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A NOTE ON FUNCTIONAL LIMIT THEOREM FOR THE INCREMENTS OF FBM IN SUP-NORM

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Abstract. In this paper, using large deviation results for Gaussian processes, we establish some functional limit theorems for increments of a fractional Brownian motion in the usual sup-norm via estimating large deviation probabilities for increments of a fractional Brownian motion.

1. Introduction and Results

The analogue of Strassen law of functional limit theorems is known for many Gaussian processes which have suitable scaling properties. Mueller [10] and Chen and Csörgő [4] studied functional modulus of continuity of a Wiener process. Révész [12] obtained functional large increment theorem which generalized Strassen's functional LIL of a Wiener process. Wang [13] obtained functional limit results for increments of a fractional Brownian motion (FBM). Goodman and Kuelbs [5], Kuelbs, Li and Talagrand [8] and Monrad and Rootzén [9] investigated functional LIL for general Gaussian process. Let $\{X(t), t \geq 0\}$ be FBM of order 2α with $0 < \alpha < 1$ and $X(0) = 0$, then $\{X(t), t \geq 0\}$ has a covariance function

$$\Gamma(s,t) = E(X(s)X(t)) = \frac{1}{2}\left(s^{2\alpha} + t^{2\alpha} - |s-t|^{2\alpha}\right)$$

for $s, t \geq 0$, and we have the representation

$$X(t) = \int_{\mathbb{R}^1} \frac{1}{q_\alpha}\left\{|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2}\right\} dW(x),$$

where

(a) $q_\alpha^2 = \displaystyle\int_{\mathbb{R}^1} \left\{|x-1|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2}\right\}^2 dx,$

(b) $\{W(t), -\infty < t < \infty\}$ is a Wiener process,

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(c) $\dfrac{1}{q_\alpha}\big\{|x-t|^{(2\alpha-1)/2}-|x|^{(2\alpha-1)/2}\big\}$ is interpreted to be $I_{(0,t]}$ when $\alpha = 1/2$.

$\{X(t), t \geq 0\}$ has stationary increments with $E\big(X(s+t)-X(s)\big)^2 = t^{2\alpha}$, $t \geq 0$ and is a Wiener process when $\alpha = 1/2$.

Let $C_0[0,1]$ be the space of continuous functions on $[0,1]$ with value zero at the origin and $\|\cdot\|_\infty$ be the usual sup-norm on $C_0[0,1]$. The limit set associated with functional LIL for $\{X(t), t \geq 0\}$ is K_α, the subset of functions in $C_0[0,1]$ with the form

$$f(t) = \int_{\mathbb{R}^1} \frac{1}{q_\alpha}\big\{|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2}\big\} g(x)dx, \quad 0 \leq t \leq 1.$$

Here the function $g(\cdot)$ range over the unit ball of $L^2(\mathbb{R}^1)$, and hence $\int_{\mathbb{R}^1} g^2(x)dx \leq 1$ (cf. [7]). Let $H_\alpha \subseteq C_0$ be the reproducing kernel Hilbert space (RKHS) of the kernel $\Gamma(s,t)$, $0 \leq s,t \leq 1$. H_α is the RKHS corresponding to the centered Gaussian measure μ on separable Banach space $C_0[0,1]$ under the sup-norm induced by $\{X(t), 0 \leq t \leq 1\}$, then the set K_α is the unit ball of H_α. If $f \in H_\alpha$, then $\|f\|_\alpha$ denotes the H_α-norm of f, and $|f(t)-f(s)| \leq |t-s|^\alpha \|f\|_\alpha$ for all $s,t \in [0,1]$. Throughout this paper put $K = K_\alpha$ and $\log x = \log_e(x \vee 1)$.

For any $\varepsilon > 0$, let P_ε be probability measure on $C_0[0,1]$ corresponding to $W =: \{\sqrt{\varepsilon}W(t), 0 \leq t < \infty\}$. Define the mapping $I : C_0[0,1] \to [0,\infty)$ by

$$I(f) = \begin{cases} \frac{1}{2}\int_0^1 \big(f'(x)\big)^2 dx, & \text{if } f \text{ is an absolutely continuous function} \\ & \text{ with a square integrable } f'(x), \\ \infty, & \text{otherwise,} \end{cases}$$

and the subset $\mathcal{S}$ of $C_0[0,1]$ by $\mathcal{S} = \big\{f \in C_0[0,1] : I(f) \leq 1/2\big\}$, then it is easy to show that $I(f)$ is lower semi-continuous and $\{f \in C_0[0,1] : I(f) \leq a\}$ is compact for any fixed $a > 0$.

Let a_T $(0 < a_T \leq T)$ be a continuous function such that

(i) a_T is nondecreasing,

(ii) T/a_T is nondecreasing,

(iii) $\displaystyle\lim_{T\to\infty} \frac{T/a_T}{\log\log T} = \infty$.

For $t \in [0, T-a_T]$, define

$$Y_{t,T}(x) = \frac{W(t+xa_T)-W(t)}{\sqrt{a_T}\,\gamma_T}, \qquad 0 \leq x \leq 1,$$
$$Z_{t,T}(x) = \frac{X(t+xa_T)-X(t)}{a_T^\alpha\,\gamma_T}, \qquad 0 \leq x \leq 1,$$

where

$$\gamma_T = \Big(2\log\frac{T}{a_T\log\log T}\Big)^{1/2}.$$

The following theorem is in [3].

Theorem A. *Let* $\{W(t), 0 \le t < \infty\}$ *be a standard Wiener process. Assume that* $0 < a_T \le T$ *satisfies conditions* (i), (ii) *and* (iii). *Then*

$$\liminf_{T\to\infty} \sup_{0\le t\le T-a_T} \inf_{f\in\mathcal{S}} \|Y_{t,T}(\cdot) - f(\cdot)\|_\infty = 0 \qquad \text{a.s.}$$

and for any $f \in \mathcal{S}$

$$\lim_{T\to\infty} \inf_{0\le t\le T-a_T} \|Y_{t,T}(\cdot) - f(\cdot)\|_\infty = 0 \qquad \text{a.s.}$$

Using large deviation estimates for Wiener processes in Hölder norm (see [1]), Wei [14] generalized to k-dimensional Wiener processes. Wang [13] investigated functional limit results for small and large increments of FBM in the sup-norm, but he only obtained the results related to Csörgő and Révész's type limit theorems. In this paper, we will consider the functional limit theorem with the normalizing factor of Csáki and Révész's type. Using large deviation results for Gaussian processes, we establish some functional limit theorems for increments of FBM in the sup-norm. Our main result is as follows:

Theorem 1.1. *Let* $\{X(t), t \ge 0\}$ *be FBM of order* 2α *with* $0 < \alpha < 1$ *and* $X(0) = 0$. *Assume that* $0 < a_T \le T$ *satisfies conditions* (i), (ii) *and* (iii). *Then we have*

$$\liminf_{T\to\infty} \sup_{0\le t\le T-a_T} \inf_{f\in K} \|Z_{t,T}(\cdot) - f(\cdot)\|_\infty = 0 \qquad \text{a.s.} \tag{1.1}$$

and for any $f \in K$

$$\lim_{T\to\infty} \inf_{0\le t\le T-a_T} \|Z_{t,T}(\cdot) - f(\cdot)\|_\infty = 0 \qquad \text{a.s.} \tag{1.2}$$

2. Some lemmas

Let B denote a real separable Banach space with norm $\|\cdot\|$ and topological dual B^*, and let μ be a centered Gaussian measure on B. Let H_μ be the Hilbert space which generates the Gaussian measure μ on B under the norm $\|\cdot\|$. Define continuous operators

$$\Pi_d(x) = \sum_{k=1}^{d} \alpha_k(x)\Delta_{\alpha_k} \qquad \text{and} \qquad Q_d(x) = x - \Pi_d(x) \qquad (d \ge 1) \tag{2.1}$$

taking B into B. Here $\{\alpha_k; k \ge 1\}$ is a sequence in B^* orthonormal in $L^2(\mu)$, and $\{\Delta_{\alpha_k}; k \ge 1\}$ is a complete orthonormal system in H_μ defined by the Bochner integral $\Delta_{\alpha_k} = \int_B x\alpha_k(x)\mu(dx)$. If ξ is a B-valued random vector with a mean zero Gaussian measure μ, then it is well known [6] that $\lim_{d\to\infty} \|Q_d(x)\| = 0$ with probability one, and

$$E\|Q_d(\xi)\| \downarrow 0 \quad \text{as} \quad d \uparrow \infty.$$

The following Lemma 2.1 is from [13] (cf. [5]).

Lemma 2.1. *Let $\{\xi(t), t \geq 0\}$ be a centered Gaussian process with values in B. Let P be the probability measure generated by $\xi(\cdot)$. Let $Q_d (d \geq 1)$ be the linear operators of* (2.1), *$U_a = \{f \in H_\mu : \|f\|_\mu \leq a\}$, where $a > 0$, and H_μ the RKHS with $\mu = \mathcal{L}(\xi)$. Let $\lambda \geq 1$ and d_λ be an integer such that*

$$d_\lambda \geq \inf\{m \geq 1 : E\|Q_m(\xi)\|/m \leq 2\tau \log \lambda/\lambda\},$$

and $\varepsilon_\lambda = \gamma d_\lambda \log \lambda/\lambda^2$ with some constant $\gamma > 3\tau$, where $\tau = \sup_{x \in U_a} \|x\|$. Then for every $\varepsilon \geq \varepsilon_\lambda$ (where ε may depend on λ) we have

$$P\Big(\inf_{f \in U_a} \Big\|\frac{\xi}{\lambda} - f\Big\| \geq \varepsilon\Big) \leq \frac{1}{\sqrt{2\pi(d_\lambda + 1)}} \exp\Big(-\frac{(a\lambda(1+\varepsilon))^2}{2} + \frac{d_\lambda - 1}{2} \log \frac{(a\lambda(1+\varepsilon))^2 e}{d_\lambda - 1}\Big)$$

for any $\lambda \geq \lambda_0$ with some $\lambda_0 > 0$.

In order to prove Lemma 2.3, we need the following lemmas. The detailed proofs of the following Lemma 2.2 are similar in [11].

Lemma 2.2. *For any $\varepsilon > 0$ there exists a positive constant $C = C(\varepsilon)$ such that*

$$P\Big\{\sup_{0 \leq s < t \leq T,\ |t-s| \leq h} \frac{|X(t) - X(s)|}{(t-s)^\alpha} \geq x\Big\} \leq C\frac{T}{h} \exp\big(-x^2/(2+\varepsilon)\big)$$

for any $x \geq x_0$ with some $x_0 > 0$.

The following Lemma 2.3 is a generalization of Lemma 2.1 in the case of $B = C_0[0,1]$ with the usual sup-norm $\|\cdot\|_\infty$ and ξ is FBM of order 2α with $0 < \alpha < 1$ and $X(0) = 0$. For the proofs we refer the reader to [13].

Lemma 2.3. *Let $\{X(t), t \geq 0\}$ be FBM of order 2α with $0 < \alpha < 1$ and $X(0) = 0$, and K be defined as in Section 1. Then for any $\varepsilon > 0$ there exists a positive number $\lambda_0 = \lambda_0(\varepsilon)$ such that*

$$P\Big(\sup_{0 \leq t \leq T-h} \inf_{f \in K} \Big\|\frac{X(t+h\cdot) - X(t)}{\lambda h^\alpha} - f(\cdot)\Big\|_\infty \geq \varepsilon\Big) \leq C\frac{T}{h} \exp\Big(-\frac{(\lambda(1+\varepsilon/3))^2}{2+\varepsilon}\Big)$$

for any $\lambda > \lambda_0$ and every $0 < h \leq T$.

Remark 2.4. From Lemma 2.3 we have

$$P\Big\{\sup_{0 \leq t \leq 1} \inf_{f \in K} \Big\|\frac{X(t+h\cdot) - X(t)}{\lambda h^\alpha} - f(\cdot)\Big\|_\infty \geq \varepsilon\Big\} \leq C \exp\Big(-\frac{(\lambda(1+\varepsilon/3))^2}{2+\varepsilon}\Big). \tag{2.2}$$

The following inequalities are well known (see [2], [9]).

Lemma 2.5. *Let V be a convex, symmetric, measurable subset of B. Then for all $f \in H_\mu$*

$$\mu(V) \geq \mu(f+V) \geq \mu(V) \exp\big\{ -\frac{1}{2}\|f\|_\mu^2 \big\}.$$

In order to prove our result, we need the following lemma. The detailed proof can be found in [7].

Lemma 2.6. *Let $0 < \alpha < 1$ and fix $0 < \eta < \alpha$. Let $s_k = k^{-k}$ and $d_k = k^{k+1-r}$ for $k \geq 1$. Set for $0 \leq t \leq 1$*

$$H^{(k)}(s_k,t) = \int_{|x| \notin (s_k d_{k-1}, s_k d_k]} \frac{1}{q_\alpha} \big\{ |x - s_k t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2} \big\} dW(x), \tag{2.3}$$

where $\{W(x), -\infty < x < \infty\}$ is a standard Wiener process. Let $0 < \beta < r$. Then for $\delta = \min\{2\beta(\alpha-\eta), r-\beta, (1-r)(2-2\alpha), (2\alpha-2\eta)r\}$ there exists a positive constant C depending on α such that uniformly in t, h, k

$$\sigma_k^2(t,h) = E\big(H^{(k)}(s_k, t+h) - H^{(k)}(s_k,t)\big)^2 \leq C h^{2\eta} s_k^{2\alpha} k^{-\delta}.$$

3. Proof of Theorem 1.1.

Proof of (1.1). If $\limsup_{T\to\infty} \dfrac{\log\big(T/a_T\big)}{\log\log\log T} = \infty$, then for any $M > 0$ there exists a sequence of positive numbers $\{T_n\}$ such that

$$\frac{T_n}{a_{T_n}} \geq (\log\log T_n)^M \tag{3.1}$$

for large n. For any $\varepsilon > 0$ we have by Lemma 2.3 and (3.1)

$$\begin{aligned} &P\Big\{ \sup_{0 \leq t \leq T_n - a_{T_n}} \inf_{f \in K} \big\| Z_{t,T_n}(x) - f(x) \big\|_\infty \geq \varepsilon \Big\} \\ &\leq C \frac{T_n}{a_{T_n}} \exp\Big(-\frac{2(1+\varepsilon/3)^2}{2+\varepsilon} \log \frac{T_n}{a_{T_n} \log\log T_n} \Big) \\ &\leq C (\log\log T_n)^{-1} \longrightarrow 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Hence, by the arbitrariness of ε, we get (1.1).

Let $T_n = e^{e^n}$ with $n \geq 1$. If $\limsup_{T\to\infty} \dfrac{\log\big(T/a_T\big)}{\log\log\log T} < \infty$, then for any $\varepsilon > 0$ there exists constant $r_0 > 1$ with $r_0 \varepsilon > 4$ such that

$$\big(\log\log T_n\big)^{r_0 - \varepsilon} \leq \frac{T_n}{a_{T_n}} \leq \big(\log\log T_n\big)^{r_0+\varepsilon} \tag{3.2}$$

for large n. It suffices to show that

$$\limsup_{n\to\infty} \sup_{0 \leq t \leq T_n - a_{T_n}} \inf_{f \in K} \|Z_{t,T_n}(\cdot) - f(\cdot)\|_\infty = 0 \qquad \text{a.s.} \tag{3.3}$$

Let d_k be as in Lemma 2.5, and set for $0 \le t \le 1$

$$X^{(k)}(t) = \int_{|x|\in(d_{k-1},d_k)} \frac{1}{q_\alpha}\big\{|x-t|^{(2\alpha-1)/2} - |x|^{(2\alpha-1)/2}\big\} dW(x), \tag{3.4}$$

where $W(\cdot)$ is a Wiener process, then $\tilde{X}^{(k)}(t) = X(t) - X^{(k)}(t)$ and $\{X^{(k)}(t)\}$, $k = 1, 2, \cdots$ are independent. Note that by self-similarity of FBM

$$\begin{aligned}
&\sup_{0\le t\le T_n - a_{T_n}} \inf_{f\in K} \big\|Z_{t,T_n}(\cdot) - f(\cdot)\big\|_\infty \\
&\le \sup_{0\le t\le T_n - a_{T_n}} \inf_{f\in K} \Big\| \frac{X^{(n)}(t + a_{T_n}\cdot) - X^{(n)}(t)}{a_{T_n}^\alpha \gamma_{T_n}} - f(\cdot)\Big\|_\infty \\
&\quad + \sup_{0\le t\le T_n - a_{T_n}} \Big\| \frac{\tilde{X}^{(n)}(t + a_{T_n}\cdot) - \tilde{X}^{(n)}(t)}{a_{T_n}^\alpha \gamma_{T_n}} \Big\|_\infty \\
&=: L_1 + L_2.
\end{aligned} \tag{3.5}$$

For any positive number t and integer $k > 0$, let $t_0 = 0$ and $t_k = a_{T_n}\big[t2^k/a_{T_n}\big]/2^k$, then we define

$$\tilde{X}^{(n)}(t) \overset{\text{a.s.}}{=} \sum_{k=1}^{\infty} (\tilde{X}^{(n)}(t_k) - \tilde{X}^{(n)}(t_{k-1})).$$

For $0 \le t \le T_n - a_{T_n}$, we write

$$\begin{aligned}
&\sup_{0\le x\le 1} |\tilde{X}^{(n)}(t_k/a_{T_n}) - \tilde{X}^{(n)}(t_{k-1}/a_{T_n} + x)| \\
&\quad = \sup_{0\le x\le 1} \sum_{k=1}^{\infty} |\tilde{X}^{(n)}(t_k/a_{T_n} + x) - \tilde{X}^{(n)}(t_{k-1}/a_{T_n} + x) \\
&\qquad - \tilde{X}^{(n)}(t_k/a_{T_n}) + \tilde{X}^{(n)}(t_{k-1}/a_{T_n})| \\
&\quad \le 2 \sup_{0\le x\le 1} \sum_{k=1}^{\infty} \Big|\tilde{X}^{(n)}(t_k/a_{T_n} + x) - \tilde{X}^{(n)}(t_{k-1}/a_{T_n} + x)\Big|.
\end{aligned}$$

We now apply the similar arguments as in Lemma 2.5, with k replaced to n, to obtain that

$$E\big(H^{(n)}(s_n, t+h) - H^{(n)}(s_n, t)\big)^2 \le Ch^{2\eta} s_n^{2\alpha} n^{-\delta},$$

where δ, η and α are in Lemma 2.5. There exists a positive number $k_0 = k_0(\alpha, \varepsilon)$ such that for $k \ge k_0$

$$\frac{\varepsilon 2^{\alpha k+1}}{\sqrt{C}(k+1)^2} \ge \sqrt{(1+\varepsilon/2)(k+1)}.$$

Letting $x_k = (k+1)^{-2}$, $k \geq 1$, we have

$$
\begin{aligned}
&P\{L_2 \geq \varepsilon\} \\
&\leq P\Big\{ \sup_{0 \leq t \leq T_n - a_{T_n}} \sup_{0 \leq x \leq 1} \sum_{k=1}^{\infty} \big| \tilde{X}^{(n)}(t_k/a_{T_n} + x) - \tilde{X}^{(n)}(t_{k-1}/a_{T_n} + x) \big| \\
&\qquad\qquad\qquad\qquad\qquad\qquad \geq 2\varepsilon \gamma_{T_n} \sum_{k=1}^{\infty} x_k \Big\} \\
&\leq \sum_{k=1}^{\infty} 2^k \frac{T_n - a_{T_n}}{a_{T_n}} \sup_{0 \leq t \leq T_n - a_{T_n}} \\
&\qquad P\Big\{ \sup_{0 \leq x \leq 1} \big| \tilde{X}^{(n)}(t_k/a_{T_n} + x) - \tilde{X}^{(n)}(t_{k-1}/a_{T_n} + x) \big| \geq 2\varepsilon \gamma_{T_n} x_k \Big\} \\
&= \sum_{k=1}^{\infty} 2^k \frac{T_n - a_{T_n}}{a_{T_n}} \sup_{0 \leq t \leq T_n - a_{T_n}} \\
&\quad P\Big\{ \sup_{0 \leq x \leq 1} \frac{|H^{(n)}(s_n, t_k/a_{T_n} + x) - H^{(n)}(s_n, t_{k-1}/a_{T_n} + x)|}{\sqrt{C(2^{-k})^{2\eta} s_n^{2\alpha} n^{-\delta}}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \geq \frac{2\varepsilon 2^{\alpha k}}{\sqrt{C n^{-\delta}}} x_k \gamma_{T_n} \Big\} \qquad (3.6) \\
&\leq C \sum_{k=k_0}^{\infty} 2^k \frac{T_n - a_{T_n}}{a_{T_n}} \sup_{0 \leq t \leq T_n - a_{T_n}} \\
&\quad P\Big\{ \sup_{0 \leq x \leq 1} \frac{|H^{(n)}(s_n, t_k/a_{T_n} + x) - H^{(n)}(s_n, t_{k-1}/a_{T_n} + x)|}{\sqrt{C(2^{-k})^{2\eta} s_n^{2\alpha} n^{-\delta}}} \\
&\qquad\qquad\qquad\qquad\qquad \geq \sqrt{(1+\varepsilon/2)(k+1)} \gamma_{T_n} \Big\} \\
&\leq C \sum_{k=k_0}^{\infty} 2^k \frac{T_n - a_{T_n}}{a_{T_n}} \exp\Big(-(k+1) n^{\delta} \log\big(\frac{T_n}{a_{T_n} \log\log T_n} \big) \Big) \\
&\leq C \frac{T_n - a_{T_n}}{a_{T_n}} \exp\Big(-n^{\delta} \log\big(\frac{T_n}{a_{T_n} \log\log T_n} \big) \Big) \sum_{k=k_0}^{\infty} (2/e)^k \\
&\leq C \big(\log\log T_n \big)^{(r_0+\varepsilon) - n^{\delta}(r_0 - \varepsilon - 1)} \\
&\leq C n^{-(n^{\delta}(r_0 - \varepsilon - 1) - (r_0 + \varepsilon))}
\end{aligned}
$$

for large n, which implies

$$\sum_{n=1}^{\infty} P\{L_2 \geq \varepsilon\} < \infty.$$

By the Borel-Cantelli lemma and the arbitrariness of ε, we obtain

$$\limsup_{n \to \infty} L_2 = 0 \qquad \text{a.s.} \tag{3.7}$$

Consider L_1. We have for any $\varepsilon > 0$

$$
\begin{aligned}
L_3 :&= P\Big\{L_1 \le 3\varepsilon\Big\} \\
&\ge P\Big\{\sup_{0\le t\le T_n - a_{T_n}} \inf_{f\in K} \Big\| \frac{X(t+a_{T_n}\cdot) - X(t)}{a_{T_n}^{\alpha}\gamma_{T_n}} - f(\cdot)\Big\|_\infty \le 2\varepsilon\Big\} \\
&\qquad - P\Big\{\sup_{0\le t\le T_n - a_{T_n}} \Big\| \frac{\tilde{X}^{(n)}(t+a_{T_n}\cdot) - \tilde{X}^{(n)}(t)}{a_{T_n}^{\alpha}\gamma_{T_n}}\Big\|_\infty \ge 2\varepsilon\Big\} \\
&=: L_4 - L_5.
\end{aligned}
\tag{3.8}
$$

It is clear from (3.6) that

$$
\sum_{n=1}^{\infty} L_5 < \infty. \tag{3.9}
$$

We need only consider L_4. By the properties of FBM we have

$$
\begin{aligned}
L_4 &\ge P\Big\{\max_{0\le j\le T_n/a_{T_n}} \sup_{2j\le t\le 2j+1} \inf_{f\in K} \Big\| \frac{X(t+\cdot) - X(t)}{\gamma_{T_n}} - f(\cdot)\Big\|_\infty \le 2\varepsilon\Big\} \\
&\ge P\Big\{\max_{0\le j\le T_n/a_{T_n}} \sup_{0\le t\le 1} \inf_{f\in K} \Big\| \frac{X^{(j)}(t+\cdot) - X^{(j)}(t)}{\gamma_{T_n}} - f(\cdot)\Big\|_\infty \le \varepsilon\Big\} \\
&\quad - P\Big\{\max_{0\le j\le T_n/a_{T_n}} \sup_{0\le t\le 1} \Big\| \frac{\tilde{X}^{(j)}(t+\cdot) - \tilde{X}^{(j)}(t)}{\gamma_{T_n}}\Big\|_\infty \ge \varepsilon\Big\} \\
&=: L_6 - L_7.
\end{aligned}
\tag{3.10}
$$

We have by Lemmas 2.2 and 2.5

(3.11)

$$
\begin{aligned}
L_7 &\le \sum_{j=0}^{T_n/a_{T_n}} P\big\{\sup_{0\le t\le 1}\sup_{0\le x\le 1} \big|\tilde{X}^{(j)}(t+\cdot) - \tilde{X}^{(j)}(t)\big| \ge \varepsilon\gamma_{T_n}\big\} \\
&= \sum_{j=0}^{T_n/a_{T_n}} P\Big\{\sup_{0\le t\le 1}\sup_{0\le x\le 1} \frac{|H^{(j)}(s_j, t+x) - H^{(j)}(s_j, t)|}{\sqrt{Cx^{2\eta}s_j^{2\alpha}j^{-\delta}}} \ge \frac{\varepsilon}{\sqrt{Cj^{-\delta}}}\gamma_{T_n}\Big\} \\
&\le \sum_{j=0}^{T_n/a_{T_n}} \exp\Big\{-\frac{2\varepsilon^2}{2+\varepsilon}\frac{j^\delta}{C}\log\big(\frac{T_n}{a_{T_n}\log\log T_n}\big)\Big\} \\
&\le C\sum_{j=j_0}^{T_n/a_{T_n}} \big(\log\log T_n\big)^{-\varepsilon^2(r_0-\varepsilon-1)j_0^\delta/(2C)} \\
&\le C\sum_{j=j_0}^{n^{r_0+\varepsilon}} n^{-\varepsilon^2(r_0-\varepsilon-1)j_0^\delta/(2C)} \\
&\le Cn^{-(\varepsilon^2(r_0-\varepsilon-1)j_0^\delta/(2C)-(r_0+\varepsilon))}
\end{aligned}
$$

for n large enough. Taking j_0 to be large enough such that $\varepsilon^2(r_0-\varepsilon-1)j_0^\delta/(2C)-(r_0+\varepsilon)>1$, we have

$$\sum_{n=1}^{\infty} L_7 < \infty. \tag{3.12}$$

Using the fact that $-\log(1-x) \le m_\nu x$ for $x \in (0, 1-\nu)$ and for some $m_\nu > 0$, we have by (2.2) and independence of $X^{(j)}$

$$\begin{aligned}
&L_6\\
&= \Big(1 - P\Big\{\sup_{0\le t\le 1}\inf_{f\in K}\big\|(X^{(j)}(t+\cdot)-X^{(j)}(t)) - f(\cdot)\gamma_{T_n}\big\|_\infty \ge \varepsilon\gamma_{T_n}\Big\}\Big)^{\frac{T_n}{a_{T_n}}+1}\\
&\ge \Big(1 - C\exp\Big\{-\frac{(1+\varepsilon/3)^2}{2+\varepsilon}\log\frac{T_n}{a_{T_n}\log\log T_n}\Big\}\Big)^{\frac{T_n}{a_{T_n}}+1}\\
&\ge \exp\Big(-m_\nu C\frac{T_n}{a_{T_n}}\Big(\frac{a_{T_n}\log\log T_n}{T_n}\Big)^{1+\varepsilon/3}\Big)\\
&\ge \exp\big(-C_\nu(\log\log T_n)^{-\varepsilon(r_0+\varepsilon)/3+(1+\varepsilon/3)}\big)\\
&\ge \exp\big(-C_\nu n^{1-\varepsilon(r_0+\varepsilon-1)/3}\big)
\end{aligned}$$

for sufficiently large n, by hence

$$\sum_{n=1}^{\infty} L_6 = \infty. \tag{3.13}$$

In combination with (3.12) and (3.13), it follows that

$$\sum_{n=1}^{\infty} L_4 = \infty. \tag{3.14}$$

Combining (3.8) with (3.9) and (3.14), we have

$$\sum_{n=1}^{\infty} L_3 = \infty.$$

Since the events of L_3 are independent, by the Borel Cantelli lemma and the arbitrariness of ε, we have

$$\limsup_{n\to\infty} L_1 = 0 \qquad \text{a.s} \tag{3.15}$$

From (3.7) and (3.15) we obtain (3.3), and (1.1) is proved.

Proof of (1.2). Let $f \in K$, put $T_n = \theta^n$ with $\theta > 1$. We have that for $T_n \le T \le T_{n+1}$

$$
\begin{aligned}
&\inf_{0\le t\le T-a_T} \|Z_{t,T}(\cdot) - f(\cdot)\|_\infty \\
&\le \sup_{T_n\le T\le T_{n+1}} \inf_{0\le t\le T_n-a_{T_n}} \|Z_{t,T_{n+1}}\big(\cdot\, a_T/a_{T_{n+1}}\big) - f\big(\cdot\, a_T/a_{T_{n+1}}\big)\|_\infty \\
&\quad + \sup_{T_n\le T\le T_{n+1}} \sup_{0\le t\le T_{n+1}-a_{T_{n+1}}} \|Z_{t,T}\big(\cdot\big) - Z_{t,T_{n+1}}\big(\cdot\, a_T/a_{T_{n+1}}\big)\|_\infty \\
&\quad + \sup_{T_n\le T\le T_{n+1}} \|f\big(\cdot\, a_T/a_{T_{n+1}}\big) - f\big(\cdot\big)\|_\infty \\
&\le \inf_{0\le t\le T_n-a_{T_n}} \|Z_{t,T_{n+1}}(\cdot) - f(\cdot)\|_\infty \\
&\quad + \Big|\frac{a^\alpha_{T_{n+1}}\gamma_{T_{n+1}}}{a^\alpha_{T_n}\gamma_{T_n}} - 1\Big| \sup_{T_n\le T\le T_{n+1}} \sup_{0\le t\le T_{n+1}-a_{T_{n+1}}} \|Z_{t,T_{n+1}}(\cdot a_T/a_{T_{n+1}})\|_\infty \\
&\quad + \sup_{T_n\le T\le T_{n+1}} \|f\big(\cdot\, a_T/a_{T_{n+1}}\big) - f(\cdot)\|_\infty \\
&=: L_8 + L_9 + L_{10}.
\end{aligned} \tag{3.16}
$$

Consider I_8. Let $\rho_n = \big[(T_n - a_{T_n})/a_{T_{n+1}}\big]$, then it is sufficient to show that

$$
\limsup_{n\to\infty} \min_{0\le j\le\rho_n} \Big\|\frac{X(j+\cdot) - X(j)}{\gamma_{T_{n+1}}} - f(\cdot)\Big\|_\infty = 0 \qquad \text{a.s.} \tag{3.17}
$$

Let $d_n, X^{(n)}$ and $\tilde{X}^{(n)}$ be as in (3.4). By standard Borel-Cantelli arguments, (3.17) follows if we prove that

$$
\sum_{n=1}^\infty P\Big\{\min_{0\le j\le\rho_n} \Big\|\frac{X^{(n)}(j+\cdot) - X^{(n)}(j)}{\gamma_{T_{n+1}}} - f(\cdot)\Big\|_\infty \le \varepsilon\Big\} = \infty \tag{3.18}
$$

and

$$
\sum_{n=1}^\infty P\Big\{\max_{0\le j\le\rho_n} \Big\|\frac{\tilde{X}^{(n)}(j+\cdot) - \tilde{X}^{(n)}(j)}{\gamma_{T_{n+1}}}\Big\|_\infty \ge \varepsilon\Big\} < \infty, \tag{3.19}
$$

since the events in (3.18) are independent.

Similar to (3.6) and (3.11), it is easily seen (3.19). Hence it remains to show (3.18). For any $\varepsilon > 0$ we have

$$
\begin{aligned}
&P\Big\{\min_{0\le j\le \rho_n}\Big\|\frac{X^{(n)}(j+\cdot)-X^{(n)}(j)}{\gamma_{T_{n+1}}}-f(\cdot)\Big\|_\infty \le 4\varepsilon\Big\}\\
&\ge P\Big\{\min_{0\le j\le \rho_n}\Big\|\frac{X(j+\cdot)-X(j)}{\gamma_{T_{n+1}}}-f(\cdot)\Big\|_\infty \le 2\varepsilon\Big\}\\
&\quad - P\Big\{\max_{0\le j\le \rho_n}\Big\|\frac{\tilde X^{(n)}(j+\cdot)-\tilde X^{(n)}(j)}{\gamma_{T_{n+1}}}\Big\|_\infty \ge 2\varepsilon\Big\}\\
&\ge P\Big\{\min_{0\le j\le \rho_n}\Big\|\frac{X^{(j)}(j+\cdot)-X^{(j)}(j)}{\gamma_{T_{n+1}}}-f(\cdot)\Big\|_\infty \le \varepsilon\Big\}\\
&\quad - P\Big\{\max_{0\le j\le \rho_n}\Big\|\frac{\tilde X^{(j)}(j+\cdot)-\tilde X^{(j)}(j)}{\gamma_{T_{n+1}}}\Big\|_\infty \ge \varepsilon\Big\}\\
&\quad - P\Big\{\max_{0\le j\le \rho_n}\Big\|\frac{\tilde X^{(n)}(j+\cdot)-\tilde X^{(n)}(j)}{\gamma_{T_{n+1}}}\Big\|_\infty \ge 2\varepsilon\Big\}\\
&=: L_{11}-L_{12}-L_{13}.
\end{aligned}
\tag{3.20}
$$

It is clear from (3.19) that

$$
\sum_{n=1}^{\infty}(L_{12}+L_{13})<\infty. \tag{3.21}
$$

We now turn to the case L_{11}. Put $f^{(\varepsilon)} = (1-\varepsilon/2)f$ $(0<\varepsilon<1)$ for $f \in K$, then $f^{(\varepsilon)} \in K$ and $\|f-f^{(\varepsilon)}\|_\infty < \varepsilon/2$. We have by Lemma 2.4 and independence of $X^{(j)}$

$$
\begin{aligned}
&P\Big\{\min_{0\le j\le \rho_n}\Big\|\frac{X^{(j)}(j+\cdot)-X^{(j)}(j)}{\gamma_{T_{n+1}}}-f(\cdot)\Big\|_\infty \le \varepsilon\Big\}\\
&= \sum_{j=0}^{\rho_n} P\big\{\big\|(X^{(j)}(j+\cdot)-X^{(j)}(j))-f(\cdot)\gamma_{T_{n+1}}\big\|_\infty \le \frac{\varepsilon}{2}\gamma_{T_{n+1}}\big\}\\
&\ge \sum_{j=0}^{\rho_n}\exp\Big(-\frac12\|f^{(\varepsilon)}\|_\alpha^2\gamma_{T_{n+1}}^2\Big)P\big\{\big\|X^{(j)}(j+\cdot)-X^{(j)}(j)\big\|_\infty\le\frac{\varepsilon}{2}\gamma_{T_{n+1}}\big\}\\
&\ge C\sum_{j=0}^{\rho_n}\exp\big\{-\frac{(1+\varepsilon/2)^2}{2}\|f\|_\alpha^2\gamma_{T_{n+1}}^2\big\}\\
&\ge C\frac{T_n}{a_{T_n}}\exp\Big(-(1+\varepsilon/2)^2\log\frac{T_{n+1}}{a_{T_{n+1}}\log\log T_{n+1}}\Big)\\
&\ge C(\log\log T_{n+1})^{(1+\varepsilon/2)^2} = C(\log((n+1)\log\theta))^{(1+\varepsilon/2)^2}
\end{aligned}
$$

for large enough n, which implies

$$\sum_{n=1}^{\infty} L_{11} = \infty. \tag{3.22}$$

Combining (3.20) with (3.21) and (3.22), we obtain (3.18).

Since $\|f\|_\infty \le 1$ for all $f \in K$, we have by (3.17)

$$L_9 \le 2(\theta^{\alpha+1} - 1) \qquad \text{a.s.} \tag{3.23}$$

for large n. Next, for $T_n \le T \le T_{n+1}$ and $0 \le x \le 1$, we have

$$|f(xa_T/a_{T_{n+1}}) - f(x)| \le |a_T/a_{T_{n+1}} - 1|^\alpha \le |\theta - 1|^\alpha,$$

thus we get

$$L_{10} \le |\theta - 1|^\alpha. \tag{3.24}$$

Combining (3.16) with (3.17), (3.23) and (3.24), and letting $\theta \downarrow 1$, we obtain (1.2), and the proof is complete. □

Since K is a compact subset of $C_0[0,1]$, Theorem 1.1 implies:

Corollary 1. *With probability one,* $\{Z_{t,T}(x) : 0 \le x \le 1, 0 \le t \le T - a_T, T \ge 3\}$ *(as* $T \to \infty$*) is relatively compact in* $C_0[0,1]$*, and the set of its limit point is* K*.*

Corollary 2. *We have*

$$\begin{aligned}\liminf_{T\to\infty} \sup_{0\le t\le T-a_T} \sup_{0\le s\le a_T} & \frac{|X(t+s) - X(t)|}{a_T^\alpha \gamma_T} \\ &= \liminf_{T\to\infty} \sup_{0\le t\le T-a_T} \frac{|X(t+a_T) - X(t)|}{a_T^\alpha \gamma_T} = 1 \qquad \text{a.s.}\end{aligned}$$

References

[1] P. Baldi, G. Ben Arous and G. Kerkyacharian, *Large deviations and the Strassen theorem in Hölder norm*, Stoch. Proc. Appl. **42** (1992), 170-180.

[2] C. Borel, *A note on Gauss measure which agree on small balls*, Ann. Inst. Henri Poincaré Sect. B **13** (1977), no. 3, 231-238.

[3] B. Chen, Ph. D. Dissertation, Univ. Carleton of Canada, 1998.

[4] B. Chen and M. Csörgő, *A functional modulus of continuity for a Wiener process*, Statist. Probab. Lett. **51** (2001), 215-223.

[5] V. Goodman and J. Kuelbs, *Rates of clustering for some Gaussian self-similar processes*, Probab. Theory Rel. Fields **88** (1991), 47-75.

[6] J. Kuelbs, *A strong convergence theorem for Banach space valued random variables*, Ann. Probab. **4** (1976), 744-771.

[7] J. Kuelbs, W. V. Li and Q. M. Shao, *Small ball probabilities for Gaussian processes with stationary increments under Hölder norms* J. Theore. Probab. **8** (1995), no. 2, 361-386.

[8] J. Kuelbs, W. V. Li and M. Talagrand, *Lim inf results for Gaussian samples and Chung's functional LIL*, Ann. Probab. **22** (1994), 1879-1903.

[9] D. Monrad and H. Rootzén, *Small values of Gaussian processes and functional laws of the iterated logarithm*, Probab. Theory Rel. Fields **101** (1995), 173-192.

[10] C. Mueller, *A unification of Strassen's law and Lévy modulus of continuity*, Z. Wahrsch. verw. Gebiete **56** (1981), 163-179.
[11] J. Ortega, *On the size of the increments of non-stationary Gaussian processes*, Stoch. proc. Appl. **18** (1984), 47-56.
[12] P. Révész, *A Generalization of Strassen's functional law of iterated logarithm*, Z. Wahrsch. verw. Gebiete **50** (1979), 257-264.
[13] W. Wang, *On a functional limit result for increments of a fractional Brownian motion*, Acta Math. Hungar. **93** (2001), no. 1-2, 153-170.
[14] Q. Wei, *Functional limit theorems for C-R increments of k-dimensional Brownian motion in Hölder norm*, Acta Math. Sinica, English Series **16** (2000), no. 4, 637-654.
[15] L. X. Zhang, *A note on liminfs on increments of a fractional Brownian motion*, Acta Math. Hungar. **76** (1997), no. 1-2, 145-154.

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