East Asian Math. J. **24** (2008), No. 4, pp. 339–345

ON THE EQUIVALENCE PROBLEMS FOR THE CONVERGENCE OF ITERATIVE SEQUENCES FOR SET-VALUED CONTRACTION MAPPINGS IN BANACH SPACES

Kwang Pak Park and Kyung Soo Kim

Abstract. Some equivalence conditions for the convergence of iterative sequences for set-valued contraction mapping in Banach spaces are obtained.

1. Preliminaries

Definition 1.1. Let E be a Banach space and $CB(E)$ be the family of all bounded closed subsets of E. A set-valued mapping $T : E \to CB(E)$ is said to be *contraction* if there exists a constant $k \in (0,1)$ such that

$$H(Tx, Ty) \leq k\|x - y\|, \quad x, y \in E, \tag{1.1}$$

where H is the Hausdorff metric on $CB(E)$, i.e.,

$$H(A, B) = \max\Big\{ \sup_{x\in A} d(x, B), \sup_{y\in B} d(A, y)\Big\},$$

for any given $A, B \in CB(E)$.

Definition 1.2 ([1], [2], [4])**.** Let B be a nonempty closed convex subset of E, $T : B \to 2^B$ be a mapping, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $[0,1]$ satisfying some conditions and $\{e_n\}$, $\{f_n\}$ and $\{g_n\}$ are three bounded sequences in E. Then the following sequences $\{w_n\}$, $\{u_n\}$, $\{r_n\}$, $\{x_n\}$ are called *Picard, Mann, Ishikawa, three-step iterative sequence with perturbed errors*, respectively:

$$\begin{cases} w_0 \in B \\ w_{n+1} \in Tw_n, \quad \forall\, n \geq 0; \end{cases} \tag{1.2}$$

Received November 15, 2007; Accepted July 9, 2008.
2000 *Mathematics Subject Classification.* 47H04, 47H10, 47H14.
Key words and phrases. Set-valued contraction mapping, Picard iteration, Mann iteration, Ishikawa iteration with perturbed errors, three-step iteration with perturbed errors, fixed point.

$$\begin{cases} u_0 \in B \\ u_{n+1} \in (1-\alpha_n)u_n + \alpha_n T u_n, \quad \forall\, n \geq 0; \end{cases} \tag{1.3}$$

$$\begin{cases} r_0 \in B \\ r_{n+1} \in (1-\alpha_n)r_n + \alpha_n T s_n + \alpha_n e_n \\ s_n \in (1-\beta_n)r_n + \beta_n T r_n + \beta_n f_n, \quad \forall\, n \geq 0; \end{cases} \tag{1.4}$$

$$\begin{cases} x_0 \in B \\ x_{n+1} \in (1-\alpha_n)x_n + \alpha_n T y_n + \alpha_n e_n \\ y_n \in (1-\beta_n)x_n + \beta_n T z_n + \beta_n f_n \\ z_n \in (1-\gamma_n)x_n + \gamma_n T x_n + \gamma_n g_n, \quad \forall\, n \geq 0; \end{cases} \tag{1.5}$$

where the sequence $\{\alpha_n\}$ appeared in (1.3)-(1.5) is the same.

In this paper, we are going to study the equivalence between the convergence of Picard, three-step iterative sequence with perturbed errors defined by (1.2), (1.5) for set-valued contraction mapping in Banach spaces.

2. Main Results

In order to prove our main results, we need the following key lemmas.

Lemma 2.1 ([3]). *Let $\{a_n\}$, $\{b_n\}$, $\{d_n\}$ and $\{t_n\}$ be nonnegative real sequence satisfying the following conditions:*

(1) $t_n \in [0,1]$ *and* $\sum_{n=0}^{\infty} t_n = \infty$;

(2) $\sum_{n=0}^{\infty} b_n < \infty$ *and* $\sum_{n=0}^{\infty} d_n < \infty$.

If

$$a_{n+1} \leq (1-t_n)a_n + b_n a_n + d_n, \quad \forall\ n \geq 0,$$

then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2 ([5]). *Let (E,d) be a complete metric space and let $T : E \to CB(E)$ be a set-valued mapping. Then for any given $\varepsilon > 0$ and for any given $u, v \in E$, $w \in Tu$, there exists $y \in Tv$ such that*

$$d(w,y) \leq (1+\varepsilon)H(Tu,Tv),$$

where $H(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$.

Theorem 2.1. *Let E be a real Banach space, D be a nonempty closed convex subset of E, $T : D \to 2^D$ be a contraction mapping with $k < \frac{1}{1+\varepsilon}$ and a constant L satisfying $\sup_{\omega\in Tx} \|\omega\| \leq L$, for all $x \in D$. Let $\{w_n\}$ and $\{x_n\}$ be the Picard and three-step iterative sequence with perturbed errors defined by* (1.2) *and* (1.5) *respectively and satisfying the following conditions:*

(i) $\alpha_n, \beta_n, \gamma_n \in [0,1], \quad \forall\, n \geq 0$;

(ii) $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \|e_n\| = 0$;

(iii) $\sum\limits_{n=0}^{\infty} \alpha_n\beta_n < \infty, \quad \sum\limits_{n=0}^{\infty} \alpha_n\|e_n\| < \infty, \quad \sum\limits_{n=0}^{\infty} \alpha_n = \infty.$

If $w_0 = x_0$, then the following statements are equivalent:

(1) *the Picard iterative sequence $\{w_n\}$ converges strongly to $x^* \in F(T)$;*

(2) *the three-step iterative sequence with perturbed errors $\{x_n\}$ converges strongly to $x^* \in F(T)$.*

Furthermore, x^* is the unique fixed point of T.

Proof. From Nadler ([5]), there exists a fixed point x^* in $F(T)$. Since $\{e_n\}$, $\{f_n\}$ and $\{g_n\}$ are bounded, there exist a constants $L' > 0$ such that

$$\sup_{n\geq 0}\{\|e_n\|, \|f_n\|, \|g_n\|\} \leq L'.$$

Put

$$M' = L + L' + \|x_0\|,$$

by induction, it is easy to prove that

$$\sup_{n\geq 0}\Big\{\|\mu_n\|, \|\eta_n\|, \|\xi_n\|, \|x_n\|, \|y_n\|, \|z_n\|\Big\} \leq M',$$

for $\mu_n \in Tx_n$, $\eta_n \in Ty_n$ and $\xi_n \in Tz_n$, $n \geq 0$. By hypothesis, let

$$M'' = \|w_0\| + \|w_1\| < \infty.$$

Put $M = \max\{M', M''\}$. Since $\{w_n\}$ be the Picard iterative sequence defined by (1.2), there exists $\nu_n \in Tw_n$ such that

$$w_{n+1} = \nu_n, \quad \forall\, n \geq 0. \tag{2.1}$$

By Lemma 2.2, we have

$$\begin{aligned}
\|\nu_{n-1} - \nu_n\| &\leq (1+\varepsilon)H(Tw_{n-1}, Tw_n)\\
&\leq (1+\varepsilon)k\|w_{n-1} - w_n\| = (1+\varepsilon)k\|\nu_{n-2} - \nu_{n-1}\|\\
&\leq (1+\varepsilon)k(1+\varepsilon)H(Tw_{n-2}, Tw_{n-1})\\
&\leq \big((1+\varepsilon)k\big)^2\|w_{n-2} - w_{n-1}\|\\
&\leq \cdots\\
&\leq \Big((1+\varepsilon)k\Big)^n\|w_0 - w_1\|\\
&\leq \Big((1+\varepsilon)k\Big)^n M,
\end{aligned} \tag{2.2}$$

for any given $\varepsilon > 0$. Since $\{x_n\}$ be the three-step iterative sequence with perturbed errors defined by (1.5), for each n, $n = 0, 1, 2, \cdots$, there exist $\eta_n \in Ty_n$, $\xi_n \in Tz_n$ and $\mu_n \in Tx_n$ such that

$$\begin{cases}
x_{n+1} = (1-\alpha_n)x_n + \alpha_n\eta_n + \alpha_n e_n,\\
y_n = (1-\beta_n)x_n + \beta_n\xi_n + \beta_n f_n,\\
z_n = (1-\gamma_n)x_n + \gamma_n\mu_n + \gamma_n g_n, \quad n \geq 0.
\end{cases} \tag{2.3}$$

It follows from (2.1), (2.2) and (2.3) that

$$\begin{aligned}\|x_{n+1}-w_{n+1}\| &= \|(1-\alpha_n)x_n+\alpha_n\eta_n+\alpha_ne_n-\nu_n\| \\ &\leq (1-\alpha_n)\|x_n-\nu_n\|+\alpha_n\|\eta_n-\nu_n\|+\alpha_n\|e_n\| \\ &\leq (1-\alpha_n)(\|x_n-w_n\|+\|w_n-\nu_n\|) \\ &\quad +\alpha_nk\|y_n-w_n\|+\alpha_n\|e_n\| \\ &\leq (1-\alpha_n)\Big(\|x_n-w_n\|+\Big((1+\varepsilon)k\Big)^nM\Big) \\ &\quad +\alpha_nk\|y_n-w_n\|+\alpha_n\|e_n\|,\end{aligned} \tag{2.4}$$

$$\begin{aligned}\|y_n-w_n\| &= \|(1-\beta_n)x_n+\beta_n\xi_n+\beta_nf_n-w_n\| \\ &\leq (1-\beta_n)\|x_n-w_n\|+\beta_n\|\xi_n-\nu_{n-1}\|+\beta_n\|f_n\| \\ &\leq (1-\beta_n)\|x_n-w_n\|+\beta_n(\|\xi_n-\nu_n\| \\ &\quad +\|\nu_n-\nu_{n-1}\|)+\beta_n\|f_n\| \\ &\leq (1-\beta_n)\|x_n-w_n\|+\beta_nk\|z_n-w_n\| \\ &\quad +\beta_n\Big((1+\varepsilon)k\Big)^nM+\beta_n\|f_n\|,\end{aligned} \tag{2.5}$$

$$\begin{aligned}\|z_n-w_n\| &= \|(1-\gamma_n)x_n+\gamma_n\mu_n+\gamma_ng_n-w_n\| \\ &\leq (1-\gamma_n)\|x_n-w_n\|+\gamma_n\|\mu_n-\nu_{n-1}\|+\gamma_n\|g_n\| \\ &\leq (1-\gamma_n)\|x_n-w_n\|+\gamma_nk\|x_n-w_{n-1}\|+\gamma_n\|g_n\| \\ &\leq (1-\gamma_n)\|x_n-w_n\| \\ &\quad +\gamma_nk(\|x_n-w_n\|+\|w_n-w_{n-1}\|)+\gamma_n\|g_n\| \\ &\leq (1-\gamma_n)\|x_n-w_n\| \\ &\quad +\gamma_nk\Big(\|x_n-w_n\|+\Big((1+\varepsilon)k\Big)^{n-1}M\Big)+\gamma_n\|g_n\|.\end{aligned} \tag{2.6}$$

Substituting (2.6) into (2.5), we have

$$\begin{aligned}\|y_n-w_n\| &\leq (1-\beta_n)\|x_n-w_n\|+\beta_nk\Big\{(1-\gamma_n)\|x_n-w_n\| \\ &\quad +\gamma_nk\Big(\|x_n-w_n\|+((1+\varepsilon)k)^{n-1}M\Big)+\gamma_n\|g_n\|\Big\} \\ &\quad +\beta_n((1+\varepsilon)k)^nM+\beta_n\|f_n\| \\ &\leq (1-\beta_n)\|x_n-w_n\|+\beta_nk\Big\{(1-\gamma_n(1-k))\|x_n-w_n\| \\ &\quad +\gamma_n((1+\varepsilon)k)^nM+\gamma_n\|g_n\|\Big\}+\beta_n((1+\varepsilon)k)^nM \\ &\quad +\beta_n\|f_n\|.\end{aligned} \tag{2.7}$$

Substituting (2.7) into (2.4), we can obtain

$$\begin{aligned}
&\|x_{n+1} - w_{n+1}\| \\
&\leq (1-\alpha_n)\Big(\|x_n - w_n\| + ((1+\varepsilon)k)^n M\Big) \\
&\quad + \alpha_n k\Big[(1-\beta_n)\|x_n - w_n\| + \beta_n k\Big\{(1-\gamma_n(1-k))\|x_n - w_n\| \\
&\quad + \gamma_n((1+\varepsilon)k)^n M + \gamma_n\|g_n\|\Big\} + \beta_n((1+\varepsilon)k)^n M \\
&\quad + \beta_n\|f_n\|\Big] + \alpha_n\|e_n\| \\
&\leq \Big[1-\alpha_n(1-k(1-\beta_n))\Big]\|x_n - w_n\| + (1-\alpha_n)((1+\varepsilon)k)^n M \\
&\quad + \alpha_n\beta_n k^2\Big\{(1-\gamma_n(1-k))\|x_n - w_n\| + \gamma_n((1+\varepsilon)k)^n M \\
&\quad + \gamma_n\|g_n\|\Big\} + \alpha_n\beta_n((1+\varepsilon)k)^{n+1} M + \alpha_n\beta_n k\|f_n\| + \alpha_n\|e_n\| \\
&\leq (1-\alpha_n(1-k))\|x_n - w_n\| + ((1+\varepsilon)k)^n M \\
&\quad + \alpha_n\beta_n k^2\|x_n - w_n\| + \alpha_n\beta_n\Big[\gamma_n k^2\Big\{((1+\varepsilon)k)^n M + \|g_n\|\Big\} \\
&\quad + ((1+\varepsilon)k)^{n+1} M + k\|f_n\|\Big] + \alpha_n\|e_n\| \\
&\leq (1-\alpha_n(1-k))\|x_n - w_n\| + \alpha_n\beta_n\|x_n - w_n\| \\
&\quad + ((1+\varepsilon)k)^n M + \alpha_n\beta_n\Big[\gamma_n k^2\Big\{((1+\varepsilon)k)^n M + \|g_n\|\Big\} \\
&\quad + ((1+\varepsilon)k)^{n+1} M + k\|f_n\|\Big] + \alpha_n\|e_n\| \\
&\leq (1-\alpha_n(1-k))\|x_n - w_n\| + \alpha_n\beta_n\|x_n - w_n\| + d_n,
\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
d_n =& ((1+\varepsilon)k)^n M + \alpha_n\beta_n\Big[\gamma_n k^2\Big\{((1+\varepsilon)k)^n M + \|g_n\|\Big\} \\
&+ ((1+\varepsilon)k)^{n+1} M + k\|f_n\|\Big] + \alpha_n\|e_n\|.
\end{aligned}$$

Take $a_n = \|x_n - w_n\|$ and $t_n = \alpha_n(1-k)$, $b_n = \alpha_n\beta_n$ in (2.8). Since $\{e_n\}$, $\{f_n\}$, $\{g_n\}$ are bounded and $(1+\varepsilon)k < 1$, $\sum\limits_{n=0}^{\infty} \alpha_n\beta_n < \infty$, we have

$$\sum_{n=0}^{\infty} t_n = \infty, \quad \sum_{n=0}^{\infty} b_n < \infty, \quad \sum_{n=0}^{\infty} d_n < \infty.$$

By Lemma 2.1, we know that

$$\|x_n - w_n\| \to 0 \quad (n \to \infty).$$

If $w_n \to x^* \in F(T)$ $(n \to \infty)$, we have

$$\|x_n - x^*\| \leq \|x_n - w_n\| + \|w_n - x^*\| \to 0 \quad (n \to \infty).$$

If $x_n \to x^* \in F(T)$ $(n \to \infty)$, we have

$$\|w_n - x^*\| \le \|w_n - x_n\| + \|x_n - x^*\| \to 0 \quad (n \to \infty).$$

The equivalence between the statement (1) and (2) was proved.
Finally, we prove that $x^* \in E$ is the unique fixed point of T. In fact, let $x^*, y^* \in E$ be two fixed points of T. Since T is a set-valued contraction with constant $0 < k < 1$, we have

$$\|x^* - y^*\| \le (1+\varepsilon)H(Tx^*, Ty^*) \le (1+\varepsilon)k\|x^* - y^*\|.$$

Since ε is arbitrary, this implies that $\|x^* - y^*\| = 0$, i.e., $x^* = y^*$. This completes the proof. □

Corollary 2.1. *Let E be a real Banach space, D be a nonempty closed convex subset of E, $T : D \to 2^D$ be a contraction mapping with $k < \frac{1}{1+\varepsilon}$ and a constant L satisfying $\sup_{\omega \in Tx} \|\omega\| \le L$, for all $x \in D$. Let $\{w_n\}$ and $\{r_n\}$ be the Picard and Ishikawa iterative sequence with perturbed errors defined by* (1.2) *and* (1.4) *respectively and satisfying the following conditions:*

(i) $\alpha_n, \beta_n \in [0,1], \quad \forall\, n \ge 0$;

(ii) $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \|e_n\| = 0$;

(iii) $\sum_{n=0}^{\infty} \alpha_n\beta_n < \infty, \quad \sum_{n=0}^{\infty} \alpha_n\|e_n\| < \infty, \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$

If $w_0 = r_0$, then the following statements are equivalent:

(1) *The Picard iterative sequence $\{w_n\}$ converges strongly to $x^* \in F(T)$;*

(2) *The Ishikawa iterative sequence with perturbed errors $\{r_n\}$ converges strongly to $x^* \in F(T)$.*

Furthermore, x^* is the unique fixed point of T.

If $\beta_n = 0$ and $e_n = 0$, in (1.4), then (1.4) reduces (1.3). So we have the following.

Corollary 2.2. *Let E be a real Banach space, D be a nonempty closed convex subset of E, $T : D \to 2^D$ be a contraction mapping with $k < \frac{1}{1+\varepsilon}$ and a constant L satisfying $\sup_{\omega \in Tx} \|\omega\| \le L$, for all $x \in D$. Let $\{w_n\}$ and $\{u_n\}$ be the Picard and Mann iterative sequence defined by* (1.2) *and* (1.3) *respectively and satisfying the following conditions:*

(i) $\alpha_n \in [0,1], \quad \forall\, n \ge 0$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

If $w_0 = u_0$, then the following statements are equivalent:

(1) *The Picard iterative sequence $\{w_n\}$ converges strongly to $x^* \in F(T)$;*

(2) *The Mann iterative sequence $\{u_n\}$ converges strongly to $x^* \in F(T)$.*

Furthermore, x^* is the unique fixed point of T.

References

[1] N. J. Huang and M. R. Bai, *A perturbed iterative procedure for multivalued pseudo-contractive mappings and multivalued accretive mappings in Banach spaces*, Comp. Math. Appl. **37** (1999), 7-15.

[2] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **44** (1974), no. 1, 147-150.

[3] Y. Q. Li and L. W. Liu, *On iterative process for Lipschitz strongly accretive operators*, Acta Math. Sinica **41** (1998), 845-850.

[4] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506-510.

[5] S. B. Nadler, *Multi-valued contraction mappings*, Pacific J. Math. **30** (1969), 475-488.

Kwang Pak Park
Department of Mathematics, Kyungnam University, Masan, Kyungnam, 631-701, Korea
E-mail address: kppark@kyungnam.ac.kr

Kyung Soo Kim
Department of Mathematics, Kyungnam University, Masan, Kyungnam, 631-701, Korea
E-mail address: kksmj@kyungnam.ac.kr