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LIMSUP RESULTS FOR THE INCREMENTS OF PARTIAL SUMS OF A RANDOM SEQUENCE

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Abstract. Let $\{\xi_j; j \geq 1\}$ be a centered strictly stationary random sequence defined by $S_0 = 0$, $S_n = \sum_{j=1}^{n} \xi_j$ and $\sigma(n) = \sqrt{ES_n^2}$, where $\sigma(t)$, $t > 0$, is a nondecreasing continuous regularly varying function. Suppose that there exists $n_0 \geq 1$ such that, for any $n \geq n_0$ and $0 \leq \varepsilon < 1$, there exist positive constants c_1 and c_2 such that $c_1 e^{-(1+\varepsilon)x^2/2} \leq P\left\{\frac{|S_n|}{\sigma(n)} \geq x\right\} \leq c_2 e^{-(1-\varepsilon)x^2/2}$, $x \geq 1$. Under some additional conditions, we investigate some limsup results for the increments of partial sum processes of the sequence $\{\xi_j; j \geq 1\}$.

1. Introduction

Let $\{X, X_n, n \geq 1\}$ be a sequence of nondegenerate centered independent and identically distributed (i.i.d.) random variables on an underlying probability space $(\Omega, \mathfrak{F}, P)$ such that $EX^2 I\{|X| \leq x\}$ is slowly varying as $x \to \infty$. Put

$$S_n = \sum_{i=1}^{n} X_i, \qquad V_n^2 = \sum_{i=1}^{n} X_i^2, \qquad n \geq 1.$$

Shao [18] proved the following: For arbitrary $0 < \varepsilon < 1/2$, there exist $0 < \delta < 1$, $x_0 > 1$ and n_0 such that, for any $n \geq n_0$ and $x_0 < x < \delta\sqrt{n}$,

$$e^{-(1+\varepsilon)x^2/2} \leq P\left\{\frac{S_n}{V_n} \geq x\right\} \leq e^{-(1-\varepsilon)x^2/2} \tag{1.1}$$

in Remark 4.1 of the just mentioned paper [18]. Further, Csörgő et al. [4] established a weak invariance principle related to the inequality (1.1) for self-normalized partial sum processes under the assumption that X belongs to the domain of attraction of the normal law.

On the other hand, consider a sequence of dependent random variables $\{Y_n; n \geq 1\}$. The sequence $\{Y_n; n \geq 1\}$ is said to be *positively associated*

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(PA) if, for any finite subsets A, B of $\{1, 2, \cdots\}$ and coordinatewise increasing functions f and g, we have $\mathrm{Cov}(f(Y_i; i \in A), g(Y_j; j \in B)) \geq 0$, while $\{Y_n; n \geq 1\}$ is said to be *negatively associated* (NA) if, for any disjoint finite subsets A, B of $\{1, 2, \cdots\}$ and coordinatewise increasing functions f and g, we have $\mathrm{Cov}(f(Y_i; i \in A), g(Y_j; j \in B)) \leq 0$. The concept of PA was introduced in [5], while that of NA in [8].

Newman and Wright [12] and Su et al. [19] obtained the central limit theorem (CLT) for partial sums of PA or NA random variables as follows. Let $\{Y_n; n \geq 1\}$ be a sequence of strictly stationary PA or NA random variables with $E(Y_1) = 0$, $0 < \mathrm{Var}(Y_1) < \infty$ and $\mathbf{S}_n := \sum_{i=1}^n Y_i$. If

$$\sigma^2 := \mathrm{Var}(Y_1) + 2 \sum_{j=2}^{\infty} \mathrm{Cov}(Y_1, Y_j) < \infty, \tag{1.2}$$

then

$$\frac{\mathbf{S}_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0, \sigma^2) \qquad \text{as} \quad n \to \infty. \tag{1.3}$$

This suggests that, for any $0 \leq \varepsilon < 1$, there exist positive constants k_1 and k_2 such that

$$k_1 \, e^{-(1+\varepsilon)x^2/2} \leq P\left\{ \frac{|\mathbf{S}_n|}{\sqrt{n}\sigma} \geq x \right\} \leq k_2 \, e^{-(1-\varepsilon)x^2/2}, \qquad x \geq 1, \tag{1.4}$$

for sufficiently large n. The inequality (1.4) represents upper and lower bounds of the tail probability (cf. Lemma 2 in page 175 of [6]).

Next, consider the case of mixing random variables. For any two σ-fields $\mathcal{A}$ and $\mathcal{B}$ in $(\Omega, \mathfrak{F}, P)$, define the correlation

$$\rho(\mathcal{A}, \mathcal{B}) := \sup \frac{|E(VW) - E(V)E(W)|}{(EV^2)^{1/2}(EW^2)^{1/2}},$$

where the sup is taken over all square-integrable random variables V and W which are $\mathcal{A}$-measurable and $\mathcal{B}$-measurable, respectively. Let now $\{Y_n; n \geq 1\}$ be a sequence of strictly stationary random variables with $E(Y_1) = 0$ and $0 < \mathrm{Var}(Y_1) < \infty$. For any nonempty disjoint sets S and D of $\{1, 2, \cdots\}$, denote

$$\rho(S, D) = \rho\left(\sigma[Y_i; i \in S], \ \sigma[Y_j; j \in D]\right),$$

where $\sigma[\,;\,]$ is the σ-field generated by Y_i's. The "distance" between any two disjoint nonempty subsets S, D of $\{1, 2, \cdots\}$ will be denoted by $\mathit{dist}(S, D) := \min_{j \in S, k \in D} \|j - k\|$, where $\|\cdot\|$ is the usual Euclidean norm. For each $n \geq 1$, define $\rho_n^* = \sup \rho(S, D)$, where the sup is taken over all pairs of nonempty disjoint subsets S, D of $\{1, 2, \cdots\}$ such that $\mathit{dist}(S, D) \geq n$. Let again $\mathbf{S}_n = \sum_{i=1}^n Y_i$ and put $\sigma_n^2 = \mathrm{Var}(\mathbf{S}_n)$.

Peligrad [15] proved the following result: If $\rho_n^* \to 0$ (say, ρ^*-*mixing*) and $\sigma_n^2 \to \infty$ as $n \to \infty$, then we have (1.3) and (1.4) in this case as well under the condition (1.2).

In the next section, we study asymptotic properties for increments of partial sum processes of dependent random sequences under the assumption (2.2) in Section 2 which involves (1.1) for i.i.d. random variables and (1.4) for ρ^*-mixing, PA or NA dependent random variables.

2. Main Results

In this paper, we develop some limit results for increments of partial sum processes of iid random sequences given as in [3, 9, 10] to the case of dependent random sequences as follows. Let $\{\xi_j; j \geq 1\}$ be a centered strictly stationary random sequence with $E\xi_1^2 = 1$. Define

$$S_0 = 0,\ S_n = \sum_{j=1}^{n} \xi_j \ \text{ and } \ \sigma(n) = \sqrt{ES_n^2}. \tag{2.1}$$

Assume that $\sigma(n)$ can be extended to a continuous function $\sigma(t)$ of $t > 0$ which is nondecreasing and regularly varying with exponent α at ∞ for some $0 < \alpha < 1$.

A positive function $\sigma(t)$, $t > 0$, is said to be *regularly varying* with exponent $\alpha > 0$ at $b \geq 0$ if $\lim_{t\to b}\{\sigma(xt)/\sigma(t)\} = x^\alpha$ for all $x > 0$.

On the basis of the result (1.4) obtained above for ρ^*-mixing, PA or NA random fields, in this paper, we suppose that there exists $n_0 \geq 1$ such that, for any $n \geq n_0$ and $0 \leq \varepsilon < 1$, there exist positive constants c_1 and c_2 such that

$$c_1\, e^{-(1+\varepsilon)x^2/2} \leq P\left\{\frac{|S_n|}{\sigma(n)} \geq x\right\} \leq c_2\, e^{-(1-\varepsilon)x^2/2}, \quad x \geq 1. \tag{2.2}$$

It is well-known that, as $n \to \infty$, $V_n/\sigma(n) \xrightarrow{p} 1$ in (1.1) and (2.2) for centered independent random variables under the Lindeberg condition (cf. [4]), and that $\sigma(n)/\sqrt{n}\sigma \to 1$ holds for standard deviations of S_n in (2.2) and $\mathbf{S}_n$ in (1.4) (cf. [14, 16, 20]).

Suppose that $\{a_n, n \geq 1\}$ is a nondecreasing sequence of positive integers such that

(i) $1 \leq a_n \leq n$.

Denote

$$\beta_n = \left\{2\Big(\log(n/a_n) + \log\log n\Big)\right\}^{1/2}, \quad n > e.$$

The main results are as follows.

Theorem 2.1. *Let $\{\xi_j; j \geq 1\}$ be a centered strictly stationary random sequence with $E\xi_1^2 = 1$ and condition* (2.2)*, and let $\{a_n, n \geq 1\}$ be a nondecreasing sequence of positive integers satisfying condition* (i)*. Then we have*

$$\limsup_{n\to\infty} \sup_{0\leq i\leq n} \sup_{1\leq j\leq a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n)\beta_n} \leq 1 \qquad a.s. \tag{2.3}$$

In order to obtain the opposite inequality of (2.3), the conditions on a_n and $\{\xi_j; j \geq 1\}$ are a little bit restricted as in Theorems 2.2 and 2.3 below.

A random sequence $\{\xi_j; j \geq 1\}$ is said to be *linearly negative quadrant dependent* (LNQD) if, for any positive number λ_j and disjoint subsets A, B of $\mathbb{Z}_+$, the inequality

(2.4)
$$P\Big\{\sum_{j\in A}\lambda_j\xi_j \geq x, \sum_{k\in B}\lambda_k\xi_k \geq y\Big\} \leq P\Big\{\sum_{j\in A}\lambda_j\xi_j \geq x\Big\}P\Big\{\sum_{k\in B}\lambda_k\xi_k \geq y\Big\}$$

holds for all real numbers x and y. This definition of LNQD was introduced by Newman [11].

In general the NA sequence is obviously LNQD, but the LNQD sequence does not imply NA (cf. [13], [17]).

Theorem 2.2. *Let* $\{\xi_j; j \geq 1\}$ *and* $\{a_n, n \geq 1\}$ *be as in* Theorem 2.1. *Further assume that*

(ii) *the random sequence* $\{\xi_j, j \geq 1\}$ *is LNQD*

and

(iii) $\limsup_{n\to\infty} a_n/n =: \rho < 1$. *Then we have*

$$\limsup_{n\to\infty} \frac{|S_{n+a_n} - S_n|}{\sigma(a_n)\beta_n} \geq 1 \qquad a.s. \tag{2.5}$$

Combining Theorems 2.1 and 2.2 yields the following lim sup result.

Corollary 2.1. *Under the assumptions of* Theorem 2.2*, we have*

$$\begin{aligned} \limsup_{n\to\infty} \sup_{0\leq i\leq n} \sup_{1\leq j\leq a_n} \frac{|S_{i+j} - S_i|}{\sigma(a_n)\beta_n} &= 1 \qquad a.s., \\ \limsup_{n\to\infty} \frac{|S_{n+a_n} - S_n|}{\sigma(a_n)\beta_n} &= 1 \qquad a.s. \end{aligned} \tag{2.6}$$

Example 2.1. Let $\{\xi_j, j \geq 1\}$ be an NA Gaussian random sequence in Corollary 2.1. Then the condition (2.2) is satisfied. Set $a_n = [\log n]$. Then, the sequence $\{a_n, n \geq 1\}$ satisfies all the conditions of Corollary 2.1 with

$$\beta_n = \Big\{2\Big(\log(n/[\log n]) + \log\log n\Big)\Big\}^{1/2}.$$

Thus we have, from (2.6),

$$\begin{aligned} \limsup_{n\to\infty} \sup_{0\leq i\leq n} \sup_{1\leq j\leq [\log n]} \frac{|S_{i+j} - S_i|}{\sigma([\log n])\beta_n} &= 1 \qquad a.s., \\ \limsup_{n\to\infty} \sup_{0\leq i\leq n} \frac{|S_{i+[\log n]} - S_i|}{\sigma([\log n])\sqrt{2\log n}} &= 1 \qquad \text{a.s.} \end{aligned}$$

3. Proofs of Theorems 2.1 and 2.2

The proof of Theorem 2.1 is based on the following Lemmas 3.1 and 3.2.

Lemma 3.1. *Let $\mathbb{D}$ be a compact subset of $\mathbb{R}^d$ with the Euclidean norm $\|\cdot\|$ and let $\{X(\mathbf{t}), \mathbf{t} \in \mathbb{D}\}$ be a real-valued separable and centered strictly stationary random field. Suppose that*

$$\begin{aligned} &0 < \Gamma := \sup_{\mathbf{t}\in\mathbb{D}}\{E(X(\mathbf{t}))^2\}^{1/2} < \infty \quad \textit{and} \\ &\sigma^2(\|\mathbf{t}-\mathbf{s}\|) := E\{X(\mathbf{t}) - X(\mathbf{s})\}^2 \le \varphi^2(\|\mathbf{t}-\mathbf{s}\|) \quad \text{for} \quad \mathbf{t} \ne \mathbf{s} \in \mathbb{D}, \end{aligned}$$

where $\varphi(h)$ is a nondecreasing continuous function of $h > 0$. Assume that, for any $0 \le \varepsilon < 1$, there exists a positive constant c_2 such that

$$P\left\{\frac{|X(\mathbf{t})|}{\sigma(\|\mathbf{t}\|)} \ge x\right\} \le c_2\, e^{-(1-\varepsilon)x^2/2}, \quad \mathbf{t} \in \mathbb{D},\ x \ge 1.$$

Then, for $\lambda > 0$ and $K_1 > (2\sqrt{2}+2)\sqrt{1+2d\,(1-\varepsilon)^{-1}\log 2}$, we have

(3.1)
$$P\left\{\sup_{\mathbf{t}\in\mathbb{D}} |X(\mathbf{t})| \ge x\Big(\Gamma + K_1 \int_0^\infty \varphi(\sqrt{d}\,\lambda\, 2^{-y^2})\, dy\Big)\right\} \le c\frac{m(\mathbb{D})}{\lambda^d} e^{-(1-\varepsilon)x^2/2},$$

where c is a positive constant and $m(\mathbb{D})$ denotes the Lebesgue measure of $\mathbb{D}$.

Proof. For each $n = 0, 1, 2, \cdots$, put $\varepsilon_n = \lambda 2^{-2^n}$, $\lambda > 0$. Denote a diameter of $\mathbb{D}$ by $d(\mathbb{D})$. Let $\{S_i^{(n)}, i = 1, 2, \cdots, N_{\varepsilon_n}(\mathbb{D})\}$ be a minimal ε_n-net of $\mathbb{D}$, where $N_{\varepsilon_n}(\mathbb{D}) = \min\big\{k : \mathbb{D} \subset \bigcup_{i=1}^k S_i^{(n)}, d(S_i^{(n)}) \le \varepsilon_n\big\}$. Then there is a positive constant c such that $N_{\varepsilon_n}(\mathbb{D}) \le c\,\frac{m(\mathbb{D})}{\varepsilon_n^d}$. Set $\Delta_n = \bigcup_{i=1}^{N_{\varepsilon_n}(\mathbb{D})} \big\{t_i^{(n)}\big\}$ for $t_i^{(n)} \in S_i^{(n)}$. Let $K_2 > \sqrt{1+2d\,(1-\varepsilon)^{-1}\log 2}$ and $K_1 = (2\sqrt{2}+2)K_2$. For $x \ge 1$, set

$$x_k = xK_2\varphi(\sqrt{d}\varepsilon_{k-1})2^{k/2}, \quad k \ge 1.$$

Let $\delta_k = 2^{(k-1)/2}$ for $k \ge 0$. Then

$$2^{k/2} = (2\sqrt{2}+2)(\delta_k - \delta_{k-1}).$$

Thus we have

$$\begin{aligned} \sum_{k=1}^\infty x_k &= xK_1 \sum_{k=1}^\infty \varphi(\sqrt{d}\lambda 2^{-\delta_k{}^2})(\delta_k - \delta_{k-1}) \\ &\le xK_1 \sum_{k=1}^\infty \int_{\delta_{k-1}}^{\delta_k} \varphi(\sqrt{d}\lambda 2^{-y^2})dy \\ &\le xK_1 \int_0^\infty \varphi(\sqrt{d}\lambda 2^{-y^2})dy. \end{aligned}$$

Therefore, we conclude

$$
\begin{aligned}
&P\left\{\sup_{\mathbf{t}\in\mathbb{D}}|X(\mathbf{t})|\geq x\left(\Gamma+K_1\int_0^\infty\varphi\big(\sqrt{d}\lambda 2^{-y^2}\big)dy\right)\right\}\\
&\leq P\left\{\sup_{\mathbf{t}\in\mathbb{D}}|X(\mathbf{t})|\geq x\Gamma+\sum_{k=1}^{\infty}x_k\right\}\\
&\leq \lim_{n\to\infty}P\left\{\sup_{\mathbf{t}\in\Delta_n}|X(\mathbf{t})|\geq x\Gamma+\sum_{k=1}^{n}x_k\right\}.
\end{aligned}
$$

Let

$$
\begin{aligned}
B_0&=\left\{\sup_{\mathbf{t}\in\Delta_0}|X(\mathbf{t})|\geq x\Gamma\right\},\\
B_n&=\left\{\sup_{\mathbf{t}\in\Delta_n}|X(\mathbf{t})|\geq x\Gamma+\sum_{k=1}^{n}x_k\right\},\quad n\geq 1.
\end{aligned}
$$

By induction, we have

$$
\begin{aligned}
P(B_n)&=P(B_n\cap B_{n-1})+P(B_n\cap B_{n-1}^c)\\
&\leq P(B_0)+\sum_{n=1}^{\infty}P(B_n\cap B_{n-1}^c)
\end{aligned}
$$

and, for a large n,

$$
\begin{aligned}
&P(B_n\cap B_{n-1}^c)\\
&\leq P\left\{\bigcup_{\mathbf{t}\in\Delta_n}\left\{|X(\mathbf{t})|\geq x\Gamma+\sum_{k=1}^{n}x_k\right\}\cap\bigcap_{\mathbf{s}\in\Delta_{n-1}}\left\{|X(\mathbf{s})|<x\Gamma+\sum_{k=1}^{n-1}x_k\right\}\right\}\\
&\leq P\left\{\bigcup_{\mathbf{t}\in\Delta_n}\bigcup_{\substack{\mathbf{s}\in\Delta_{n-1}\\ \|\mathbf{t}-\mathbf{s}\|\leq\sqrt{d}\,\varepsilon_{n-1}}}\big\{|X(\mathbf{t})|-|X(\mathbf{s})|\geq x_n\big\}\right\}\\
&\leq \sum_{\mathbf{t}\in\Delta_n}\sum_{\substack{\mathbf{s}\in\Delta_{n-1}\\ \|\mathbf{t}-\mathbf{s}\|\leq\sqrt{d}\,\varepsilon_{n-1}}}P\big\{|X(\mathbf{t})-X(\mathbf{s})|\geq x_n\big\}\\
&\leq c\frac{m(\mathbb{D})}{\varepsilon_n^d}P\left\{\frac{|X(\mathbf{t})-X(\mathbf{s})|}{\sigma(\|\mathbf{t}-\mathbf{s}\|)}\geq\frac{x_n}{\varphi(\|\mathbf{t}-\mathbf{s}\|)}\right\}\\
&\leq c\frac{m(\mathbb{D})}{\varepsilon_n^d}P\left\{\frac{|X(\mathbf{t})|}{\sigma(\|\mathbf{t}\|)}\geq xK_2 2^{n/2}\right\}\leq c_2\frac{m(\mathbb{D})}{\lambda^d}2^{d2^n}\exp\left(-\frac{1-\varepsilon}{2}x^2K_2^2 2^n\right)\\
&\leq c_2 2^{d2^n}e^{-\frac{1-\varepsilon}{2}(K_2^2 2^n-1)}e^{-\frac{1-\varepsilon}{2}x^2}\frac{m(\mathbb{D})}{\lambda^d},
\end{aligned}
$$

and

$$
\sum_{n=1}^{\infty}P\left(B_n\cap B_{n-1}^c\right)\leq c_3\frac{m(\mathbb{D})}{\lambda^d}e^{-(1-\varepsilon)x^2/2}
$$

for some $c_3 > 0$. On the other hand, we have

$$P(B_0) \le c\frac{m(\mathbb{D})}{\varepsilon_0^d} P\left\{|X(\mathbf{t})| \ge x\Gamma\right\} \le c_2 \frac{m(\mathbb{D})}{\lambda^d} \exp\left(-\frac{(1-\varepsilon)x^2}{2}\right).$$

This proves our Lemma 3.1. □

For $\theta > 1$, let

$$\mathbb{D}_{k,l} = \{(i,j) : 0 \le i \le \theta^k, 1 \le j \le \theta^l\}, \quad k \ge 1, \quad l \ge 1. \tag{3.2}$$

From Lemma 3.1, we can estimate an upper bound of the following large deviation probability.

Lemma 3.2. *Let* $\{\xi_j\}$ *and* $\sigma(\cdot)$ *be as in* Theorem 2.1 *with condition* (2.2). *Then, for any* $0 \le \varepsilon < 1$, *there exists a positive constant* C_ε *depending only on* ε *such that*

$$P\bigg\{\sup_{(i,j)\in\mathbb{D}_{k,l}} \frac{|S_{i+j} - S_i|}{\sigma(\theta^l)} \ge u\bigg\} \le C_\varepsilon \theta^{k-l} \exp\Big(-\frac{(1-\varepsilon)u^2}{2+\varepsilon}\Big)$$

for all $u > 1$.

Proof. Set

$$X(i,j) = \frac{S_{i+j} - S_i}{\sigma(\theta^l)}, \qquad (i,j) \in \mathbb{D}_{k,l}$$

and

$$\varphi(z) = \frac{2\sigma(\sqrt{2}z)}{\sigma(\theta^l)}, \qquad z > 0.$$

Clearly, $EX(i,j) = 0$ and $\Gamma = \sup_{(i,j)\in\mathbb{D}_{k,l}} \sqrt{EX^2(i,j)} = 1$. For $(i,j) \ne (i',j') \in \mathbb{D}_{k,l}$, we have

$$\begin{aligned}
E\{X(i,j) - X(i',j')\}^2 &= \frac{1}{\sigma^2(\theta^l)} E\{S_{i+j} - S_i - (S_{i'+j'} - S_{i'})\}^2 \\
&\le \frac{2}{\sigma^2(\theta^l)} E\{(S_{i+j} - S_{i'+j'})^2 + (S_i - S_{i'})^2\} \\
&\le \frac{4}{\sigma^2(\theta^l)} \sigma^2\big(\sqrt{2}\sqrt{(i-i')^2 + (j-j')^2}\,\big) \\
&= \varphi^2(\|(i,j) - (i',j')\|).
\end{aligned}$$

Also, by (2.2) and (3.2), we have

$$P\left\{\frac{|X(i,j)|}{\sigma(\|(i,j)\|)} \ge x\right\} \le P\bigg\{\frac{|S_{i+j} - S_i|}{\sigma(j)} \ge \sigma(1)\,x\bigg\} \le c_2\, e^{-(1-\varepsilon)x^2/2}$$

for all $x \ge 1$, since $\sigma(\|(i,j)\|) \ge \sigma(1) = 1$. Therefore, $X(i,j)$ defined above satisfies all the conditions of Lemma 3.1.

On the other hand, noting that $\sigma(\cdot)$ is regularly varying, for any $\varepsilon > 0$ there exists a constant $c_\varepsilon > 0$ such that

$$K_1 \int_0^\infty \varphi(\sqrt{2}\, c_\varepsilon\, \theta^l\, 2^{-y^2})\, dy < \varepsilon/8$$

for all $l \geq 1$, where $K_1 > 2(\sqrt{2}+1)(1+4(1-\varepsilon)^{-1}\log 2)^{1/2}$. Set $u = x(1+\varepsilon/8)$ for $x \geq 1$. Then it follows from Lemma 3.1 that

$$\begin{aligned}
&P\Big\{ \sup_{(i,j)\in\mathbb{D}_{k,l}} \frac{|S_{i+j}-S_i|}{\sigma(\theta^l)} \geq u \Big\} = P\Big\{ \sup_{(i,j)\in\mathbb{D}_{k,l}} |X(i,j)| \geq u \Big\} \\
&\leq P\Big\{ \sup_{(i,j)\in\mathbb{D}_{k,l}} |X(i,j)| \geq x\left(1 + K_1 \int_0^\infty \varphi(\sqrt{2}\, c_\varepsilon \theta^l\, 2^{-y^2})\, dy\right) \Big\} \\
&\leq C_\varepsilon \theta^{k-l} \exp\left(- \frac{(1-\varepsilon)u^2}{2+\varepsilon} \right),
\end{aligned}$$

where C_ε is a positive constant. This completes the proof of Lemma 3.2. □

Proof of Theorem 2.1. For $\theta > 1$, let

$$\mathbb{A}_{k,l} = \{ n : \theta^{k-1} \leq n \leq \theta^k,\ \theta^{l-1} \leq a_n \leq \theta^l \}, \quad k \geq 1,\ l \geq 1.$$

By condition (i), we have $1 \leq l \leq k-1$ and

$$\begin{aligned}
\inf_{n\in\mathbb{A}_{k,l}} \beta_n &\geq \big\{ 2\log\big((\theta^{k-1}/\theta^l)\log\theta^{k-1}\big)\big\}^{1/2} \\
&\geq \theta^{-1}\big\{ 2\log\big((\theta^k/\theta^l)\log\theta^k\big)\big\}^{1/2} \\
&=: \theta^{-1}\beta_{kl}
\end{aligned} \tag{3.3}$$

for all large k. Hence, by (i) and the regularity of $\sigma(\cdot)$,

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{0\leq i\leq n} \sup_{1\leq j\leq a_n} \frac{|S_{i+j}-S_i|}{\sigma(a_n)\beta_n} \\
&\quad\leq \limsup_{k\to\infty} \sup_{1\leq l\leq k-1} \sup_{n\in\mathbb{A}_{k,l}} \sup_{0\leq i\leq n} \sup_{1\leq j\leq a_n} \frac{|S_{i+j}-S_i|}{\sigma(a_n)\beta_n} \\
&\quad\leq \theta^2 \limsup_{k\to\infty} \sup_{1\leq l\leq k-1} \sup_{0\leq i\leq\theta^k} \sup_{1\leq j\leq\theta^l} \frac{|S_{i+j}-S_i|}{\sigma(\theta^l)\beta_{kl}}.
\end{aligned} \tag{3.4}$$

Now, applying Lemma 3.2, it follows that, for any small $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} P\Big\{ & \sup_{1\le l\le k-1}\sup_{0\le i\le\theta^k}\sup_{1\le j\le\theta^l}\frac{|S_{i+j}-S_i|}{\sigma(\theta^l)\beta_{kl}} \ge \sqrt{1+2\varepsilon}\Big\} \\ &\le \sum_{l=1}^{k-1} P\Big\{\sup_{0\le i\le\theta^k}\sup_{1\le j\le\theta^l}\frac{|S_{i+j}-S_i|}{\sigma(\theta^l)} \ge \sqrt{1+2\varepsilon}\,\beta_{kl}\Big\} \\ &\le C_\varepsilon \sum_{l=1}^{k-1}\theta^{k-l}\exp\Big(-\frac{2+4\varepsilon}{2+\varepsilon}(1-\varepsilon)\log(\theta^{k-l}\log\theta^k)\Big) \\ &\le C_\varepsilon\, k^{-1-\varepsilon'} \end{aligned}$$

for all large k, where $\varepsilon' = \varepsilon/(4+2\varepsilon)$. By the Borel-Cantelli lemma, we get

$$\limsup_{k\to\infty}\sup_{1\le l\le k-1}\sup_{0\le i\le\theta^k}\sup_{1\le j\le\theta^l}\frac{|S_{i+j}-S_i|}{\sigma(\theta^l)\beta_{kl}} \le 1 \qquad \text{a.s.}$$

Combining this inequality with (3.4) yields (2.3) by the arbitrariness of θ. This completes the proof. □

The following Lemma 3.3 is a well-known version of the second Borel-Cantelli lemma, which is used to prove Theorem 2.2.

Lemma 3.3. *Let* $\{A_k,\ k\ge 1\}$ *be any sequence of events in* $(\Omega, \mathcal{F}, P)$. *If*

(a) $\displaystyle\sum_{k=1}^{\infty} P(A_k) = \infty$

and

(b) $\displaystyle\liminf_{n\to\infty}\sum_{1\le j<k\le n}\frac{P(A_j\cap A_k)-P(A_j)P(A_k)}{\big(\sum_{j=1}^n P(A_j)\big)^2} \le 0,$

then $P(\limsup_{k\to\infty} A_k) = 1$.

Proof of Theorem 2.2. Let $\{n_k\}_{k=1}^\infty$ be a subsequence of $\{n\}_{n=1}^\infty$ such that $n_1 = 1$ and $n_k = n_{k-1} + a_{n_{k-1}}$ $(k\ge 2)$, due to (iii). Set

$$Z_k = \frac{S_{n_k+a_{n_k}} - S_{n_k}}{\sigma(a_{n_k})} \quad \text{and} \quad A_k = \{Z_k > (1-\varepsilon)\beta_{n_k}\}$$

for $0<\varepsilon<1$. Then, by (2.2),

$$\begin{aligned} \sum_{k=1}^\infty P(A_k) &\ge c_1\sum_{k=1}^\infty \exp\Big(-(1+\varepsilon)(1-\varepsilon)^2\log\Big(\frac{n_k\log n_k}{a_{n_k}}\Big)\Big) \\ &\ge c_1\sum_{k=1}^\infty\frac{n_{k+1}-n_k}{n_k\log n_k} \ge c_1\sum_{k=1}^\infty\int_{n_k}^{n_{k+1}}\frac{1}{x\log x}\,dx = \infty. \end{aligned}$$

Now, it suffices to show that condition (b) of Lemma 3.3 is satisfied. By the definition of $\{n_k\}$, two sets $\{n_i+1, n_i+2, \cdots, n_i+a_{n_i}\}$ and $\{n_j+1, n_j+2, \cdots, n_j+a_{n_j}\}$ for $i<j$, are disjoint. So, by condition (ii),

$$\begin{aligned}
&P(A_i \cap A_j)\\
&= P\left\{\frac{\xi_{n_i+1}+\cdots+\xi_{n_i+a_{n_i}}}{\sigma(a_{n_i})\beta_{n_i}} > 1-\varepsilon,\ \frac{\xi_{n_j+1}+\cdots+\xi_{n_j+a_{n_j}}}{\sigma(a_{n_j})\beta_{n_j}} > 1-\varepsilon\right\}\\
&\le P\left\{\frac{\xi_{n_i+1}+\cdots+\xi_{n_i+a_{n_i}}}{\sigma(a_{n_i})\beta_{n_i}} > 1-\varepsilon\right\} P\left\{\frac{\xi_{n_j+1}+\cdots+\xi_{n_j+a_{n_j}}}{\sigma(a_{n_j})\beta_{n_j}} > 1-\varepsilon\right\}\\
&= P(A_i)P(A_j).
\end{aligned}$$

This implies the condition (b) of Lemma 3.3 and hence (2.5) holds true. □

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