East Asian Math. J. **24** (2008), No. 4, pp. 347–358

SOME NEW INTEGRAL MEANS INEQUALITIES AND INCLUSION PROPERTIES FOR A CLASS OF ANALYTIC FUNCTIONS INVOLVING CERTAIN INTEGRAL OPERATORS

R. K. Raina and Deepak Bansal

Abstract. In this paper we investigate integral means inequalities for the integral operators Q_λ^μ and P_λ^μ applied to suitably normalized analytic functions. Further, we obtain some neighborhood and inclusion properties for a class of functions $G_\alpha(\phi, \psi)$ (defined below). Several corollaries exhibiting the applications of the main results are considered in the concluding section.

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by $f(0) = f'(0) - 1 = 0$, and analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C},\ |z| < 1\}$, then $f(z)$ can be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

We denote by $\mathcal{M}(\alpha)$, $\mathcal{N}(\alpha)$ and $\Lambda_\alpha(\lambda)$ the three subclasses of the class $\mathcal{A}$, which are defined (for $\alpha > 1$) as follows (see [9]):

$$\mathcal{M}(\alpha) = \left\{ f : f \in \mathcal{A},\ \Re\left(\frac{zf'(z)}{f(z)}\right) < \alpha \ \ (\alpha > 1;\ z \in \mathbb{U}) \right\}, \tag{1.2}$$

$$\mathcal{N}(\alpha) = \left\{ f : f \in \mathcal{A}\,, \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \alpha \ \ (\alpha > 1; z \in \mathbb{U}) \right\} \tag{1.3}$$

and

$$\Lambda_\alpha(\lambda) = \left\{ f : f \in \mathcal{A}\,, \Re\left(\frac{D^{\lambda+1}f(z)}{D^\lambda f(z)}\right) < \alpha \ (\alpha > 1; \lambda > -1;\ z \in \mathbb{U}) \right\}, \tag{1.4}$$

where the operator D^λ involved in (1.4) is the familiar Ruscheweyh operator [10]. The classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were studied recently by Owa and Nishiwaki

Received November 21, 2007; Accepted August 27, 2008.
2000 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Hadamard product, subordination, Ruscheweyh derivative, analytic functions, starlike functions and convex functions.

[6], and also by Owa and Srivastava [8]. In fact, for $1 < \alpha \leqq 4/3$, these classes were investigated earlier by Uralegaddi *et al.* [15], and the class $\Lambda_\alpha(\lambda)$ was recently studied by Raina and Bansal [9].

It follows from (1.2) and (1.3) that

$$f(z) \in \mathcal{N}(\alpha) \Leftrightarrow zf'(z) \in \mathcal{M}(\alpha)\,. \tag{1.5}$$

If $f, h \in \mathcal{A}$, where $f(z)$ is given by (1.1), and $h(z)$ is defined by

$$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, \tag{1.6}$$

then their Hadamard product (or convolution) $f * h$ is defined (as usual) by

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n c_n z^n = (h * f)(z). \tag{1.7}$$

For two functions f and g analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$ (denoted by $f \prec g$), if there exists a function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$, and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(w(z))$.

In order to prove our main results, we need the following definitions and lemmas.

Definition 1 (Raina and Bansal [9, p. 3686])**.** Let the functions $\phi(z)$ and $\psi(z)$ be given by

$$\phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n, \tag{1.8}$$

and

$$\psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n, \tag{1.9}$$

where $\lambda_n \geqq \mu_n > 0$ $(\forall\, n \in \mathbb{N}\backslash\{1\})$. Then, we say that $f \in \mathcal{A}$ is in $\mathcal{S}_\alpha(\phi, \psi)$ if

$$\Re\left\{\frac{(f * \phi)(z)}{(f * \psi)(z)}\right\} < \alpha \quad (\alpha > 1;\ z \in \mathbb{U}), \tag{1.10}$$

provided that $(f * \psi)(z) \neq 0$.

Several new and known subclasses can be obtained from the class $\mathcal{S}_\alpha(\phi, \psi)$ by suitably choosing the functions $\phi(z)$ and $\psi(z)$. We mention below some of these subclasses of $\mathcal{S}_\alpha(\phi, \psi)$ consisting of functions $f(z) \in \mathcal{A}$.

For example, using (1.8) to (1.10), it evidently follows that

$$\mathcal{S}_\alpha\left(\frac{z}{(1-z)^{\lambda+2}}, \frac{z}{(1-z)^{\lambda+1}}\right) = \Lambda_\alpha(\lambda) \tag{1.11}$$

$$\left(\text{where } \lambda_n = \frac{\Gamma(n+\lambda+1)}{\Gamma(n)\Gamma(\lambda+2)}; \mu_n = \frac{\Gamma(n+\lambda)}{\Gamma(n)\Gamma(\lambda+1)}\right)$$

$$\mathcal{S}_\alpha\left(\frac{z}{(1-z)^2}, \frac{z}{(1-z)}\right) = \mathcal{M}(\alpha) \quad (\text{where } \lambda_n = n;\ \mu_n = 1) \tag{1.12}$$

and

$$\mathcal{S}_\alpha\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}\right) = \mathcal{N}_(\alpha) \quad \left(\text{where } \lambda_n = n^2; \mu_n = n\right). \tag{1.13}$$

Definition 2 (Jung-Kim-Srivastava [3])**.** Let $f(z) \in \mathcal{A}$ be defined by (1.1), then

$$Q_\lambda^\mu f(z) = \binom{\lambda+\mu}{\lambda} \frac{\mu}{z^\lambda} \int_0^z t^{\lambda-1}\left(1-\frac{t}{z}\right)^{\mu-1} f(t)dt$$

$$= z + \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+1)} \sum_{n=2}^{\infty} \frac{\Gamma(n+\lambda)}{\Gamma(n+\lambda+\mu)} a_n z^n. \tag{1.14}$$

$$(\lambda > -1;\ \mu > 0;\ f \in \mathcal{A})$$

For $\mu = 1$, (1.14) reduces to the generalized Libera operator [7] given by

$$Q_\lambda^1 f(z) = B_\lambda f(z) = z + \sum_{n=2}^{\infty}\left(\frac{\lambda+1}{\lambda+n}\right) a_n z^n. \tag{1.15}$$

Definition 3 (Komatu [4])**.** Let $f(z) \in \mathcal{A}$ be defined by (1.1), then

$$P_\lambda^\mu f(z) = \frac{(\lambda+1)^\mu}{z^\lambda \Gamma(\mu)} \int_0^z t^{\lambda-1}\left(\log\frac{z}{t}\right)^{\mu-1} f(t)dt$$

$$= z + \sum_{n=2}^{\infty}\left(\frac{\lambda+1}{\lambda+n}\right)^\mu a_n z^n. \tag{1.16}$$

$$(\lambda > -1;\ \mu > 0;\ f \in \mathcal{A})$$

The operators (1.14) and (1.16) contain the familiar Jung-Kim-Srivastava and Komatu operator (see the details in [3], [4]).

Lemma 1 (Raina & Bansal [9, Theorem 2.1, p. 3687])**.** *If $f(z) \in \mathcal{A}$ and satisfies*

$$\sum_{n=2}^{\infty} L(\lambda_n, \mu_n, k, \alpha)\,|a_n| \leqq 2(\alpha-1), \tag{1.17}$$

where

$$L(\lambda_n, \mu_n, k, \alpha) = \{(\lambda_n - k\mu_n) + |\lambda_n + (k-2\alpha)\mu_n|\}, \tag{1.18}$$

for some k $(0 \leqq k \leqq 1)$, and some α $(\alpha > 1)$, then $f(z) \in \mathcal{S}_\alpha(\phi, \psi)$.

Lemma 2. *Let $L(\lambda_n, \mu_n, k, \alpha)$ be defined by (1.18), then $\{L(\lambda_n, \mu_n, k, \alpha)\}_{n=2}^\infty$ is a nonvanishing, positive and nondecreasing sequence provided that the sequences $\langle \mu_n \rangle$ and $\left\langle \frac{\lambda_n}{\mu_n} \right\rangle$ are nondecreasing, and*

$$(\lambda_n > \mu_n > 0\ ; n \in \mathbb{N}\backslash\{1\};\ \alpha > 1;\ 0 \leqq k \leqq 1). \tag{1.19}$$

Proof. See details in [9, p. 3692]. □

Lemma 3 (Littlewood [5]). *If $f(z)$ and $h(z)$ are analytic in $\mathbb{U}$ with $f(z) \prec h(z)$, then for $p > 0$ and $z = re^{i\theta}(0 < r < 1)$:*

$$\int_0^{2\pi} |f(z)|^p d\theta \leq \int_0^{2\pi} |h(z)|^p d\theta. \tag{1.20}$$

Corresponding to the neighborhood definition given by Frasin and Darus [2], let $f \in \mathcal{A}$ be of the form (1.1) and $s \geqq 0$, then a $(q-s)$ neighborhood of the function f is defined by

$$M_s^q(f) = \left\{ h \in \mathcal{A} : h(z) = z + \sum_{n=2}^{\infty} c_n z^n, \ \sum_{n=2}^{\infty} n^{q+1} |a_n - c_n| \leqq s \right\}. \tag{1.21}$$

For

$$e(z) = z,$$

we observe that

$$M_s^q(e) = \left\{ h \in \mathcal{A} : h(z) = z + \sum_{n=2}^{\infty} c_n z^n, \ \sum_{n=2}^{\infty} n^{q+1} |c_n| \leqq s \right\}, \tag{1.22}$$

where $q \in \mathbb{N} \cup \{0\}$. We note that $M_s^0(f) = N_s(f)$ and $M_s^1(f) = M_s(f)$, where $N_s(f)$ denotes the s-neighborhood of f introduced by Ruscheweyh [11], and $M_s(f)$ is the neighborhood defined by Silverman [12].

In view of Lemma 1, we further define the following subclasses of the class $\mathcal{S}_\alpha(\phi, \psi)$.

Definition 4. Let $G_\alpha(\phi, \psi)$ denote the class of functions $f \in \mathcal{S}_\alpha(\phi, \psi)$(defined by (1.10)) whose coefficients satisfy the coefficient inequality (1.17).

Corresponding to the subclasses $\Lambda_\alpha(\lambda)$, $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ defined by (1.11) to (1.13), we also have the following set of subclasses of the class $G_\alpha(\phi, \psi)$(see [9, p. 3691]):

$$G_\alpha\left(\frac{z}{(1-z)^{\lambda+2}}, \frac{z}{(1-z)^{\lambda+1}}\right) \equiv \Lambda_\alpha^*(\lambda) \tag{1.23}$$

$$G_\alpha\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}\right) \equiv \mathcal{M}^*(\alpha) \tag{1.24}$$

and

$$G_\alpha\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}\right) \equiv \mathcal{N}^*(\alpha). \tag{1.25}$$

Obviously, we have the relatioships

$$\Lambda_\alpha^*(\lambda) \subset \Lambda_\alpha(\lambda); \ \mathcal{M}^*(\alpha) \subset \mathcal{M}(\alpha); \quad \mathcal{N}^*(\alpha) \subset \mathcal{N}(\alpha).$$

Among many others, the classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were studied recently by Choi [1], Srivastava and Attiya [13] and Owa and Nishiwaki [6]. In this paper we investigate the integral means inequalities for the integral operators Q_λ^μ and P_λ^μ involving suitably normalized analytic functions. We also derive some neighborhood and inclusion relationships for the class $G_\alpha(\phi, \psi)$ (defined above). Several corollaries depicting some interesting consequences of the main results are also mentioned.

2. THE MAIN RESULTS

In this section we give integral means inequalities in Theorems 1 and 2, a neighborhood property for the class $G_\alpha(\phi, \psi)$ in Theorem 3, and some inclusion properties for class $G_\alpha(\phi, \psi)$ in Theorem 4, involving the integral operators Q_λ^μ and P_λ^μ (defined above by (1.14) and (1.16), respectively).

Theorem 1. *Let $f(z) \in \mathcal{A}$ and $g(z)$ be defined by*

$$g(z) = z + b_j z^j \ (b_j \neq 0; j \geqq 2) \tag{2.1}$$

and suppose that

$$\sum_{n=2}^{\infty} L(\lambda_n, \mu_n, k, \alpha)\, |a_n| \leqq \frac{|b_j|\, \Gamma(\delta+\eta+1)\Gamma(\delta+j)L(\lambda_2, \mu_2, k, \alpha)(\lambda+\mu+1)}{\Gamma(\delta+1)\Gamma(j+\delta+\eta)(\lambda+1)}, \tag{2.2}$$

where $L(\lambda_n, \mu_n, k, \alpha)$ is given by (1.18). If $\langle\lambda_n\rangle$, $\langle\mu_n\rangle$ and $\langle\lambda_n/\mu_n\rangle$ are non-decreasing sequences and $\lambda_n > \mu_n > 0$ $(n \in \mathbb{N}\backslash\{1\})$, $0 \leqq k \leqq 1$, then for $\lambda > -1$, $\delta > -1$, $\mu > 0$, $\eta > 0$, $p > 0$ and $z = re^{i\theta}(0 < r < 1)$:

$$\int_0^{2\pi} |Q_\lambda^\mu f(z)|^p d\theta \quad \leqq \quad \int_0^{2\pi} |Q_\delta^\eta g(z)|^p d\theta. \tag{2.3}$$

Proof. Let $f(z)$ be given by (1.1). In view of (1.14), we obtain

$$Q_\lambda^\mu f(z) = z\left[1 + \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+1)} \sum_{n=2}^{\infty} \frac{\Gamma(n+\lambda)}{\Gamma(n+\lambda+\mu)} a_n z^{n-1}\right]$$

and

$$Q_\delta^\eta g(z) = z\left[1 + \frac{\Gamma(\delta+\eta+1)\Gamma(j+\delta)}{\Gamma(\delta+1)\Gamma(j+\delta+\eta)} b_j z^{j-1}\right].$$

To prove (2.3), it is sufficient to show by means of Lemma 3 that

$$1 + \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+1)} \sum_{n=2}^{\infty} \frac{\Gamma(n+\lambda)}{\Gamma(n+\lambda+\mu)} a_n z^{n-1} \prec 1 + \frac{\Gamma(\delta+\eta+1)\Gamma(j+\delta)}{\Gamma(\delta+1)\Gamma(j+\delta+\eta)} b_j z^{j-1}. \tag{2.4}$$

By setting

$$1 + \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+1)} \sum_{n=2}^{\infty} \frac{\Gamma(n+\lambda)}{\Gamma(n+\lambda+\mu)} a_n z^{n-1} = 1 + \frac{\Gamma(\delta+\eta+1)\Gamma(j+\delta)}{\Gamma(\delta+1)\Gamma(j+\delta+\eta)} b_j \left[w(z)\right]^{j-1}$$

we find that

$$[w(z)]^{j-1} = \frac{\Gamma(\lambda+\mu+1)\Gamma(\delta+1)\Gamma(j+\delta+\eta)}{b_j\ \Gamma(\delta+\eta+1)\Gamma(\lambda+1)\Gamma(j+\delta)} \sum_{n=2}^{\infty} L(\lambda_n,\mu_n,k,\alpha)\theta(n)a_n z^{n-1}, \tag{2.5}$$

where

$$\theta(n) = \frac{\Gamma(n+\lambda)}{L(\lambda_n,\mu_n,k,\alpha)\Gamma(n+\lambda+\mu)} \tag{2.6}$$

$$(\lambda_n > \mu_n > 0\ (\forall n \in \mathbb{N}\backslash\{1\}),\ 0 \leqq k \leqq 1,\ \alpha > 1,\ \lambda > -1,\ \mu > 0).$$

If $\langle \mu_n \rangle$ and $\langle \lambda_n/\mu_n \rangle$ are nondecreasing sequences, then by virtue of Lemma 2 we observe that $\frac{1}{L(\lambda_n,\mu_n,k,\alpha)}$ is a positive nonincreasing sequence. Also, $\frac{\Gamma(n+\lambda)}{\Gamma(n+\lambda+\mu)}$ is a nonincreasing positive sequence. Thus, $\theta(n)$ $(n \in \mathbb{N}\backslash\{1\})$ is also a nonincreasing sequence of n (being the product of two positive nonincreasing sequences). It readily follows that

$$0 < \theta(n) \leqq \theta(2) = \frac{\Gamma(\lambda+2)}{L(\lambda_2,\mu_2,k,\alpha)\Gamma(\lambda+\mu+2)},$$

and from (2.5), we infer that $w(0) = 0$, and therefore, we are lead to

$$\begin{aligned} &|w(z)|^{j-1} \\ &\leqq \frac{\Gamma(\lambda+\mu+1)\Gamma(\delta+1)\Gamma(j+\delta+\eta)}{|b_j|\,\Gamma(\delta+\eta+1)\Gamma(\lambda+1)\Gamma(j+\delta)} \sum_{n=2}^{\infty} L(\lambda_n,\mu_n,k,\alpha)\theta(n)\,|a_n|\,|z|^{n-1} \\ &\leqq \frac{|z|\,\Gamma(\lambda+\mu+1)\Gamma(\delta+1)\Gamma(j+\delta+\eta)}{|b_j|\,\Gamma(\delta+\eta+1)\Gamma(\lambda+1)\Gamma(j+\delta)}\theta(2) \sum_{n=2}^{\infty} L(\lambda_n,\mu_n,k,\alpha)\,|a_n| \\ &\leqq |z| < 1, \end{aligned}$$

on making use of the hypothesis (2.2) of Theorem 1. Evidently, the last inequality above establishes the subordination (2.4), which consequently proves our Theorem 1. □

Theorem 2. *Let $f(z) \in \mathcal{A}$ and $g(z)$ be defined by (2.1), and suppose that*

$$\sum_{n=2}^{\infty} L(\lambda_n,\mu_n,k,\alpha)\,|a_n| \leqq \left(\frac{\delta+1}{\delta+j}\right)^{\eta} \left(\frac{\lambda+2}{\lambda+1}\right)^{\mu} L(\lambda_2,\mu_2,k,\alpha)\,|b_j|\,, \tag{2.7}$$

where $L(\lambda_n,\mu_n,k,\alpha)$ is given by (1.18). If $\langle \lambda_n \rangle$, $\langle \mu_n \rangle$ and $\langle \lambda_n/\mu_n \rangle$ are nondecreasing sequences, $\lambda_n > \mu_n > 0$ ($\forall\ n \in \mathbb{N}\backslash\{1\}$), $0 \leqq k \leqq 1$, then for $\lambda > -1$, $\delta > -1$, $\mu > 0$, $\eta > 0$, $p > 0$ and $z = re^{i\theta}(0 < r < 1)$:

$$\int_0^{2\pi} |P_\lambda^\mu f(z)|^p d\theta \quad \leqq \quad \int_0^{2\pi} |P_\delta^\eta g(z)|^p d\theta. \tag{2.8}$$

Proof. Let $f(z)$ be given by (1.1). Using (1.16), we obtain

$$P_{\lambda}^{\mu} f(z) = z \left[1 + \sum_{n=2}^{\infty} \left(\frac{\lambda+1}{\lambda+n} \right)^{\mu} a_n z^{n-1} \right]$$

and

$$P_{\delta}^{\eta} g(z) = z \left[1 + \left(\frac{\delta+1}{\delta+j} \right)^{\eta} b_j z^{j-1} \right].$$

To establish (2.8), it is sufficient to show that (in view of Lemma 3)

$$1 + \sum_{n=2}^{\infty} \left(\frac{\lambda+1}{\lambda+n} \right)^{\mu} a_n z^{n-1} \prec 1 + \left(\frac{\delta+1}{\delta+j} \right)^{\eta} b_j z^{j-1}. \tag{2.9}$$

Putting

$$1 + \sum_{n=2}^{\infty} \left(\frac{\lambda+1}{\lambda+n} \right)^{\mu} a_n z^{n-1} = 1 + \left(\frac{\delta+1}{\delta+j} \right)^{\eta} b_j \left[w(z) \right]^{j-1},$$

we obtain

$$\left[w(z) \right]^{j-1} = \left(\frac{\delta+j}{\delta+1} \right)^{\eta} \frac{1}{b_j} \sum_{n=2}^{\infty} L(\lambda_n, \mu_n, k, \alpha) \sigma(n) a_n z^{n-1}, \tag{2.10}$$

where

$$\sigma(n) = \left(\frac{\lambda+1}{\lambda+n} \right)^{\mu} \frac{1}{L(\lambda_n, \mu_n, k, \alpha)}. \tag{2.11}$$

$$(\lambda_n > \mu_n > 0 \, (\forall\, n \in \mathbb{N}\backslash\{1\}),\ 0 \leqq k \leqq 1,\ \alpha > 1,\ \lambda > -1,\ \mu > 0)$$

If $\langle \lambda_n \rangle$, $\langle \mu_n \rangle$ and $\langle \lambda_n / \mu_n \rangle$ are nondecreasing sequences then by the application of Lemma 2, we observe that $\frac{1}{L(\lambda_n, \mu_n, k, \alpha)}$ is a nonincreasing sequence. Also, $\left(\frac{\lambda+1}{\lambda+n} \right)^{\mu}$ is a nonincreasing sequence. Thus, $\sigma(n)$ is a nonincreasing sequence (being the product of two positive nonincreasing sequences).

Now $\sigma(n)$ being a nonincreasing sequence of n implies that

$$0 < \sigma(n) \leqq \sigma(2) = \left(\frac{\lambda+1}{\lambda+2} \right)^{\mu} \frac{1}{L(\lambda_2, \mu_2, k, \alpha)},$$

and from (2.10), we note that $w(0) = 0$, and hence, we obtain

$$\begin{aligned} |w(z)|^{j-1} &\leqq \left(\frac{\delta+j}{\delta+1} \right)^{\eta} \frac{1}{|b_j|} \sum_{n=2}^{\infty} L(\lambda_n, \mu_n, k, \alpha) \sigma(n) \left| a_n \right| |z|^{n-1} \\ &\leqq \frac{|z|}{|b_j|} \left(\frac{\delta+j}{\delta+1} \right)^{\eta} \sigma(2) \sum_{n=2}^{\infty} L(\lambda_n, \mu_n, k, \alpha) \left| a_n \right| \\ &\leqq |z| < 1, \end{aligned}$$

by virtue of (2.7) of Theorem 2. The last inequality above establishes the subordination (2.9), which completes the proof of Theorem 2. □

Theorem 3. *If* $\left\{\frac{L(\lambda_n,\mu_n,k,\alpha)}{n^{q+1}}\right\}_{n=2}^{\infty}$ *is a nondecreasing sequence, then* $G_\alpha(\phi,\psi)\subset M_s^q(e)$, *where*

$$s=\frac{2^{q+2}(\alpha-1)}{L(\lambda_2,\mu_2,k,\alpha)} \tag{2.12}$$

and $L(\lambda_n,\mu_n,k,\alpha)$ *is given by (1.18).*

Proof. It follows from (1.17) that if $f(z)\in G_\alpha(\phi,\psi)$, then

$$\frac{L(\lambda_2,\mu_2,k,\alpha)}{2^{q+1}}\sum_{n=2}^{\infty}n^{q+1}\,|a_n|\le\sum_{n=2}^{\infty}L(\lambda_n,\mu_n,k,\alpha)\,|a_n|\leqq 2(\alpha-1)$$

which at once gives

$$\sum_{n=2}^{\infty}n^{q+1}\,|a_n|\le\frac{2^{q+2}(\alpha-1)}{L(\lambda_2,\mu_2,k,\alpha)}$$

and the result follows on using (1.22). □

Theorem 4. *Let* $f(z)\in G_\alpha(\phi,\psi)$, *then* $Q_\lambda^\mu f(z)\in G_\alpha(\phi,\psi)$ *and* $P_\lambda^\mu f(z)\in G_\alpha(\phi,\psi)$ $(\lambda>-1,\mu>0)$.

Proof. Let $f(z)\in G_\alpha(\phi,\psi)$, then $f(z)$ satisfies the coefficient inequality (1.17), and

$$Q_\lambda^\mu f(z)=z+\frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\lambda+n)}{\Gamma(\lambda+\mu+n)}a_nz^n.$$

To show that $Q_\lambda^\mu f(z)\in G_\alpha(\phi,\psi)$, we need simply to show that

$$\sum_{n=2}^{\infty}L(\lambda_n,\mu_n,k,\alpha)\frac{\Gamma(\lambda+\mu+1)\Gamma(\lambda+n)}{\Gamma(\lambda+1)\Gamma(\lambda+\mu+n)}\,|a_n|\le 2(\alpha-1),$$

which is true in view of coefficient inequality (1.17), because evidentally

$$\frac{\Gamma(\lambda+n)}{\Gamma(\lambda+\mu+n)}\le\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\mu+1)}\ \ (\forall\ n=\ 2,\ 3,...).$$

The proof of second part, viz. that $f(z)\in G_\alpha(\phi,\psi)$ implies that $P_\lambda^\mu f(z)\in G_\alpha(\phi,\psi)$ is similar to the first part, and is hence omitted. □

3. APPLICATIONS OF MAIN RESULTS

In this section we consider some applications of our main results (Theorems 1 to 3).

Let us set

$$\eta=\mu,\ \delta=\lambda,\ b_j=\frac{2(\beta-1)}{L(\lambda_j,\mu_j,k,\alpha)}\ (j\ge 2), \tag{3.1}$$

in Theorem 1, and suppose that $f(z) \in G_\alpha(\phi, \psi)$ (which is given by Definition 4), then the inequality (2.3) holds if the following coefficient inequality holds true:

$$\sum_{n=2}^{\infty} L(\lambda_n, \mu_n, k, \alpha)\, |a_n| \leq \frac{2(\beta-1)\Gamma(\lambda+\mu+2)\Gamma(\lambda+j)L(\lambda_2, \mu_2, k, \alpha)}{L(\lambda_j, \mu_j, k, \alpha)\Gamma(\lambda+2)\Gamma(\lambda+\mu+j)}. \quad (3.2)$$

To show that (3.2) is true, let us choose β such that

$$\beta \geq 1 + \frac{L(\lambda_j, \mu_j, k, \alpha)\Gamma(\lambda+2)\Gamma(\lambda+\mu+j)}{L(\lambda_2, \mu_2, k, \alpha)\Gamma(\lambda+j)\Gamma(\lambda+\mu+2)}(\alpha-1)$$

then (3.2) reduces to

$$\sum_{n=2}^{\infty} L(\lambda_n, \mu_n, k, \alpha)\, |a_n| \leq 2(\alpha-1)$$

which is true in view of (1.17) of Lemma 1.

In view of the above parametric substitutions (3.1), Theorem 1 finally reduces to the following result.

Corollary 1. *Let $f(z) \in G_\alpha(\phi, \psi)$ and $g(z)$ be given by*

$$g(z) = z + \frac{2(\beta-1)}{L(\lambda_n, \mu_n, k, \alpha)} z^n \ \ (n \geq 2) \quad (3.3)$$

satisfying the conditions given by (1.19), then for $z = re^{i\theta}$ $(0 < r < 1)$:

$$\int_0^{2\pi} |Q_\lambda^\mu f(z)|^p d\theta \quad \leqq \quad \int_0^{2\pi} |Q_\lambda^\mu g(z)|^p d\theta \quad (3.4)$$

$$(\lambda > -1, \mu > 0, p > 0)$$

provided that there exists β such that

$$\beta \geqq 1 + \frac{L(\lambda_n, \mu_n, k, \alpha)\Gamma(\lambda+2)\Gamma(\lambda+\mu+n)}{L(\lambda_2, \mu_2, k, \alpha)\Gamma(\lambda+n)\Gamma(\lambda+\mu+2)}(\alpha-1)\ (n \geq 2) \quad (3.5)$$

where $L(\lambda_n, \mu_n, k, \alpha)$ is given by (1.18).

Next, let us choose $n = 2$ in Corollary 1, then from (3.5) we get $\beta \geqq \alpha$. Consequently, Corollary 1 gives

Corollary 2. *Let $f(z) \in G_\alpha(\phi, \psi)$ and $g(z)$ be given by*

$$g(z) = z + \frac{2(\beta-1)}{L(\lambda_2, \mu_2, k, \alpha)} z^2 \ (\beta \geqq \alpha) \quad (3.6)$$

satisfying the coditions corresponding to those given by (1.19), then for $z = re^{i\theta}$ $(0 < r < 1)$:

$$\int_0^{2\pi} |Q_\lambda^\mu f(z)|^p d\theta \quad \leqq \quad \int_0^{2\pi} |Q_\lambda^\mu g(z)|^p d\theta, \tag{3.7}$$

$$(\lambda > -1, \mu > 0, p > 0)$$

where $L(\lambda_2, \mu_2, k, \alpha)$ *is given by (1.18).*

Making similar substitutions as given by (3.1) in Theorem 2, we shall arrive at the following result:

Corollary 3. *Let* $f(z) \in G_\alpha(\phi, \psi)$ *and* $g(z)$ *be given by*

$$g(z) = z + \frac{2(\beta - 1)}{L(\lambda_n, \mu_n, k, \alpha)} z^n \ (n \geq 2) \tag{3.8}$$

satisfying the conditions given by (1.19), then for $z = re^{i\theta}$ $(0 < r < 1)$:

$$\int_0^{2\pi} |P_\lambda^\mu f(z)|^p d\theta \quad \leqq \quad \int_0^{2\pi} |P_\lambda^\mu g(z)|^p d\theta \tag{3.9}$$

$$(\lambda > -1, \mu > 0, p > 0)$$

provided that there exists β *such that*

$$\beta \geqq 1 + \left(\frac{\lambda + n}{\lambda + 2}\right)^\mu \frac{L(\lambda_n, \mu_n, k, \alpha)}{L(\lambda_2, \mu_2, k, \alpha)} (\alpha - 1) \ (n \geq 2) \tag{3.10}$$

where $L(\lambda_n, \mu_n, k, \alpha)$ *is given by (1.18).*

For $n = 2$, Corollary 3 reduces to

Corollary 4. *Let* $f(z) \in G_\alpha(\phi, \psi)$ *and* $g(z)$ *be given by*

$$g(z) = z + \frac{2(\beta - 1)}{L(\lambda_2, \mu_2, k, \alpha)} z^2 \ (\beta \geqq \alpha) \tag{3.11}$$

satisfying the conditions corresponding to those given by (1.19), then for $z = re^{i\theta}$ $(0 < r < 1)$:

$$\int_0^{2\pi} |P_\lambda^\mu f(z)|^p d\theta \quad \leqq \quad \int_0^{2\pi} |P_\lambda^\mu g(z)|^p d\theta \tag{3.12}$$

$$(\lambda > -1, \mu > 0, p > 0)$$

where $L(\lambda_2, \mu_2, k, \alpha)$ *is given by (1.18).*

If we set the arbitrary functions ϕ and ψ in Corollary 2 in accordance with (1.24) and choose $\mu = 1$, then in view of (1.15) we obtain the following result involving generalized Libera operator [7].

Corollary 5. *Let* $f(z) \in \mathcal{M}^*(\alpha)$ *and* $g(z)$ *be given by*

$$g(z) = z + \frac{2(\beta - 1)}{\rho(k, \alpha)} z^2 \quad (\beta \geqq \alpha) \tag{3.13}$$

where

$$\rho(k, \alpha) = \{(2 - k) + |2 + k - 2\alpha|\}. \tag{3.14}$$

satisfying the conditions that $0 \leq k \leq 1$, $\alpha > 1$, *then for* $z = re^{i\theta}$ $(0 < r < 1)$:

$$\int_0^{2\pi} |B_\lambda f(z)|^p d\theta \quad \leqq \quad \int_0^{2\pi} |B_\lambda g(z)|^p d\theta. \tag{3.15}$$

$$(\lambda > -1, p > 0)$$

where the operator B_λ *is defined by (1.15).*

Making use of the relation (1.24) to reduce the class $G_\alpha(\phi, \psi)$ to $\mathcal{M}^*(\alpha)$ in Theorem 3, we obtain

Corollary 6. *If* $\left\{\frac{\Omega(n,k,\alpha)}{n^{q+1}}\right\}_{n=2}^{\infty}$ *is a nondecreasing sequence, then* $\mathcal{M}^*(\alpha) \subset M_s^q(e)$, *where*

$$s = \frac{2^{q+2}(\alpha - 1)}{\Omega(2, k, \alpha)} \tag{3.16}$$

and

$$\Omega(n, k, \alpha) = \{(n - k) + |n + k - 2\alpha|\}. \tag{3.17}$$

provided that $0 \leq k \leq 1, \alpha > 1$ *and* $q \in \mathbb{N} \cup \{0\}$.

Similarly, if we use the relation (1.25) to reduce the class $G_\alpha(\phi, \psi)$ to $\mathcal{N}^*(\alpha)$ in Theorem 3, we get the following result.

Corollary 7. *If* $\left\{\frac{\Delta(n,k,\alpha)}{n^{q+1}}\right\}_{n=2}^{\infty}$ *is a nondecreasing sequence, then* $\mathcal{N}^*(\alpha) \subset M_s^q(e)$, *where*

$$s = \frac{2^{q+2}(\alpha - 1)}{\Delta(2, k, \alpha)} \tag{3.18}$$

and

$$\Delta(n, k, \alpha) = n\{(n - k) + |n + k - 2\alpha|\}. \tag{3.19}$$

provided that $0 \leq k \leq 1, \alpha > 1$ *and* $q \in \mathbb{N} \cup \{0\}$.

References

[1] J. H. Choi, *Univalent functions with positive coefficients involving a certain fractional integral operator and its partial sums*, Fract. Cal. Appl. Anal. **1** (1998), no. 3, 311–318.

[2] B. A. Frasin and M. Darus, *Integral means inequalities and neighborhoods for analytic univalent functions with negative coefficients*, Soochow J. Math. **30** (2004), no. 2, 217–223.

[3] I. B. Jung, Y. C. Kim and H. M. Srivastava, *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*, J. Math. Anal. Appl. **176** (1993), 138–147.

[4] Y. Komatu, *On analytic prolongation of a family of integral operators*, Mathematica (cluj) **32** (55), 141–145.

[5] J. E. Littlewood, *On inequalities in the theory of functions*, Proc. London Math. Soc. **23** (1925), 481–519.

[6] S. Owa and J. Nishiwaki, *Coefficients estimates for certain classes of analytic functions*, J. Inequal. Pure Appl. Math. **3** (2002), no. 5, 1–5 (electronic).

[7] S. Owa and H. M. Srivastava, *Some applications of the generalized Libera integral operator*, Proc. Japan. Acad. Ser. A Math. Sci. **62** (1986), 125–128.

[8] S. Owa and H. M. Srivastava, *Some generalized convolution properties associated with certain subclasses of analytic functions*, J. Inequal. Pure Appl. Math. **3** (2002), no. 3, 1–13 (electronic).

[9] R. K. Raina and Deepak Bansal, *Coefficient estimates and subordination properties associated with certain classes of functions*, Internat. J. Math. Math. Sci. **22** (2005), 3685–3695.

[10] S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), 109–115.

[11] S. Ruscheweyh, *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc. **81** (1981), no. 4, 521–527.

[12] H. Silverman, *Neighborhoods of class of analytic functions*, Far East J. Math. Sci., **3** (1995), no. 2, 165–169.

[13] H. M. Srivastava and A. Attiya, *Some subordination results associated with certain subclasses of analytic functions*, J. Inequal. Pure Appl. Math., **5** (2004), no. 4, 1–6 (electronic).

[14] H. M. Srivastava, and S. Owa, *Current Topics in Analytic Functions Theory*, World Scientific Publishing Company, Singapore, New Jersey, London and Hongkong, 1992.

[15] B. A. Uralegaddi, M. D. Gangi and S. M. Sarangi, *Univalent function with positive coefficients*, Tamkang J. Math. **25** (1994), 225–230.

R. K. Raina
10/11, Ganpati Vihar, Opposite Sector 5, Udaipur 313002, Rajasthan, India
E-mail address: rainark_7@hotmail.com

Deepak Bansal
Department of Mathematics, College of Engineering and Technology, BIKANER-334004, RAJASTHAN, INDIA
E-mail address: deepakbansal_79@yahoo.com