{x},\hat{u})\Big)dt.$$

Since $\sum_{i\in I_0} \hat{\mu}^i\left(h^i(t,\hat{x},\hat{u})+\dot{\hat{x}}^i\right)=0$ and $\sum_{j\in J_0} \hat{\lambda}^j(t)g^j(t,\hat{x},\hat{u})$, which are consequence of feasibility of $(\bar{x},\bar{u})$ for (CP) and $(\hat{x},\hat{u},\hat{\lambda},\hat{\mu})$ for (Mix CD), we have

$$\begin{aligned}&\int_a^b f(t,\bar{x},\bar{u})dt\\ > &\int_a^b \left(f(t,\hat{x},\hat{u})+\sum_{i\in I_0}\hat{\mu}^i(t)\left(h^i(t,\hat{x},\hat{u})-\dot{\hat{x}}^i\right)+\sum_{j\in J_0}\hat{\lambda}^j(t)g^j(t,\hat{x},\hat{u})\right)dt.\end{aligned}$$

This is a contradiction to (17). Hence $(\hat{x},\hat{u})=(\bar{x},\bar{u})$, i.e., $(\hat{x},\hat{u})$ must be an optimal solution of (CP).

We now write

$$\begin{aligned}\psi_1 &\equiv \psi_1(t,x,u,\lambda,\mu)\\ &= f_x+\mu(t)^T(h_x(t,x,u)+\dot{\mu})+\lambda^T(t)g_x(t,x,u),\\ \psi_2 &\equiv \psi_2(t,x,u,\lambda,\mu,\mu)\\ &= f_x+\mu(t)^Th_u+\lambda^T(t)g_u \text{ where } f_x=f_x(t,x,u),\\ f_u &= f_u(t,x,u), g_x=g_x(t,x,u) \text{ and } h_x=h_x(t,x,u).\end{aligned}$$

Consider $\psi_1(t,x(t),u(t),\lambda(\cdot),\mu(\cdot),\dot{\mu}(\cdot))$ as defining a mapping $Q_1: X\times U\times Y\times Z\to B$, where Y is the space of piecewise smooth functions $\lambda: I\to R^k, Z$ is the space of differentiable function $\mu: I\to R^n$ and B is a banach space; X and U are already defined. Also consider $\psi_2(t,x(\cdot),u(\cdot),\lambda(\cdot),\mu(\cdot))$ as defining a mapping $Q_2: X\times U\times Y\times Z\to C$ where C is another banach space. In order to apply Proposition 1 to the problem (CD), some assumptions on $\psi_1(\cdot)=0$ and $\psi_2(\cdot)=0$ are in order. For this it suffices to assume that Frèchèt derivatives.

$$\begin{aligned}Q_1' &= \left(Q_{1x}(\bar{x},\bar{u},\bar{\lambda},\bar{\mu}),Q_{1u}(\bar{x},\bar{u},\bar{\lambda},\bar{\mu}),Q_{1\lambda}(\bar{x},\bar{u},\bar{\lambda},\bar{\mu}),Q_{1\mu}(\bar{x},\bar{u},\bar{\lambda},\bar{\mu})\right)\\ Q_2' &= \left(Q_{2x}(\bar{x},\bar{u},\bar{\lambda},\bar{\mu}),Q_{2u}(\bar{x},\bar{u},\bar{\lambda},\bar{\mu}),Q_{2\lambda}(\bar{x},\bar{u},\bar{\lambda},\bar{\mu}),Q_{2\mu}(\bar{x},\bar{u},\bar{\lambda},\bar{\mu})\right)\end{aligned}$$

have weak *closed range. For notational convenience, we shall write in the sequel $\bar{f}=f(t,\bar{x},\bar{u})$, $\bar{g}=g(t,\bar{x},\bar{u})$, $\bar{h}=h(t,\bar{x},\bar{u})$, $\bar{f}_x=f_x(t,\bar{x},\bar{u})$, $\bar{g}_x=g_x(t,\bar{x},\bar{u})$, $\bar{h}_x=h_x(t,\bar{x},\bar{u})$, etc.

Theorem 4 (Converse duality). *Let $f,g,$ and h be twice continuously differentiable and $(\bar{x},\bar{u},\bar{\lambda},\bar{\mu})$ be an optimal solution of (Mix CD). Let the Frèchèt derivatives Q_1' and Q_2' have weak closed range. Assume that*

$$(H_1): \int_a^b \sigma(t)^T(M(t)\sigma(t)dt=0\to\sigma(t)=0,\ t\in I$$

where $\sigma(t)\in R^{n+m}$ and

$$M(t)=\begin{bmatrix}\bar{f}_{xx}+\bar{\mu}(t)^Th_{xx}+\bar{\lambda}(t)^Tg_{xx}, \bar{f}_{ux}+\bar{\mu}(t)^Th_{ux}+\bar{\lambda}(t)^Tg_{ux}\\ \bar{f}_{xu}+\bar{\mu}(t)^Th_{xu}+\bar{\lambda}(t)^Tg_{xu}, \bar{f}_{uu}+\bar{\mu}(t)^Th_{uu}+\bar{\lambda}(t)^Tg_{uu}\end{bmatrix}$$

$$(H_2)\ :\ \left\{\sum_{i\in I_\alpha}\left(\mu^i(t)h_x^i(t,\bar{x},\bar{u})+\dot{\mu}^i(t)\right)+\sum_{j\in J_\alpha}\lambda^j(t)g_x^j(t,\bar{x},\bar{u}),\alpha=1,2,\ldots,r\right\}$$

and

$$\left\{ \sum_{i\in I_\alpha} \mu^i(t)h^i_u(t,\bar{x},\bar{u}) + \dot{\mu}^i(t) + \sum_{j\in J_\alpha} \lambda^j(t)g^j_u(t,\bar{x},\bar{u}), \alpha = 1,2,\ldots,r \right\}$$

are linearly independent, there exist corresponding to (14) a piecewise smooth Lagrange multiplier $\beta : I \to R^n$ with $\beta(t) \geq 0, t \in I$ with $\beta(a) = 0 = \beta(b)$.

If, for all feasible $(\bar{x},\bar{u},x,u,\lambda,\mu)$, $\int_a^b \left(f + \sum_{i\in I_0} \mu^i(t)\left(h^i - \dot{x}^i\right) + \sum_{j\in J_0} \lambda^j(t)g^j\right)dt$ is pseudoinvex and $\int_a^b \left(\sum_{i\in I_\alpha} \mu^i(t)\left(h^i - \dot{x}^i\right) + \sum_{j\in J_\alpha} \lambda^j(t)g^j\right)dt$ is quasi-invex with respect to the same η and ξ, then $(\bar{x},\bar{u})$ is an optimal solution of (CD).

Proof. Since $(\bar{x},\bar{u},\bar{\lambda},\bar{\mu})$ is an optimal solution to (CP), therefore, by Proposition 1, there exist $\tau \in R, \gamma_\alpha \in R, \alpha = 1,2,,\ldots,r$, and piecewise smooth $\beta : I \to R^n$ and $\theta : I \to R^m$ such that

$$\tau\left(f_x + \sum_{i\in I_0}\left(\mu^i(t)h^i_x + \dot{\mu}^i(t)\right) + \sum_{j\in J_0} \lambda^j(t)g^j_x\right)$$
$$+\beta(t)^T\left(f_{xx} + \lambda(t)^T g_{xx} + \mu(t)^T h_{xx}\right) + \theta(t)^T\left(f_{ux} + \lambda(t)^T g_{ux} + \mu(t)^T h_{ux}\right)$$
$$+\sum_{\alpha=1}^r \gamma_\alpha\left\{\sum_{i\in I_\alpha}\left(\mu^i(t)h^i_x + \dot{\mu}^i(t)\right) + \sum_{i\in J_\alpha} \lambda^j(t)g^i_x\right\} = 0, t\in I, \tag{26}$$

$$\tau\left(f_u + \sum_{i\in I_0} \mu^i(t)h^i_u + \sum_{j\in J_0} \lambda^j(t)g^j_u\right) + \beta(t)^T\left(f_{xu} + \lambda(t)^T g_{xu} + \mu(t)^T h_{xu}\right)$$
$$+\theta(t)^T\left(f_{uu} + \lambda(t)^T g_{uu} + \mu(t)^T h_{uu}\right)$$
$$+\sum_{\alpha=1}^r \gamma_\alpha\left\{\sum_{i\in I_\alpha} \mu^i(t)h^i_u + \sum_{i\in J_\alpha} \lambda^j(t)g^i_u\right\} = 0, t\in I, \tag{27}$$

$$\tau(h^i - \dot{x}^i) + \beta(t)^T h^i_x - \dot{\beta}^i(t) + \theta^T(t)h^i_u = 0, i\in I_0, \tag{28}$$
$$\beta(t)^T h^i_x - \beta^i(t) + \theta(t)^T h^i_x + \gamma_\alpha(h^i - \dot{x}^i) = 0, i\in I_\alpha, \alpha = 1,2,\ldots,r, \tag{29}$$
$$\tau g^i + \beta(t)g^i_x + \theta(t)g^i_u + \eta^i(t) = 0, i\in I_o, \tag{30}$$
$$\beta(t)^T g^i_x + \theta(t)^T g^i_u + \gamma_\alpha g^i + \eta^i(t) = 0, i\in J_\alpha, \alpha = 1,2,\ldots,r, \tag{31}$$
$$\gamma_\alpha\int_a^b\left(\sum_{i\in I_\alpha} \mu^i(t)\left(h^i - \dot{x}^i\right) + \sum_{j\in J_\alpha} \lambda^j(t)g^j\right)dt = 0, \alpha = 1,2,\ldots,r, \tag{32}$$
$$\eta(t)^T\lambda(t) = 0, t\in I, \tag{33}$$
$$(\tau,\gamma_1,\ldots,\gamma_r,\eta(t)) \geq 0, t\in I, \tag{34}$$
$$(\tau,\beta(t),\theta(t),\gamma_r\ldots\gamma_r,\eta(t)) \neq 0, t\in I. \tag{35}$$

Multiplying (28) by $\mu^i(t), i \in I_0$ and $t \in I$, and summing over $i \in I_0$ and then integrating, we have

$$\tau \int_a^b \sum_{i \in I_0} \mu^i(t)(h^i - \dot{x}^i)dt + \int_a^b \Big\{ \beta(t)^T \sum_{i \in I_0} \Big(\mu^i(t) h^i_x + \dot{\mu}^i(t) \Big)$$
$$+\theta(t)^T \Big(\sum_{i \in I_0} \mu^i(t) h^i_u \Big) \Big\} dt - \sum_{i \in I_0} \mu^i(t) \beta^i(t) \Big|_{t=a}^{t=b} = 0.$$

Using $\hat{\beta}(a) = 0 = \hat{\beta}(b)$, we have

$$\tau \int_a^b \sum_{i \in I_0} \mu^i(t)(h^i - \dot{x}^i)dt$$
$$+ \int_a^b \{ \beta(t)^T \sum_{i \in I_0} \Big(\mu^i(t) h^i_x + \dot{\mu}^i(t) \Big) + \theta(t)^T \Big(\sum_{i \in I_0} \mu^i(t) h^i_u \Big) \} dt = 0. \quad (36)$$

Multiplying (29) by $\mu^i(t), i \in I_0$ and $t \in I$, and summing over $I \in I_\alpha$ and then integrating, we have

$$\int_a^b \Big\{ \beta(t)^T \Big(\sum_{i \in I_\alpha} \Big(\mu^i(t) h^i_x + \dot{\mu}^i(t) \Big) \Big) + \theta(t)^T \Big(\sum_{i \in I_\alpha} \mu^i(t) h^i_u \Big) \Big\} dt$$
$$+\gamma_\alpha \int_a^b \Big(\sum_{i \in I_\alpha} \mu^i(t)(h^i - \dot{x}^i) \Big) dt = 0, \alpha = 1, 2, \ldots, r. \quad (37)$$

Similarly from (30) and (31) together with (33), it implies respectively

$$\tau \int_a^b \sum_{j \in J_0} \lambda^j(t) g^j dt$$
$$+ \int_a^b \Big\{ \beta(t)^T \Big(\sum_{j \in J_0} \lambda^j(t) g^j_x \Big) + \theta(t)^T \Big(\sum_{j \in J_0} \lambda^j(t) g^j_u \Big) \Big\} dt = 0, \quad (38)$$

and

$$\int_a^b \Big\{ \beta(t)^T \Big(\sum_{j \in J_\alpha} \lambda^j(t) g^j_x \Big) + \theta(t)^T \Big(\sum_{j \in J_\alpha} \lambda^j(t) g^j_u \Big) \Big\} dt$$
$$+\gamma_\alpha \int_a^b \Big(\sum_{j \in J_\alpha} \lambda^j(t) g^j \Big) dt = 0, \quad \alpha = 1, 2, \ldots, r. \quad (39)$$

Adding (36) to (38) and (37) to (39), we have

$$\tau \int_a^b \left(\sum_{i \in I_0} \mu^i(t)\Big(h^i + \dot{x}^i\Big) + \sum_{j \in J_0} \lambda^j(t) g^j \right) dt$$
$$+ \int_a^b \left(\beta(t)^T \Big(\sum_{i \in I_0} \mu^i(t) h_x^i + \dot{\mu}^i(t) + \sum_{j \in J_0} \lambda^j(t) g_x^j \Big) \right.$$
$$\left. + \theta(t)^T \Big(\sum_{i \in I_0} \mu^i(t) h_u^i + \sum_{j \in J_0} \lambda^j(t) g_u^j \Big) \right) dt = 0 \quad (40)$$

and

$$\gamma_\alpha \int_a^b \left(\sum_{i \in I_\alpha} \mu^i(t)\Big(h^i - \dot{x}^i\Big) + \sum_{j \in J_\alpha} \lambda^j(t) g^j \right) dt$$
$$+ \int_a^b \left(\beta(t)^T \Big(\sum_{i \in I_\alpha} \Big(\mu^i(t) h_x^i + \dot{\mu}^i(t)\Big) + \sum_{j \in J_\alpha} \lambda^j(t) g_x^j \Big) \right.$$
$$\left. + \theta(t)^T \Big(\sum_{i \in I_\alpha} \mu^i(t) h_u^i + \sum_{j \in J_\alpha} \lambda^j(t) g_u^j \Big) \right) dt = 0, \alpha = 1, 2, \ldots, r. \quad (41)$$

Using (32) in (41), we have

$$\int_a^b \left\{ \beta(t)^T \Big(\sum_{i \in I_\alpha} \Big(\mu^i(t) h_x^i + \dot{\mu}^i(t)\Big) + \sum_{j \in J_\alpha} \lambda^j(t) g_x^i \Big) \right.$$
$$\left. + \theta(t)^T \Big(\sum_{i \in I_\alpha} \mu^i(t) h_u^i + \sum_{j \in J_\alpha} \lambda^j(t) g_u^j \Big) \right\} dt = 0, \alpha = 1, 2, \ldots r.$$

This can be written as

$$\int_a^b \Big(\beta(t), \theta(t)\Big)^T \begin{pmatrix} \sum_{i \in I_\alpha} \Big(\mu^i(t) h_x^i + \dot{\mu}^i(t)\Big) + \sum_{j \in J_\alpha} \lambda^j(t) g_x^i \\ \sum_{i \in I_\alpha} \mu^i(t) h_u^i + \sum_{j \in J_\alpha} \lambda^j(t) g_u^i \end{pmatrix} dt = 0,$$
$$\alpha = 1, 2, \ldots, r. \quad (42)$$

Using (14) and (15) in (26) and (27) respectively

$$\sum_{\alpha=1}^r (\gamma_\alpha - \tau) \Big(\sum_{i \in I_\alpha} \Big(\mu^i(t) h_x^i + \dot{\mu}^i(t)\Big) + \sum_{j \in J_\alpha} \lambda^j(t) g_x^j \Big)$$
$$+ \beta(t)^T (f_{xx} + \lambda(t)^T g_{xx} + \mu(t)^T h_{xx})$$
$$+ \theta(t)^T (f_{ux} + \lambda(t)^T g_{ux} + \mu(t)^T h_{ux}) = 0, t \in I$$

and

$$\sum_{\alpha=1}^{r}(\gamma_\alpha-\tau)\Big(\sum_{i\in I_\alpha}\bar{\mu}^i(t)h_u^i+\sum_{j\in J_\alpha}\lambda^j(t)g_u^j\Big)$$
$$+\beta(t)^T(f_{xu}+\bar{\lambda}(t)^T g_{xu}+\bar{\mu}(t)^T h_{xu})$$
$$\theta(t)^T(f_{uu}+\lambda(t)^T g_{uu}+\mu(t)^T h_{uu})=0, t\in \mathrm{I}.$$

Combining these relations, we have

$$\sum_{\alpha=1}^{r}\left(\gamma_\alpha-\tau\right)\begin{pmatrix}\sum\limits_{i\in I_\alpha}(\mu^i(t)h_x^i+\dot{\mu}^i(t))+\sum\limits_{j\in J_\alpha}\lambda^j(t)g_x^j\\ \sum\limits_{i\in I_\alpha}\mu^i(t)h_u^i+\sum\limits_{j\in J_\alpha}\lambda^j(t)g_u^j\end{pmatrix}$$
$$+\begin{pmatrix}f_{xx}+\lambda(t)^T g_{xx}+\mu(t)^T h_{xx}, f_{ux}+\lambda(t)^T g_{ux}+\mu(t)^T h_{ux}\\ f_{xu}+\lambda(t)^T g_{xu}+\mu(t)^T h_{xu}, f_{ux}+\lambda(t)^T g_{ux}+\mu(t)^T h_{ux}\end{pmatrix}$$
$$+\begin{pmatrix}\beta(t)\\ \theta(t)\end{pmatrix}=0, t\in I. \tag{43}$$

Pre-multiplying (43) by $(\beta(t),\theta(t))^T$ and then using (42), we have

$$\int_a^b\Big(\beta(t),\theta(t)\Big)^T M(t)\begin{pmatrix}\beta(t)\\ \theta(t)\end{pmatrix}dt=0,$$

where $M(t)=\begin{pmatrix}f_{xx}+\lambda(t)^T g_{xx}+\mu(t)^T h_{xx}, f_{ux}+\lambda(t)^T g_{ux}+\mu(t)^T h_{ux}\\ f_{xu}+\lambda(t)^T g_{xu}+\mu(t)^T h_{xu}, f_{ux}+\lambda(t)^T g_{ux}+\mu(t)^T h_{ux}\end{pmatrix}$.

This, in view of (H_1), yields

$$\sigma(t)\begin{pmatrix}\beta(t)\\ \theta(t)\end{pmatrix}=0,\ t\in I.$$

That is,

$$\beta(t)=0=\theta(t), t\in I. \tag{44}$$

Using (28) in (29), we have

$$\sum_{\alpha=1}^{r}(\gamma_\alpha-\tau)\begin{pmatrix}\sum\limits_{i\in I_\alpha}(\mu^i(t)h_x^i+\dot{\mu}^i(t))+\sum\limits_{j\in J_\alpha}\lambda^j(t)g_x^j\\ \sum\limits_{i\in I_\alpha}\mu^i(t)h_u^i+\sum\limits_{j\in J_\alpha}\lambda^j(t)g_u^j\end{pmatrix}=0.$$

This, because of the hypothesis (H_2), gives

$$\gamma_\alpha=\tau, \alpha=1,2,\ldots,r \tag{45}$$

If $\tau=0$, then $\gamma_\alpha=0, \alpha=1,2,\ldots,r$ from (45), $\eta=0$ from (30) and (31), consequently, $(\tau,\gamma,\ldots,\gamma_r,\beta(t),\theta(t),\eta(t))=0, t\in I$ but this contradicts (35). Hence $\tau=\gamma_\alpha>0, \alpha=1,2,\ldots,r$.

Using (44) in (28) and (29) along with $\tau>0, \gamma_\alpha$, $(\alpha=1,2,\ldots,r)$ and $\dot{\beta}^i(t)>0$, $t\in I$ we have

$$h^i-\dot{x}^i\geq 0, i\in I_0 \text{ and } h^i-\dot{x}^i\geq 0, i\in I_\alpha, \alpha=1,2,\ldots,r.$$

This implies

$$h(t,\bar{x},\bar{u}) - \dot{\bar{x}}(t) \geq 0, t \in I. \tag{46}$$

Using (44) in (30) and (31) together with $\tau>0$, $\gamma_\alpha>0$, $\alpha = 1,2,\ldots r$, we have

$$g(t,\bar{x},\bar{u}) \leq 0, \ t \in I. \tag{47}$$

The relation (46) and (47) implies that, $(\bar{x},\bar{u})$ is feasible for (CP).

Using (44) with $\tau>0$ in (40), we have

$$\int_a^b \left(\sum_{i\in I_o} \mu^i(t)(h^i - \bar{x}^i) + \sum_{j\in J_o} \lambda^j(t) g^j(t,\bar{x},\bar{u}) \right) dt = 0.$$

This accomplishes the equality of objective values of (CP) and (Mix CD), i.e.,

$$\int_a^b f(t,\bar{x},\bar{u})dt = \int_a^b \left(f(t,\bar{x},\bar{u}) + \sum_{i\in I_o} \mu^i(t)(h^i - \dot{\bar{x}}^i) + \sum_{j\in J_o} \lambda^j(t) g^j(t,\bar{x},\bar{u}) \right) dt.$$

If, all feasible (x,u,λ,μ), $\int_a^b \left(f + \sum_{i\in I_0} \mu^i(h^i - \dot{\bar{x}}^i) + \sum_{i\in J_0} \lambda^j g^j \right) dt$ is pseudoinvex and $\int_a^b \left(\sum_{i\in I_\alpha} \mu^i h^i + \sum_{j\in J_\alpha} \lambda^j g^j \right)$ dt is quasi-invex with respect to the same η and ξ, then from Theorem 1, $(\bar{x},\bar{u})$ is an optimal solution of (CP).

4. Control Problem with Free Boundary Conditions

The duality results established in the preceding section can be applied to the control problem with free boundary conditions. If the "targets" $x(a)$ and $x(b)$ are not restricted, we have

Problem PF (Primal): Maximize $\int_a^b f(t,x,u)dt$
subject to
$h(t,x,u) = \dot{x}, t \in I$
$g(t,x,u) \leq 0, t \in I.$

This duality now includes the transversality $\mu(t) = 0$, at $t = a$ and $t = b$ as new constraints. This implies

Problem DF (Dual):

$$\text{Maximize} \int_a^b \left[f(t,x,u) + \sum_{i\in I_0} \mu^i(t) \Big(h^i(t,x,u) - \dot{x}^i \Big) + \sum_{j\in J_0} \lambda^j(t) g^j(t,x,u) \right] dt$$

Subject to $\mu(a) = 0, \mu(b) = 0,$

$f_x(t,x,u) + \mu(t)^T h_x(t,x,u) + \lambda(t)^T g_x(t,x,u) + \dot{\mu}(t) = 0, t \in I,$

$f_u(t,x,u) + \mu(t)^T h_u(t,x,u) + \lambda(t)^T g_u(t,x,u) = 0, t \in I,$

$\int_a^b \left(\sum_{i\in I_\alpha} \mu^i(t)(h^i(t,x,u) - \dot{x}^i) + \sum_{j\in J_\alpha} \lambda^j(t) g^j(t,x,u) \right) dt \geq 0,$

$\lambda(t) \geq 0, t \in I.$

5. Related Control Problems and Mathematical Programming

We now consider some special cases of (Mix CD). If $I_0 = N$ and $J_0 = K$, then (Mix CD) becomes the following Wolfe type dual, considered by Mond and Smart [9] under invexity of

$$\int_a^b f dt, \int_a^b \mu^T (h - \dot{x}) dt \text{ and } \int_a^b \lambda^T g dt$$

(WCD): $$\text{Maximize} \int_a^b \Big(f(t,x,u) + \mu(t)^T \Big(h(t,x,u) - \dot{x} \Big) + \lambda(t)^T g(t,x,u) \Big) dt$$

subject to

$$x(a) = \hat{\alpha}, \ x(b) = \hat{\beta},$$
$$f_x(t,x,u) + \mu(t)^T h_x(t,x,u) + \lambda(t)^T g_x(t,x,u) + \dot{\mu}(t) = 0, t \in I,$$
$$f_u(t,x,u) + \mu(t)^T h_u(t,x,u) + \lambda(t)^T g_u(t,x,u) = 0, t \in I,$$
$$\lambda(t) \geq 0, t \in I.$$

If $I_0 = \phi$ and $J_0 = \phi$, then (Mix CD) becomes following Mond – Weir type dual recently considered by Husain et al. [6] in order to relax invexity requirement on suitable forms of functionals involved in the formulation of the dual:

(M-WCD): $$\text{Maximize} \int_a^b f(t,x,u) dt$$

subject to

$$x(a) = \hat{\alpha}, \ x(b) = \hat{\beta},$$
$$f_x(t,x,u) + \lambda(t)^T g_x(t,x,u) + \mu(t)^T h_x + \dot{\mu}(t) = 0, t \in I,$$
$$f_u(t,x,u) + \lambda(t)^T g_u(t,x,u) + \mu(t)^T h_u = 0, t \in I,$$
$$\int_a^b \Big(\sum_{i \in I_\alpha} \mu(t)^T \Big(h - \dot{x} \Big) + \sum_{j \in J_\alpha} \Big(\lambda(t)^T g(t,x,u) \Big) \Big) dt \geq 0,$$
$$\lambda(t) \geq 0, t \in I.$$

If f, g and h are independent of t (without any loss of generality, assume $b - a = 1$), then the control problems (CP) and Mix (CD) reduce to a pair of static primal and dual of mathematical programming, consider by Mond and Weir [8] the duality results of this

Putting $z = \binom{x}{u}$, we have

Problem(PS) : Minimize $f(z)$

Subject to

$$h(z) = 0,$$
$$g(z) \leq 0.$$

Problem (Mix DS): $\text{Maximize} f(z) + \sum_{i \in I_0} \mu^i h^i(z) + \sum_{j \in J_0} \lambda^j g^j(z)$

Subject to

$$f_z(z) + \mu^T h_z(z) + \lambda^T g_z(z) = 0$$

$$\sum_{i \in I_\alpha} \mu^i h^i(z) + \sum_{j \in J_\alpha} \lambda^j g^j(z) \geq 0, \alpha = 1, 2, \ldots, r.$$

$$\lambda \geq 0,$$

where $\lambda \in R^k$ and $\mu \in R^n$.

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