An Alternative Approach for Further Approximate Optimum Inspection Intervals

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Abstract. Having previously presented an article entitled "Further approximate optimum inspection intervals" in this Journal, here the author derives an alternative set of general explicit formulae using Cardan's solution to a cubic equation and presents a modified heuristic algorithm for solving Baker's model. The examples show that this new alternative approximate solution procedure for determining near optimum inspection intervals is as accurate and computationally efficient as the one suggested in the previous article. Through the examples, the author also indicates the relative merits and demerits of the two algorithms.

Keywords: Exponential Distribution, Inspection, Replacement, Cost, Profit, Machine.

1. INTRODUCTION

Consider a single unit representing a manufacturing system composed of many components. In the following, the author will use the word "machine" to refer to such a single-unit or complex system. Now suppose that a machine follows the exponential failure distribution $F(t) = 1 - e^{-\lambda t}$ for a constant hazard $\lambda > 0$ and time $t \geq 0$, and that failures can be revealed only by periodic inspection (or testing) and then replaced. Frequent inspection increases inspection costs while infrequent inspection leads to increasing lost production costs. Thus, an economically optimum inspection interval usually exists. Recent studies in which the basic profit model, proposed by Baker (1990), of periodically inspecting a machine has been extended, generalized or modified can be found in Leung (2005). Recent articles concerning inspection problems are Yang and Klutke (2001), Lam (1995, 2003), Cui *et al*. (2004) and Zequeria and Berenguer (2006).

For easy reference, the author restates the essential equations of Baker's model in the next section. In the rest of this article, he: (1) proposes an alternative near optimum solution procedure for Baker's model; (2) gives three typical examples to show that this new alternative procedure is as accurate and computationally efficient as the one put forward by Leung (2005); (3) gives the relative merits and demerits of the two algorithms; (4) concludes with a possible application of the procedure; and (5) shows the limited applicability of a second or third degree Taylor series approximation for the factor e^{-x} arising in the maximum condition.

2. THE EXPECTED AND MAXIMUM PROFIT RATES, AND THE MAXIMUM CONDITION

Let a be the profit per unit time while the machine is operating and b be the cost of replacement if the machine is found to have failed, where $a, b \geq 0$. We assume that all replacements are equally expensive, that a failure completely halts production until the next inspection and replacement, and that each replacement restores the machine to the as-good-as new state. Let c be the cost of each periodic inspection, where $c \geq 0$. Now, suppose that the machine is inspected with periodic time T between two successive inspections. The expected profit rate (or per unit time) is given by Baker (1990):

$$z(T) = \frac{P(T)}{T} = \frac{1}{T}\left[\left(\frac{a}{\lambda} - b\right)(1 - e^{-\lambda T}) - c\right]. \quad (1)$$

The maximum condition of equation (1) is

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$$(1 + x_a)\mathrm{e}^{-x_a} = 1 - d, \qquad (2.1)$$

or equivalently the condition can be written as the logarithmic form

$$\ln(1 + x_a) - x_a = \ln(1 - d), \qquad (2.2)$$

where $x_a = \lambda T_a$ and $d = \dfrac{c}{\frac{a}{\lambda} - b}$.

With $\frac{a}{\lambda} - b - c > 0$, Hariga (1996) showed the existence and uniqueness of the optimum inspection interval T_a. However, the author thinks that his deduction is not direct and complete for the situation under study. Thus, the author states Theorem 1 below and gives its proof in the Appendix. Through economic interpretations of expressions $\alpha = \frac{a}{\lambda} - b - c$ and $\phi = \frac{a}{\lambda} - b$, the new proof also provides more insight into the model.

> **Theorem 1.** If $\alpha > 0$ or equivalently $a > \lambda(b + c)$, then there exists a unique finite optimum interval T_a that maximizes the expected profit rate $z(T)$. Otherwise, $T_a = \infty$, i.e. no inspections take place.

The maximum profit rate is

$$z(T_a) = \frac{a - (b + c)\lambda}{1 + \lambda T_a}. \qquad (3)$$

The emphasis in this article is to find accurate approximate solutions $x_a = \lambda T_a$ of equation (2.2) and then determine the maximum profit rate using equation (3).

3. MORE ACCURATE APPROXIMATE OPTIMUM INSPECTION INTERVALS

A third degree Taylor series approximation for ln(1+x) is given by

$$\ln(1 + x) \cong x - \frac{x^2}{2} + \frac{x^3}{3}. \qquad (4)$$

Vaurio (1994) used the accurate approximation

$$\ln(1 + x) \cong \frac{x(1 + \frac{x}{6})}{1 + \frac{2}{3}x} = \frac{3x + \frac{1}{2}x^2}{3 + 2x}. \qquad (5.1)$$

Equation (5.1) can be written as

$$\ln(1 + x) \cong x - \frac{x^2}{2} + \frac{x^3}{3 + 2x}. \qquad (5.2)$$

Putting equation (5.2) into equation (2.2) yields a quadratic equation with the solution

$$x_r = \frac{-2}{3}\ln(1 - d) + \sqrt{\frac{4}{9}\ln^2(1 - d) - 2\ln(1 - d)}. \qquad (5.3)$$

Note that equation (5.3) corresponds to equation (8) in Vaurio (1994).

The author deduces from equations (4) and (5.2) that the general form of approximation for ln(1+x) is given by

$$\ln(1 + x) \cong x - \frac{x^2}{2} + \frac{x^3}{3 + lx}, \quad \text{for } 2 \le l \le 3. \qquad (6.1)$$

Equation (6.1) can be written as

$$\ln(1 + x) \cong \frac{3x + (l - \frac{3}{2})x^2 - (\frac{l}{2} - 1)x^3}{3 + lx}, \qquad (6.2)$$

In particular, putting equation (6.2) with $l = 2$ into equation (2.2), we can obtain equation (5.3).

In general, putting equation (6.2) with $l > 2$ into equation (2.2) yields a cubic equation

$$(l - 2)x_l^3 + 3x_l^2 + 2\ln(1 - d)lx_l + 6\ln(1 - d) = 0. \qquad (7)$$

The solution to equation (7) is given as follows:

Equating the coefficients of terms with the same order in equations (7) and (A3), we obtain

$$a_0 = l - 2, \quad a_1 = 1, \quad a_2 = \tfrac{2}{3}\ln(1 - d)l \text{ and } a_3 = 6\ln(1 - d).$$

From equation (A5), we have

$$H = \frac{(l - 2)[\frac{2}{3}\ln(1 - d)l] - 1}{(l - 2)^2} \text{ and } G = \frac{6\ln(1 - d)(l - 2)^2 - 2\ln(1 - d)(l - 2)l + 2}{(l - 2)^3}. \qquad (8)$$

Since $G^2 + 4H^3 < 0$ (the proof is given in the Appendix), from equation (A7) we have

$$\theta = \frac{1}{3}\cos^{-1}\left[\frac{-G}{2\sqrt{(-H)^3}}\right], \qquad (9)$$

and from equations (A4) and (A6) we have

$$x_l = 2\sqrt{-H}\cos\theta - \frac{1}{l - 2}. \qquad (10)$$

In the Appendix, a brief discussion is given of the general solution of the cubic equation. For more details, see pp.131-133 in Tranter (1976).

Since x_l is a near optimum value, from equation (2.1) we have

$$(1+x_l)e^{-x_l} \cong 1-d.$$

The following two theorems provide conditions by which an heuristic algorithm, introduced below, is devised.

Theorem 2. $g(x_l)$ is a strictly decreasing function with respect to $x_l > 0$, where

$$g(x_l) = (1+x_l)e^{-x_l}. \tag{11}$$

The proof of Theorem 2 is given in the Appendix.

Figure 2 shows the curve of $g(x_l)$ versus x_l. A quick but quite inaccurate x_a (especially for $1-d$ close to 0 and 1) can be obtained from the enlarged Figure 2 which is attached at the end of the Appendix.
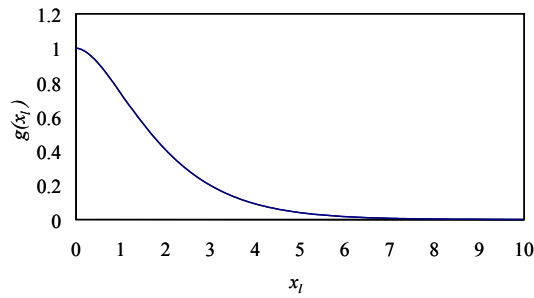

Figure 2.

Theorem 3. x_l is a strictly decreasing function with respect to l in the interval [2, 3].

The proof of Theorem 3 is given in the Appendix.

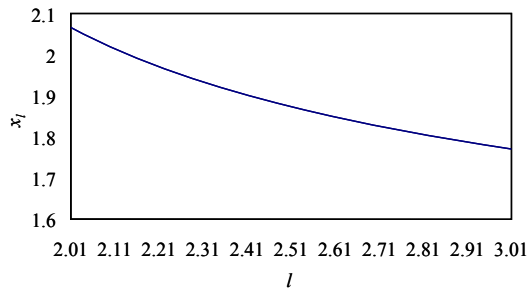Figure 3 shows the curve of x_l versus l when $d = 0.5940$, see Example 1 below.

Figure 3.

4. AN HEURISTIC ALGORITHM

The procedure the author proposes to solve equation (2.2) works as follows:

(1)   To start, we compute x_r using equation (5.3), i.e. let $l = 2$.

(2)   Compute x_l and $g(x_l)$, correct to 4 significant figures, using equations (8) to (10) and (11) respectively.

(3)   When $|g(x_l)-1+d|<10^{-3}$, stop.

(4)   When $g(x_l) < $ (or $>$) $1-d$, set l greater (or smaller) than the value assigned in step (2). This revision is due to Theorems 2 and 3. In practice, it is sufficient for the new l value to be heuristically chosen from the interval (2, 2.15). Then go to step (2).

Equations (5.3) and (8) to (11) are simple enough to be solved using a portable programmable calculator, with which the near optimum inspection interval x_l and the absolute error $|g(x_l)-1+d|$ can be computed "with the push of a button" and no tables (such as Table I in Baker 1990 and the enlarged Figure 2) have to be consulted. The algorithm is illustrated by the following three typical examples.

Although x_r always overshoots x_a when $d > 0.2642$ (see Table I in Vaurio 1994), step (1) initiates a good approximation. Consequently, step (2) usually iterates twice to give the optimum value to any desired degree of accuracy.

Example 1

Given that $d = 0.5940$; hence $1-d = 0.4060$ and $\ln(1-d) = -0.9014$.

From Table I in Vaurio (1994), we obtain

$$x_a = 2 \text{ and } x_2 = x_r = 2.072.$$

Using equation (11) with $l = 2$ yields

$$g(x_2) = (1+2.072)e^{-2.072} = 0.3869 < 0.4060.$$

Next, set $l = 2.1$. Using equation (8), we obtain

$$H = \frac{-0.1 \times \frac{2}{3} \times 0.9014 \times 2.1 - 1}{(0.1)^2} = -112.62,$$

and

$$G = \frac{-6 \times 0.9014 \times (0.1)^2 + 2 \times 0.9014 \times 0.1 \times 2.1 + 2}{(0.1)^3}$$

$$= 2324.50.$$

Using equation (9), we obtain

$$\theta = \frac{1}{3}\cos^{-1}\left[\frac{-2324.50}{2\sqrt{(112.62)^3}}\right] = 55.51^{\circ}.$$

Using equations (10) and (11) respectively, we have

$$x_{2.1} = 2\sqrt{112.62}\cos 55.51^{\circ} - \frac{1}{0.1} = 2.02,$$

and

$$g(x_{2.1}) = (1 + 2.02)e^{-2.02} = 0.4002 < 0.4060.$$

We perform a fine adjustment for l, try $l = 2.15$ and obtain

$$H = -53.06, \quad G = 728.80, \quad \theta = 53.51^{\circ},$$
$$x_{2.15} = 1.997, \quad g(x_{2.15}) = 0.4068 < 0.4060.$$

Finally, we obtain $x_a \cong 1.997$.

Example 2

Let $\lambda = 0.01$ per day, $a = \$1000$ per day, $b = \$5000$, $c = \$90,000$. Hence, $d = 0.9474$, $1 - d = 0.0526$ and $\ln(1 - d) = -2.945$. This is the third (extreme) example solved in Baker (1990), from which we know that $x_a = 4.682$.

Since $\alpha = \frac{1000}{0.01} - 5000 - 90,000 = 5000 > 0$, a unique finite optimum T_a exists, by Theorem 1.

First, setting $l = 2$, we have

$$x_2 = \frac{2 \times 2.945}{3} + \sqrt{\frac{4(-2.945)^2}{9} + 2 \times 2.945} = 5.085,$$

and

$$g(x_2) = (1 + 5.085)e^{-5.085} = 0.0377 < 0.0526.$$

Next, setting $l = 2.1$, we obtain

$$H = -141.23, \quad G = 3060.20, \quad \theta = 51.91^{\circ},$$
$$x_{2.1} = 4.662, \quad g(x_{2.1}) = 0.0535 > 0.0526.$$

We perform a fine adjustment for l, try $l = 2.09$ or 2.095 and respectively obtain

$$H = -169.05, \quad G = 4066.92, \quad \theta = 52.56^{\circ},$$
$$x_{2.09} = 4.697, \quad g(x_{2.09}) = 0.0520 < 0.0526,$$

or

$$H = -154.10, \quad G = 3513.96, \quad \theta = 52.23^{\circ},$$
$$x_{2.095} = 4.680, \quad g(x_{2.095}) = 0.0527 > 0.0526.$$

Finally, we obtain $x_a \cong 4.680$, $T_a \cong 468$ days and

$$z(T_a) \cong \frac{1000 - (5000 + 90,000) \times 0.01}{1 + 4.680} = \$8.803 \quad \text{from}$$

equation (3).

Example 3

Given that $d = 0.8009$; hence $1 - d = 0.1991$ and $\ln(1 - d) = -1.614$.
From Table I in Vaurio (1994), we obtain

$$x_a = 3 \text{ and } x_2 = x_r = 3.170,$$

implying $g(x_2) = 0.175 < 0.1991$.
Next, we set $l = 2.1$ and obtain

$$H = -122.60, \quad G = 2581.04, \quad \theta = 53.98^{\circ},$$
$$x_{2.1} = 3.023, \quad g(x_{2.1}) = 0.1957 < 0.1991.$$

Third, we set $l = 2.11$ and obtain

$$H = -103.63, \quad G = 1992.77, \quad \theta = 53.61^{\circ},$$
$$x_{2.11} = 2.988, \quad g(x_{2.11}) = 0.2009 > 0.1991.$$

Finally, we obtain $x_a \cong 2.988$.

5. CONCLUSIONS

The three typical examples show that the formulae for x_l, i.e. equations (8) to (10), are the most accurate approximation of x_a, regardless of the different values of x_a and hence of the range of d. They also show that the general logarithmic form of approximation for determining near optimum inspection intervals is as accurate and computationally efficient as the general exponential form of approximation suggested in Leung (2005). Moreover, the author deems that the proposed algorithm is more efficient and less tedious than the one proposed by Hariga (1996) on p.356.

Of these two methods of approximation, that of Leung (2005) is probably the better for ordinary users since they do not need to understand *Cardan's solution* to use it. On the other hand, if the user are not mathematically shy, they are better off adopting the algorithm proposed here because it usually requires less iterations.

The proposed algorithm can also be applied to solve equation (7) or (12) in Ben-Daya and Hariga (1998). This algorithm incorporates fixed inspection and repair times and relaxes the strict assumption of no production during the failed (regarded as an out-of-control) state. It should be more efficient and less tedious to use than the algorithm suggested on pp.484-485 of Ben-Daya and Hariga (1998) for solving equation (7) and be more accurate than

equations (13) to (15) in reaching an approximate solution of equation (12). The author have present results obtained from adopting the general logarithmic and exponential forms of approximation for solving Ben-Daya and Hariga's model in Leung (2008).

Finally, it is worth pointing out the following. (a) The application of $x_a^3 - x_a^2 + 2d \cong 0$, equation (6) in Chung (1993), for approximating x_a is very limited because its roots have a physical meaning only if $d \le 0.0740$. The explanation for this is given in the Appendix. (b) When the author attempts a third degree Taylor series approximation for e^{-x}, equation (2.1) becomes a quartic (or biquadratic) equation $x_a^4 - 2x_a^3 + 3x_a^2 - 6d \cong 0$. Using *Ferrari's solution* (see pp.133-135 in Tranter (1976)) to a quartic equation and letting $d = 0.5940$, after some troublesome arithmetical manipulations, he obtains only one positive real root $x_a \cong 1.305$. This is a poor approximate since $x_a = 2$ for the maximum condition $(1 + x_a)e^{-x_a} = 1 - 0.5940$. With this counter example, we may conclude that a third degree Taylor series approximation is not only very inaccurate but also very tedious for determining near values of x_a. The author has not attempted a fourth or higher degree Taylor series approximation because it is known that the general solution of an algebraic equation of a degree higher than the fourth is not possible.

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APPENDIX

1. *Proof of Theorem 1*

Differentiating equation (1) with respect to T and manipulating the resulting expression, we obtain

$$\frac{dz(T)}{dT} = \frac{v(T)}{T^2}, \qquad (A1)$$

where

$$v(T) = -\alpha + \phi e^{-\lambda T} + \phi \lambda T e^{-\lambda T}, \qquad (A2)$$

and $\alpha = \frac{a}{\lambda} - b - c$, $\phi = \frac{a}{\lambda} - b$.

From equation (A2), it is evident that $v(0) = \phi - \alpha = c$ and $v(\infty) = -\alpha$ (applying L'Hospita's rule).

Note that the behaviour of the function $z(T)$ depends on the sign of its derivative which in turn depends on the sign of $v(T)$. Differentiating equation (A2) with respect to T, we have

$$\frac{dv(T)}{dT} = -\phi \lambda^2 T e^{-\lambda T}.$$

To investigate the sign of $\frac{dv(T)}{dT}$, we need to consider the following three cases.

Case (a): If $\alpha > 0$ (i.e. the expected profit covers both the repair cost and inspection cost), then $\phi > \alpha > 0$,

and hence $\frac{\mathrm{d}v(T)}{\mathrm{d}T} < 0$. This implies that $v(T)$ is strictly decreasing in the interval $(0, \infty)$ from $\phi - \alpha = c > 0$ to $-\alpha < 0$. Hence, there exists a unique finite optimum inspection interval, see Figure 1 below. As expected, the expected profit can cover both repair and inspection costs.

Note that the condition for case (a) can be written as $a > \lambda(b + c)$.

Note also that $\alpha = 0$ is not included because if the costs are just covered, there is no point in wasting efforts on inspection and repair.

Case (b): If $\phi > 0$ and $\alpha \le 0$ (i.e. the expected profit covers repair cost only), then $\frac{\mathrm{d}v(T)}{\mathrm{d}T} < 0$. This implies that $v(T)$ is strictly decreasing in the interval $[0, \infty)$ from $\phi - \alpha = c > 0$ to $-\alpha \ge 0$. Hence, no solutions exist, i.e. there are no inspections at all, or the optimum inspection interval $T_a = \infty$, see Figure 1 below. This is also as expected since the machine's state cannot be revealed without inspection, even though the expected profit covers the repair cost.

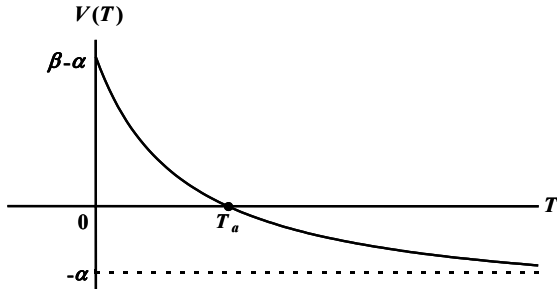Note that the condition for case (b) can be written as $\lambda b < a \le \lambda(b + c)$.
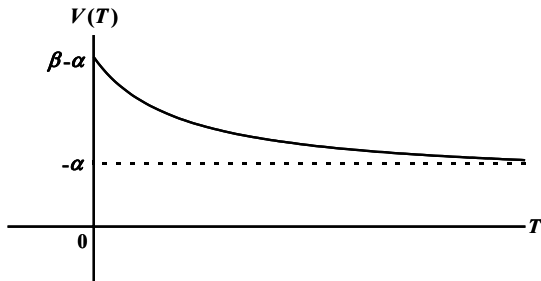

Figure 1. Case(a)

Figure 1. Case(b)

Case (c): If $\phi \le 0$ (i.e. the expected profit covers neither cost), no solutions exist, i.e. $T_a = \infty$. This result is intuitive because the machine is not profitable to use. Also, the result can be verified mathematically as follows: Consider

$$[z(T) - z(\infty)]T = \phi(1 - e^{-\lambda T}) - c \quad \text{as} \quad z(\infty) = 0$$
$$< 0 \quad \text{as} \quad e^{-\lambda T} < 1.$$

This implies $z(T) < z(\infty)$, meaning $T = \infty$. Hence, $T_a = \infty$ for $\phi \le 0$.

Note that the condition for case (c) can be written as $0 < a \le \lambda b$.

2. *The general solution of the cubic equation*

The *general* form of the cubic equation is

$$a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3 = 0. \tag{A3}$$

By writing

$$x = y - \frac{a_1}{a_0}, \tag{A4}$$

the general cubic can be transformed into the *standard* form of the cubic equation

$$y^3 + 3Hy + G = 0,$$

where

$$H = \frac{a_0 a_2 - a_1^2}{a_0^2} \quad \text{and} \quad G = \frac{a_0^2 a_3 - 3a_0 a_1 a_2 + 2a_1^3}{a_0^3}. \tag{A5}$$

Putting $y = u + v$ and $H = -uv$ into the standard form, we obtain

$$u^3 + v^3 = -G \quad \text{and} \quad u^3 v^3 = -H^3.$$

Hence, u^3 and v^3 can be regarded as the roots of the quadratic equation

$$z^2 + Gz - H^3 = 0,$$

so that

$$u^3 = \frac{-G + \sqrt{G^2 + 4H^3}}{2} \quad \text{and} \quad u^3 = \frac{-G - \sqrt{G^2 + 4H^3}}{2},$$

and a root of the standard cubic is given, through the relation $y = u + v$, by

$$y = \left(\frac{-G + \sqrt{G^2 + 4H^3}}{2} \right)^{\frac{1}{3}} + \left(\frac{-G - \sqrt{G^2 + 4H^3}}{2} \right)^{\frac{1}{3}}.$$

This solution was first published by Cardan and is usually known as *Cardan's solution* although it was first invented by Tartaglia some four hundred years ago.

Consider *Cardan's solution* in more detail:
(a) If $G^2 + 4H^3 > 0$, the cubic has one real and two complex roots, namely

$$\bar{u} + \bar{v}, \quad \omega\bar{u} + \omega^2\bar{v} \quad \text{and} \quad \omega^2\bar{u} + \omega\bar{v},$$

where $\bar{u}$ and $\bar{v}$ are the principal (or arithmetical) cube roots of u^3 and v^3 respectively, and $\omega = \frac{1}{2}(-1 + i\sqrt{3})$ is a complex cube root of unity.

(b) If $G^2 + 4H^3 = 0$, the cubic has three real roots, two of which are equal, namely

$$2\bar{u}, \quad -\bar{u} \quad \text{and} \quad -\bar{u}.$$

(c) If $G^2 + 4H^3 < 0$, the cubic has three real roots.

To deal with this case, we put

$$y = 2\sqrt{-H}\cos\theta, \tag{A6}$$

into the standard cubic and yield

$$2\sqrt{(-H)^3}\cos 3\theta = -G. \tag{A7}$$

In a numerical case, three values of θ can then be found to satisfy this equation. Nevertheless, it is sufficient to take the principal (or smallest) value of θ in determining the optimum x_a.

3. *Proof of $G^2 + 4H^3 < 0$ with expressions H and G given by equation (8)*

Denote $1 - d$ by $\bar{d}$ and remember that $\ln\bar{d} < 0$ and $2 \leq l \leq 3$. Then

$$G^2 < -4H^3$$

$$\Leftrightarrow \{6\ln\bar{d}(l-2)^2 - 2[\ln\bar{d}(l-2)l - 1]\}^2$$

$$< \frac{-4}{27}[2\ln\bar{d}(l-2)l - 3]^3$$

$$\Leftrightarrow 36\ln^2\bar{d}(l-2)^4 - 24\ln^2\bar{d}(l-2)^3l + 24\ln\bar{d}(l-2)^2$$

$$+ 4\ln^2\bar{d}(l-2)^2l^2 - 8\ln\bar{d}(l-2)l + 4$$

$$< \frac{-32}{27}\ln^3\bar{d}(l-2)^3l^3 + \frac{16}{3}\ln^2\bar{d}(l-2)^2l^2$$

$$- 8\ln\bar{d}(l-2)l + 4$$

$$\Leftrightarrow 0 < \frac{-32}{27}\ln^3\bar{d}(l-2)^3l^3 + \frac{4}{3}\ln^2\bar{d}(l-2)^2l^2$$

$$- 12\ln^2\bar{d}(l-2)^3(l-6) - 24\ln\bar{d}(l-2)^2.$$

4. *Proof of Theorem 2*

Differentiating equation (11) with respect to x_l, we have

$$\frac{\mathrm{d}g(x_l)}{\mathrm{d}x_l} = -x_l e^{-x_l},$$

which is negative for $x_l > 0$. Hence, $g(x_l)$ is a strictly decreasing function with respect to x_l.

5. *Proof of Theorem 3*

Implicitly differentiating equation (7) with respect to l and solving for $\frac{\mathrm{d}x_l}{\mathrm{d}l}$ yields

$$\frac{\mathrm{d}x_l}{\mathrm{d}l} = \frac{-x_l[x_l^2 + 2\ln(1-d)]}{3(l-2)x_l^2 + 6x_l + 2\ln(1-d)l}.$$

Since $\ln(1-d) < 0$, from equation (7) we have

$$(l-2)x_l^3 + 3x_l^2 + 2\ln(1-d)lx_l > 0$$

$$\Rightarrow (l-2)x_l^2 + 3x_l + 2\ln(1-d)l > 0$$

$$\Rightarrow 3(l-2)x_l^2 + 6x_l + 2\ln(1-d)l > 0.$$

Putting equation (6.1) into equation (2.2), we obtain

$$\frac{x_l^2}{2} - \frac{x_l^3}{3 + lx_l} = -\ln(1-d)$$

$$\Rightarrow \frac{x_l^2}{2} > -\ln(1-d), \quad \text{i.e.} \quad x_l^2 + 2\ln(1-d) > 0.$$

Consequently, $\frac{\mathrm{d}x_l}{\mathrm{d}l} < 0$ for $x_l > 0$, and x_l is a strictly decreasing function with respect to l.

6. *Limitation of application of equation (6) in Chung (1993)*

A second degree Taylor series approximation for e^x is given by

$$e^x \cong 1 + x + \frac{x^2}{2}. \tag{A8}$$

Putting $e^{-x} \cong 1 - x + \frac{x^2}{2}$ in equation (2.1) yields a cubic equation

$$x_a^3 - x_a^2 + 2d \cong 0, \qquad\qquad (A9)$$

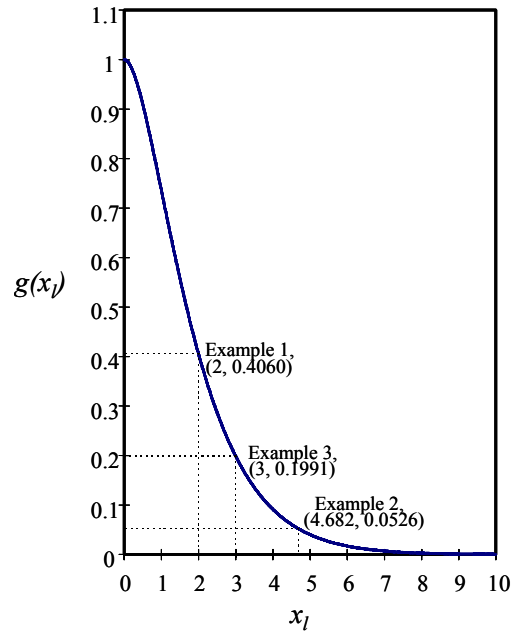which is equation (6) in Chung (1993).

Here, $a_0 = 1$, $a_1 = \frac{-1}{3}$, $a_2 = 0$, $a_3 = 2d$ and hence, $H = \frac{-1}{9}$, $G = 2d - \frac{2}{27}$.

Cases (b) and (c):

$G^2 + 4H^3 \le 0 \Leftrightarrow (2d - \frac{2}{27})^2 < 4(\frac{1}{9})^3 \Leftrightarrow d \le 0.0740$ gives roots with a physical meaning.

Case (a): If $d > 0.0740$, x has one and only one negative real root, which nobody can give a physical meaning.

For example, let $d = 0.5940$. Then $G^2 + 4H^3 = 1.236$, which is positive; i.e. case (a). Thus, there is only one real root $y = -1.136$. Hence, $x_a \cong -0.8027$.

Enlarged Figure 2.