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FIXED POINT THEORY IN FRÉCHET SPACES FOR MÖNCH INWARD AND CONTRACTIVE URYSOHN TYPE OPERATORS

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Abstract. We present new fixed point theorems for inward and weakly inward Urysohn type maps. Also we discuss Mönch Kakutani and contractive type maps.

1. Introduction

This paper presents new fixed point theorems for multivalued maps of Urysohn type between Fréchet spaces. In particular we present new fixed point theorems for weakly inward Kakutani maps and new Leray-Schauder alternatives for inward acyclic and approximable Urysohn type maps and weakly inward Kakutani maps in Fréchet spaces. Also we obtain an applicable Leray-Schauder alternative in Fréchet spaces for Kakutani Mönch type operators. Finally contractive maps will also be discussed. The proofs rely on fixed point theory in Banach spaces and viewing a Fréchet space as the projective limit of a sequence of Banach spaces. In particular our theory is partly motivated by the papers [1, 2, 4, 5, 11].

For the remainder of this section we present some definitions and some known facts. Let X and Y be subsets of Hausdorff topological vector spaces E_1 and E_2 respectively. We will look at maps $F : X \to K(Y)$; here $K(Y)$ denotes the family of nonempty compact subsets of Y. We say $F : X \to K(Y)$ is *Kakutani* if F is upper semicontinuous with convex values. A nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now $F : X \to K(Y)$ is *acyclic* if F is upper semicontinuous with acyclic values.

Given two open neighborhoods U and V of the origins in E_1 and E_2 repectively, a (U, V)–approximate continuous selection of $F : X \to K(Y)$ is a continuous function $s : X \to Y$ satisfying

$$s(x) \in (F[(x+U)\cap X] + V) \cap Y \quad \text{for every} \quad x \in X.$$

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We say $F : X \to K(Y)$ is *approximable* if it is a closed map and if its restriction $F|_K$ to any compact subset K of X admits a (U, V)–approximate continuous selection for every open neighborhood U and V of the origins in E_1 and E_2 repectively.

Let Q be a subset of a Hausdorff topological space X and $x \in X$. The *inward set* $I_Q(x)$ is defined by

$$I_Q(x) = \{x + r\,(y - x) :\ y \in Q,\ r \geq 0\}.$$

If Q is convex and $x \in Q$ then

$$I_Q(x) = x + \{r\,(y - x) :\ y \in Q,\ r \geq 1\}.$$

A mapping $F : Q \to 2^X$ (here 2^X denotes the family of all nonempty subsets of X) is said to be weakly inward with respect to Q if $F(x) \cap \overline{I_Q(x)} \neq \emptyset$ for $x \in Q$.

Existence in Section 2 is based on the following continuation theory for $Ac\,Ap$ maps. A map is said to be $Ac\,Ap$ if it is either acyclic or approximable. In our next definitions E is a Banach space, C a closed convex subset of E and U_0 a bounded open subset of E. We will let $U = U_0 \cap C$ and $0 \in U$. In our definitions $\overline{U}$ and ∂U denote the closure and the boundary of U in C respectively.

Definition 1.1. We say $F \in A(\overline{U}, E)$ if $F : \overline{U} \to K(E)$ is a closed $Ac\,Ap$ countably condensing map with $F(x) \subseteq I_C(x)$ for $x \in \overline{U}$.

Definition 1.2. A map $F \in A_{\partial U}(\overline{U}, E)$ if $F \in A(\overline{U}, E)$ with $x \notin F\,x$ for $x \in \partial U$.

Definition 1.3. A map $F \in A_{\partial U}(\overline{U}, E)$ is *essential* in $A_{\partial U}(\overline{U}, E)$ if for every $G \in A_{\partial U}(\overline{U}, E)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in G\,x$.

The following result was established in [10].

Theorem 1.1. *Let E, C, U_0, U be as above, $0 \in U$ and $F \in A(\overline{U}, E)$ with*

$$x \notin \lambda\,F x \quad \text{for } x \in \partial U \quad \text{and} \quad \lambda \in (0, 1]. \tag{1.1}$$

Then F is essential in $A_{\partial U}(\overline{U}, E)$.

Remark 1.1. The proof of Theorem 1.1 is based on the fact that the zero map is essential in $A_{\partial U}(\overline{U}, E)$ and $F \cong 0$ in $A_{\partial U}(\overline{U}, E)$.

If the map F in Theorem 1.1 was Kakutani then in fact we can obtain more general results. The following result can be found in [6, 9].

Theorem 1.2. *Let E be a Banach space and C a closed bounded convex subset of E. Suppose $F : C \to CK(E)$ is a upper semicontinuous condensing map with $F(x) \cap \overline{I_C(x)} \neq \emptyset$ for $x \in C$; here $CK(E)$ denotes the family of nonempty convex compact subsets of E. Then F has a fixed point in E.*

Again in our next definitions E is a Banach space, C a closed convex subset of E and U_0 a bounded open subset of E. We will let $U = U_0 \cap C$.

Definition 1.4. We say $F \in K(\overline{U}, E)$ if $F : \overline{U} \to CK(E)$ is a upper semicontinuous condensing map with $F(x) \cap \overline{I_C(x)} \neq \emptyset$ for $x \in \overline{U}$.

Definition 1.5. A map $F \in K_{\partial U}(\overline{U}, E)$ if $F \in K(\overline{U}, E)$ with $x \notin F\,x$ for $x \in \partial U$.

Definition 1.6. A map $F \in K_{\partial U}(\overline{U}, E)$ is *essential* in $K_{\partial U}(\overline{U}, E)$ if for every $G \in K_{\partial U}(\overline{U}, E)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in G\,x$.

Definition 1.7. Two maps $F, G \in K_{\partial U}(\overline{U}, E)$ are *homotopic* in $K_{\partial U}(\overline{U}, E)$, written $F \cong G$ in $K_{\partial U}(\overline{U}, E)$, if there exists a upper semicontinuous condensing map $N : \overline{U} \times [0,1] \to CK(E)$ such that $N_t(u) = N(t,u) : \overline{U} \to CK(E)$ belongs to $K_{\partial U}(\overline{U}, E)$ for each $t \in [0,1]$ and $N_0 = F$, $N_1 = G$.

The topological transversality theorem for weakly inward Kakutani maps was established in [9].

Theorem 1.3. *Let E, C, U_0 and U be as above. Suppose F and G are maps in $K_{\partial U}(\overline{U}, E)$ with $F \cong G$ in $K_{\partial U}(\overline{U}, E)$. Then F is essential in $K_{\partial U}(\overline{U}, E)$ iff G is essential in $K_{\partial U}(\overline{U}, E)$.*

Remark 1.2. If the map F in Definition 1.4 (and throughout) was countably condensing instead of condensing then we have to assume $F(x) \cap I_C(x) \neq \emptyset$ for $x \in \overline{U}$ instead of $F(x) \cap \overline{I_C(x)} \neq \emptyset$ for $x \in \overline{U}$ in Definition 1.4 (and throughout); see [10] for details.

Remark 1.3. If $0 \in U$ then the zero map is essential in $K_{\partial U}(\overline{U}, E)$; see [10] for details (the proof uses Theorem 1.2).

The following Krasnoselskii type result was established in [9] (there is also an obvious analogue for countably condensing maps if we note Remark 1.2).

Theorem 1.4. *Let E be a Banach space, C a closed convex subset of E, W and V are open bounded subsets of E with $U_1 = W \cap C$ and $U_2 = V \cap C$. Suppose $0 \in U_1 \subseteq \overline{U_1} \subseteq U_2$ and $F : \overline{U_2} \to CK(E)$ a upper semicontinuous, condensing, weakly inward with respect to C (i.e. $F(x) \cap \overline{I_C(x)} \neq \emptyset$ for $x \in \overline{U_2}$) map. In addition assume the following conditions are satisfied:*

$$x \notin \lambda F\,x \quad \text{for } x \in \partial U_2 \text{ and } \lambda \in [0,1] \tag{1.2}$$

$$\exists\, v \in C \setminus \{0\} \text{ with } x \notin F\,x + \delta\, v \text{ for } \delta \geq 0 \text{ and } x \in \partial U_1 \tag{1.3}$$

$$\begin{cases} F(\,.\,) + \mu\, v : \overline{U_1} \to CK(E) \text{ is a weakly inward with respect} \\ \text{to } C \text{ (i.e. } [F(x) + \mu\, v] \cap \overline{I_C(x)} \neq \emptyset \text{ for } x \in \overline{U_1}) \\ \text{map for all } \mu \geq 0. \end{cases} \tag{1.4}$$

Then F has a fixed point in $\overline{U_2} \setminus U_1$.

In this paper we also discuss Mönch type compactness conditions instead of countable condensing. In Section 2 one of our results will be based on a Leray–Schauder alternative for Kakutani Mönch maps [1, 13] which we state here for the convenience of the reader.

Theorem 1.5. *Let K be a closed convex subset of a Banach space X, U a relatively open subset of K, $x_0 \in U$ and suppose $F : \overline{U} \to CK(K)$ is a upper semicontinuous map. Also assume the following conditions hold:*

$$\left\{ \begin{array}{l} M \subseteq \overline{U},\ M \subseteq co\,(\{x_0\} \cup F(M)) \ \text{ with } \ \overline{M} = \overline{C} \ \text{ and} \\ C \subseteq M \ \text{ countable, implies } \ \overline{M} \ \text{ is compact} \end{array} \right. \tag{1.5}$$

and

$$x \notin (1-\lambda)\,\{x_0\} + \lambda\,F\,x \quad \text{for} \quad x \in \overline{U} \setminus U \quad \text{and} \quad \lambda \in (0,1). \tag{1.6}$$

Then there exists a compact set $\sum$ of $\overline{U}$ and a $x \in \sum$ with $x \in F\,x$.

Also in Section 2 we will discuss inward Kakutani Mönch maps. In our next definition and theorem E is a Banach space, C a closed convex subset of E and U_0 a bounded open subset of E. We will let $U = U_0 \cap C$ and $0 \in U$. In our definitions $\overline{U}$ and ∂U denote the closure and the boundary of U in C respectively.

Definition 1.8. We say $F \in KM(\overline{U}, E)$ if $F : \overline{U} \to CK(E)$ is upper semicontinuous, $F(\overline{U})$ is bounded, $F(x) \subseteq \overline{I_C(x)}$ for $x \in \overline{U}$, and if $D \subseteq E$ with $D \subseteq co\,(\{0\} \cup F(D \cap U))$ and $\overline{D} = \overline{B}$ with $B \subseteq D$ countable then $\overline{D \cap U}$ is compact.

The following theorem [2, 12] will be needed in Section 2.

Theorem 1.6. *Let E, C, U_0, U be as before Definition 1.8, $0 \in U$ and $F \in KM(\overline{U}, E)$ with*

$$x \notin \lambda\,F x \quad \text{for} \quad x \in \partial U \quad \text{and} \quad \lambda \in (0,1) \tag{1.7}$$

holding. Then there exists a compact set $\sum$ of $\overline{U}$ and a $x \in \sum$ with $x \in F\,x$.

Finally in Section 2 we consider contractive type maps. We recall the following two results from the literature [3, 8].

Theorem 1.7 ([8, Theorem 3.9])**.** *Let U be an open subset in a Banach space $(X, \|\,.\,\|)$ and $F : \overline{U} \to X$. Assume $0 \in U$ and suppose there exists a continuous nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that $\|F\,x - F\,y\| \le \phi(\|x - y\|)$ for all $x,\, y \in \overline{U}$. In addition assume $F(\overline{U})$ is bounded and $x \neq \lambda\,F\,x$ for $x \in \partial U$ and $\lambda \in (0,1)$. Then F has a fixed point in $\overline{U}$.*

Theorem 1.8 ([3, Theorem 2.3 (and Remark 2.1)])**.** *Let U be an open subset in a Banach space $(X, \|\,.\,\|)$ and $F : \overline{U} \to C(X)$ a closed map (i.e. has closed graph); here $C(X)$ denotes the family of nonempty closed subsets of*

X. Assume $0 \in U$ and suppose there exists a continuous strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that $H(F\,x, F\,y) \le \phi(\|x-y\|)$ for all $x, y \in \overline{U}$. In addition assume the following conditions hold:

$$\text{(1.8)} \qquad \begin{cases} \Phi : [0,\infty) \to [0,\infty), \quad \textit{given by} \;\; \Phi(x) = x - \phi(x), \\ \textit{is strictly increasing} \end{cases}$$

$$\text{(1.9)} \qquad \Phi^{-1}(a) + \Phi^{-1}(b) \le \Phi^{-1}(a+b) \quad \textit{for} \;\; a,\, b \ge 0$$

$$\text{(1.10)} \qquad \sum_{i=0}^{\infty} \phi^i(t) < \infty \quad \textit{for} \;\; t > 0$$

$$\text{(1.11)} \qquad \sum_{i=1}^{\infty} \phi^i(x - \phi(x)) \le \phi(x) \quad \textit{for} \;\; x > 0$$

$$\text{(1.12)} \qquad F(\overline{U}) \;\; \textit{is bounded}$$

and

$$\text{(1.13)} \qquad x \notin \lambda\, F\, x \;\; \textit{for} \;\; x \in \partial U \;\; \textit{and} \;\; \lambda \in (0,1).$$

Then F has a fixed point in $\overline{U}$.

Remark 1.4. In fact the assumption that F is closed can be removed in Theorem 1.8. In [3, Theorem 2.3] we assumed a more general contractive condition and the condition is needed there.

Let (X,d) be a metric space and S a nonempty subset of X. For $x \in X$ let $d(x,S) = \inf_{y \in S} d(x,y)$. Now suppose $G : S \to 2^X$. Then G is said to be hemicompact if each sequence $\{x_n\}_{n \in N}$ in S has a convergent subsequence whenever $d(x_n, G(x_n)) \to 0$ as $n \to \infty$.

Now let I be a directed set with order $\le$ and let $\{E_\alpha\}_{\alpha \in I}$ be a family of locally convex spaces. For each $\alpha \in I$, $\beta \in I$ for which $\alpha \le \beta$ let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{ x = (x_\alpha) \in \prod_{\alpha \in I} E_\alpha :\; x_\alpha = \pi_{\alpha,\beta}(x_\beta) \;\forall\, \alpha,\, \beta \in I,\, \alpha \le \beta \right\}$$

is a closed subset of $\prod_{\alpha \in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha \in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow} \{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection [9, pp. 439] $\cap_{\alpha \in I} E_\alpha$.)

2. Fixed point theory in Fréchet spaces

Let $E = (E, \{|\cdot|_n\}_{n\in N})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in N\}$; here $N = \{1, 2, ...\}$. We assume that the family of seminorms satisfies

$$|x|_1 \leq |x|_2 \leq |x|_3 \leq .... \quad \text{for every} \quad x \in E. \tag{2.1}$$

A subset X of E is bounded if for every $n \in N$ there exists $r_n > 0$ such that $|x|_n \leq r_n$ for all $x \in X$. For $r > 0$ and $x \in E$ we denote $B(x, r) = \{y \in E : |x - y|_n \leq r \, \forall n \in N\}$. To E we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in N$ we consider the equivalence relation $\sim_n$ defined by

$$x \sim_n y \quad \text{iff} \quad |x - y|_n = 0. \tag{2.2}$$

We denote by $\mathbf{E}^n = (E/\sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now since (2.1) is satisfied the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \geq n$ (again this seminorm is denoted by $|\cdot|_n$). Also (2.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m/\sim_n$ can be regarded as a subset of $\mathbf{E}_n$. Now $\mu_{n,m}\,\mu_{m,k} = \mu_{n,k}$ if $n \leq m \leq k$ and $\mu_n = \mu_{n,m}\,\mu_m$ if $n \leq m$. We now assume the following condition holds:

$$\begin{cases} \text{for each } n \in N, \quad \text{there exists a Banach space } (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } j_n : \mathbf{E}_n \to E_n. \end{cases} \tag{2.3}$$

Remark 2.1. (i). For convenience the norm on E_n is denoted by $|\cdot|_n$.
(ii). In our applications $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in N$.
(iii). Note if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$) then $x \in E$. However if $x \in E_n$ then x is not necessaily in E and in fact E_n is easier to use in applications (even though E_n is isomorphic to $\mathbf{E}_n$). For example if $E = C[0, \infty)$, then $\mathbf{E}^n$ consists of the class of functions in E which coincide on the interval $[0, n]$ and $E_n = C[0, n]$.

Finally we assume

$$\begin{cases} E_1 \supseteq E_2 \supseteq .... \quad \text{and for each} \quad n \in N, \\ |j_n\,\mu_{n,n+1}\,j_{n+1}^{-1}\,x|_n \leq |x|_{n+1} \,\forall\, x \in E_{n+1} \end{cases} \tag{2.4}$$

(here we use the notation from [9] i.e. decreasing in the generalized sense). Let $\lim_{\leftarrow} E_n$ (or $\cap_1^\infty E_n$ where $\cap_1^\infty$ is the generalized intersection [9]) denote the projective limit of $\{E_n\}_{n\in N}$ (note $\pi_{n,m} = j_n\,\mu_{n,m}\,j_m^{-1} : E_m \to E_n$ for $m \geq n$) and note $\lim_{\leftarrow} E_n \cong E$, so for convenience we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in N$ we set $X_n = j_n\,\mu_n(X)$, and we let $\overline{X_n}$, $int\,X_n$ and ∂X_n denote respectively the closure, the interior and the boundary of X_n with respect to $|\cdot|_n$ in E_n. Also the pseudo-interior of X is defined by

$$pseudo - int\,(X) = \{x \in X : \; j_n\,\mu_n(x) \in \overline{X_n} \setminus \partial X_n \quad \text{for every} \quad n \in N\}.$$

The set X is pseudo-open if $X = pseudo - int\,(X)$. For $r > 0$ and $x \in E_n$ we denote $B_n(x,r) = \{y \in E_n : |x - y|_n \leq r\}$.

We now show how easily one can extend fixed point theory in Banach spaces to applicable fixed point theory in Fréchet spaces. Our results are motivated by Urysohn type operators. In this case the map F_n will be related to F by the closure property (2.10).

Theorem 2.1. *Let E and E_n be as described in the beginning of Section 2, C a convex subset in E, V a pseudo-open bounded subset of E, $0 \in V \cap C$, and $F : Y \to 2^E$ with $Y \subseteq E$, and $\overline{U_n} = \overline{V_n \cap \overline{C_n}} \subseteq Y_n$ for each $n \in N$ (here $U_n = V_n \cap \overline{C_n}$). Also for each $n \in N$ assume $F_n : \overline{U_n} \to 2^{E_n}$ and suppose the following conditions are satisfied:*

$$\overline{U_1} \supseteq \overline{U_2} \supseteq .... \tag{2.5}$$

$$\begin{cases} \textit{for each } n \in N,\ F_n : \overline{U_n} \to K(E_n) \textit{ is a} \\ \textit{closed AcAp countably condensing map; here} \\ \overline{U_n} \textit{ denotes the closure of } U_n \textit{ in } \overline{C_n} \end{cases} \tag{2.6}$$

$$\textit{for each } n \in N,\ F_n(x) \subseteq I_{\overline{C_n}}(x) \textit{ for each } x \in \overline{U_n} \tag{2.7}$$

$$\begin{cases} \textit{for each } n \in N,\ y \notin \lambda\, F_n\, y \textit{ in } E_n \textit{ for all} \\ \lambda \in (0,1] \textit{ and } y \in \partial\, U_n; \textit{ here } \partial U_n \\ \textit{denotes the boundary of } U_n \textit{ in } \overline{C_n} \end{cases} \tag{2.8}$$

$$\begin{cases} \textit{for each } n \in N, \textit{ the map } \mathcal{K}_n : \overline{U_n} \to 2^{E_n} \textit{ given in} \\ \textit{Remark 2.2 is hemicompact} \end{cases} \tag{2.9}$$

and

$$\begin{cases} \textit{if there exists a } w \in Y \textit{ and a sequence } \{y_n\}_{n \in N} \\ \textit{with } y_n \in U_n \textit{ and } y_n \in F_n\, y_n \textit{ in } E_n \textit{ such that} \\ \textit{for every } k \in N \textit{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, ...\} \textit{ of } N \textit{ with } j_k\, \mu_{k,n}\, j_n^{-1}\,(y_n) \to w \\ \textit{in } E_k \textit{ as } n \to \infty \textit{ in } S, \textit{ then } w \in F\, w \textit{ in } E. \end{cases} \tag{2.10}$$

Then F has a fixed point in E.

Remark 2.2. The definition of $\mathcal{K}_n$ is as follows. If $y \in \overline{U_n}$ and $y \notin \overline{U_{n+1}}$ then $\mathcal{K}_n(y) = F_n(y)$. If $y \in \overline{U_{n+1}}$ and $y \notin \overline{U_{n+2}}$ then

$$\mathcal{K}_n(j_n\, \mu_{n,n+1}\, j_{n+1}^{-1}\, y) = F_n(j_n\, \mu_{n,n+1}\, j_{n+1}^{-1}\, y) \ \cup\ j_n\, \mu_{n,n+1}\, j_{n+1}^{-1}\, F_{n+1}(y)$$

whereas if $y \in \overline{U_{n+2}}$ and $y \notin \overline{U_{n+3}}$ then

$$\begin{aligned} \mathcal{K}_n(j_n\, \mu_{n,n+2}\, j_{n+2}^{-1}\, y) &= F_n(j_n\, \mu_{n,n+2}\, j_{n+2}^{-1}\, y) \\ &\cup\ j_n\, \mu_{n,n+1}\, j_{n+1}^{-1}\, F_{n+1}(j_{n+1}\, \mu_{n+1,n+2}\, j_{n+2}^{-1}\, y) \\ &\cup\ j_n\, \mu_{n,n+2}\, j_{n+2}^{-1}\, F_{n+2}(y), \end{aligned}$$

and so on.

Proof. Fix $n \in N$. We would like to apply Theorem 1.1. To do so we need to show

$$\overline{C_n} \text{ is convex} \tag{2.11}$$

and

$$V_n \text{ is a bounded open subset of } E_n \text{ and } j_n\,\mu_n\,(0) \in U_n. \tag{2.12}$$

First we check (2.11). To see this let $\hat{x},\, \hat{y} \in \mu_n(C)$ and $\lambda \in [0,1]$. Then for every $x \in \mu_n^{-1}(\hat{x})$ and $y \in \mu_n^{-1}(\hat{y})$ we have $\lambda x + (1-\lambda)y \in C$ since C is convex and so $\lambda\hat{x} + (1-\lambda)\hat{y} = \lambda\mu_n(x) + (1-\lambda)\mu_n(y)$. It is easy to check that $\lambda\mu_n(x) + (1-\lambda)\mu_n(y) = \mu_n(\lambda x + (1-\lambda)y)$ so as a result

$$\lambda\hat{x} + (1-\lambda)\hat{y} = \mu_n(\lambda x + (1-\lambda)y) \in \mu_n(C),$$

and so $\mu_n(C)$ is convex. Now since j_n is linear we have $C_n = j_n(\mu_n(C))$ is convex and as a result $\overline{C_n}$ is convex. Thus (2.11) holds.

Now since V is pseudo-open and $0 \in V$ then $j_n\,\mu_n\,(0) \in pseudo - int\,V$ so $j_n\,\mu_n(0) \in \overline{V_n} \setminus \partial V_n$ (here $\overline{V_n}$ and ∂V_n denote the closure and boundary of V_n in E_n respectively). Of course

$$\overline{V_n} \setminus \partial V_n = (V_n \cup \partial V_n) \setminus \partial V_n = V_n \setminus \partial V_n$$

so $j_n\,\mu_n\,(0) \in V_n \setminus \partial V_n$, and in particular $j_n\,\mu_n\,(0) \in V_n$ (this is easy to see anyway from the definition of V_n). Thus $j_n\,\mu_n\,(0) \in V_n \cap \overline{C_n} = U_n$. Next notice V_n is bounded since V is bounded (note if $y \in V_n$ then there exists $x \in V$ with $y = j_n\mu_n(x)$). It remains to show V_n is open. First notice $V_n \subseteq \overline{V_n} \setminus \partial V_n$ since if $y \in V_n$ then there exists $x \in V$ with $y = j_n\mu_n(x)$ and this together with $V = pseudo - int\,V$ yields $j_n\mu_n(x) \in \overline{V_n} \setminus \partial V_n$ i.e. $y \in \overline{V_n} \setminus \partial V_n$. In addition notice

$$\overline{V_n} \setminus \partial V_n = (int\,V_n \cup \partial V_n) \setminus \partial V_n = int\,V_n \setminus \partial V_n = int\,V_n$$

since $int\,V_n \cap \partial V_n = \emptyset$. Consequently

$$V_n \subseteq \overline{V_n} \setminus \partial V_n = int\,V_n, \quad \text{so} \quad V_n = int\,V_n.$$

As a result V_n is open in E_n. Thus (2.12) holds.

For each $n \in N$ (see Theorem 1.1) there exists $y_n \in U_n = V_n \cap \overline{C_n}$ with $y_n \in F_n\,y_n$. Lets look at $\{y_n\}_{n \in N}$. We claim that

$$d_1(j_1\,\mu_{1,n}\,j_n^{-1}\,(y_n), \mathcal{K}_1\,(j_1\,\mu_{1,n}\,j_n^{-1}\,(y_n))) = 0; \tag{2.13}$$

here $d_1(x, Z) = \inf_{y \in Z}\,|x - y|_1$ for $Z \subseteq E_1$. First we show (2.13) is true with $n = 1$ i.e. we show $d_1(y_1, \mathcal{K}_1(y_1)) = 0$. Note $y_1 \in U_1$. If $y_1 \notin \overline{U_2}$ then $\mathcal{K}_1(y_1) = F_1(y_1)$ so $y_1 \in F_1(y_1) = \mathcal{K}_1(y_1)$. If $y_1 \in \overline{U_2}$ and $y_1 \notin \overline{U_3}$ then

$$\mathcal{K}_1(y_1) = \mathcal{K}_1(j_1\,\mu_{1,2}\,j_2^{-1}\,(y_1)) \supseteq F_1(j_1\,\mu_{1,2}\,j_2^{-1}\,(y_1)) = F_1(y_1)$$

since $y_1 \in U_1$ and $y_1 \in \overline{U_2}$ (so we have $y_1 = j_1\,\mu_{1,2}\,j_2^{-1}\,(y_1)$). Thus $y_1 \in \mathcal{K}_1(y_1)$. If $y_1 \in \overline{U_3}$ and $y_1 \notin \overline{U_4}$ then

$$\mathcal{K}_1(y_1) = \mathcal{K}_1(j_1\,\mu_{1,3}\,j_3^{-1}\,(y_1)) \supseteq F_1(j_1\,\mu_{1,3}\,j_3^{-1}\,(y_1)) = F_1(y_1)$$

since $y_1 \in U_1$. Thus $y_1 \in \mathcal{K}_1(y_1)$. Continue this process and we see that (2.13) is true when $n = 1$. Next we show (2.13) is true with $n = 2$ i.e. we show $d_1(j_1\,\mu_{1,2}\,j_2^{-1}(y_2), \mathcal{K}_1(j_1\,\mu_{1,2}\,j_2^{-1}(y_2))) = 0$. Note $y_2 \in F_2(y_2)$ so $j_1\,\mu_{1,2}\,j_2^{-1}(y_2) \in j_1\,\mu_{1,2}\,j_2^{-1}\,F_2(y_2)$. Note $y_2 \in U_2$. If $y_2 \notin \overline{U_3}$ then

$$\mathcal{K}_1(j_1\,\mu_{1,2}\,j_2^{-1}(y_2)) \supseteq j_1\,\mu_{1,2}\,j_2^{-1}\,F_2(y_2)$$

and so

$$j_1\,\mu_{1,2}\,j_2^{-1}(y_2) \in j_1\,\mu_{1,2}\,j_2^{-1}\,F_2(y_2) \subseteq \mathcal{K}_1(j_1\,\mu_{1,2}\,j_2^{-1}(y_2)).$$

If $y_2 \in \overline{U_3}$ and $y_2 \notin \overline{U_4}$ then

$$\mathcal{K}_1(j_1\,\mu_{1,3}\,j_3^{-1}(y_2)) \supseteq j_1\,\mu_{1,2}\,j_2^{-1}\,F_2(j_2\,\mu_{2,3}\,j_3^{-1}(y_2))$$

and since $y_2 \in U_2$ and $y_2 \in \overline{U_3}$ we have $y_2 = j_2\,\mu_{2,3}\,j_3^{-1}(y_2)$ so

$$\begin{aligned} j_1\,\mu_{1,2}\,j_2^{-1}(y_2) &= j_1\,\mu_{1,2}\,j_2^{-1}\,j_2\,\mu_{2,3}\,j_3^{-1}(y_2) = j_1\,\mu_{1,2}\,\mu_{2,3}\,j_3^{-1}(y_2) \\ &= j_1\,\mu_{1,3}\,j_3^{-1}(y_2). \end{aligned}$$

Thus

$$\mathcal{K}_1(j_1\,\mu_{1,2}\,j_2^{-1}(y_2)) \supseteq j_1\,\mu_{1,2}\,j_2^{-1}\,F_2(y_2)$$

so $j_1\,\mu_{1,2}\,j_2^{-1}(y_2) \in \mathcal{K}_1(j_1\,\mu_{1,2}\,j_2^{-1}(y_2))$. Continue this process and we see that (2.13) is true when $n = 2$. Proceed as above and it is easy to see that (2.13) is true for $n \in N$.

Now (2.9) (with $n = 1$) guarantees that there exists a subsequence $N_1^\star$ of N and a $z_1 \in \overline{U_1}$ with $j_1\,\mu_{1,n}\,j_n^{-1}(y_n) \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Look at $\{y_n\}_{n \in N_1}$. Now as above it is easy to see for $n \in N_1$ that

$$d_2(j_2\,\mu_{2,n}\,j_n^{-1}(y_n), \mathcal{K}_2\,(j_2\,\mu_{2,n}\,j_n^{-1}(y_n))) = 0;$$

here $d_2(x, Z) = \inf_{y \in Z}\,|x - y|_2$ for $Z \subseteq E_2$. Also there exists a subsequence $N_2^\star$ of N_1 and a $z_2 \in \overline{U_2}$ with $j_2\,\mu_{2,n}\,j_n^{-1}(y_n) \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$. Note from (2.4) and the uniqueness of limits that $j_1\,\mu_{1,2}\,j_2^{-1}\,z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1$ (note $j_1\,\mu_{1,n}\,j_n^{-1}(y_n) = j_1\,\mu_{1,2}\,j_2^{-1}\,j_2\,\mu_{2,n}\,j_n^{-1}(y_n)$ for $n \in N_2^\star$). Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \ldots\ldots, \quad N_k^\star \subseteq \{k, k+1, \ldots.\}$$

and $z_k \in \overline{U_k}$ with $j_k\,\mu_{k,n}\,j_n^{-1}(y_n) \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \ldots\}$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Note

$$\begin{aligned} z_k &= j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,z_{k+1} = j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,j_{k+1}\,\mu_{k+1,k+2}\,j_{k+2}^{-1}\,z_{k+2} \\ &= j_k\,\mu_{k,k+2}\,j_{k+2}^{-1}\,z_{k+2} = \ldots.. = j_k\,\mu_{k,m}\,j_m^{-1}\,z_m = \pi_{k,m}\,z_m \end{aligned}$$

for every $m \geq k$. We can do this for each $k \in N$. As a result $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and also note $y \in Y$ since $z_k \in \overline{U_k} \subseteq Y_k$ for each $k \in N$. Also since $y_n \in F_n\,y_n$ in E_n for $n \in N_k$ and $j_k\,\mu_{k,n}\,j_n^{-1}(y_n) \to z_k = y$ in E_k as $n \to \infty$ in N_k we have from (2.10) that $y \in F\,y$ in E. $\square$

Remark 2.3. Note we could replace $\overline{U_n} \subseteq Y_n$ above with $\overline{U_n}$ a subset of the closure of Y_n in E_n if Y is a closed subset of E (so in this case we can take $Y = C \cap \overline{V}$ if $\overline{\overline{C_n} \cap V_n}$ is a subset of the closure of $j_n\, \mu_n\, (C \cap \overline{V})$ in E_n and if C is closed). To see this note $z_k \in \overline{U_k}$, $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and $\pi_{k,m}\,(y_m) \to z_k$ in E_k as $m \to \infty$ and we can conclude that $y \in \overline{Y} = Y$ (note $q \in \overline{Y}$ iff for every $k \in N$ there exists $(x_{k,m}) \in Y$, $x_{k,m} = \pi_{k,n}\,(x_{n,m})$ for $n \geq k$ with $x_{k,m} \to j_k\, \mu_k\,(q)$ in E_k as $m \to \infty$).

Remark 2.4. Suppose in Theorem 2.1 we have

$$(2.5)^\star \qquad U_1 \supseteq U_2 \supseteq \ldots\ldots\ldots.$$

and

$$(2.9)^\star \qquad \text{for each } n \in N, \text{ the map } \mathcal{K}_n : U_n \to 2^{E_n} \text{ is hemicompact}$$

instead of (2.5) and (2.9); here if $y \in U_n$ and $y \notin U_{n+1}$ then $\mathcal{K}_n(y) = F_n(y)$ whereas if $y \in U_{n+1}$ and $y \notin U_{n+2}$ then

$$\mathcal{K}_n(j_n\, \mu_{n,n+1}\, j_{n+1}^{-1}\, y) = F_n(j_n\, \mu_{n,n+1}\, j_{n+1}^{-1}\, y) \;\cup\; j_n\, \mu_{n,n+1}\, j_{n+1}^{-1}\, F_{n+1}(y)$$

and so on. In addition we assume $F : Y \to 2^E$ with $\overline{U_n} \subseteq Y_n$ for each $n \in N$ is replaced by $F : Y \to 2^E$ with $U_n \subseteq Y_n$ for each $n \in N$. Then the result in Theorem 2.1 is again true.

The proof follows the reasoning in Theorem 2.1 except in this case $z_k \in U_k$.

Next we present a result for weakly inward Kakutani maps using Theorem 1.2.

Theorem 2.2. *Let E and E_n be as described in the beginning of Section 2, C a convex bounded subset in E, $F : Y \to 2^E$ with $Y \subseteq E$, and $\overline{C_n} \subseteq Y_n$ for each $n \in N$. Also for each $n \in N$ assume $F_n : \overline{C_n} \to 2^{E_n}$ and suppose the following conditions are satisfied:*

$$(2.14) \qquad \overline{C_1} \supseteq \overline{C_2} \supseteq \ldots\ldots\ldots.$$

$$(2.15) \qquad \begin{cases} \text{for each } n \in N,\ F_n : \overline{C_n} \to CK(E_n) \text{ is a} \\ \text{upper semicontinuous condensing map} \end{cases}$$

$$(2.16) \qquad \text{for each } n \in N,\ F_n(x) \cap \overline{I_{\overline{C_n}}(x)} \neq \emptyset \text{ for } x \in \overline{C_n}$$

$$(2.17) \qquad \begin{cases} \text{for each } n \in N, \text{ the map } \mathcal{K}_n : \overline{C_n} \to 2^{E_n} \text{ given in} \\ \text{Remark 2.5 is hemicompact} \end{cases}$$

and

$$(2.18) \qquad \begin{cases} \text{if there exists a } w \in Y \text{ and a sequence } \{y_n\}_{n \in N} \\ \text{with } y_n \in \overline{C_n} \text{ and } y_n \in F_n\, y_n \text{ in } E_n \text{ such that} \\ \text{for every } k \in N \text{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, \ldots..\} \text{ of } N \text{ with } j_k\, \mu_{k,n}\, j_n^{-1}\,(y_n) \to w \\ \text{in } E_k \text{ as } n \to \infty \text{ in } S, \text{ then } w \in F\, w \text{ in } E. \end{cases}$$

Then F has a fixed point in E.

Remark 2.5. The definition of $\mathcal{K}_n$ is as follows. If $y \in \overline{C_n}$ and $y \notin \overline{C_{n+1}}$ then $\mathcal{K}_n(y) = F_n(y)$ whereas if $y \in \overline{C_{n+1}}$ and $y \notin \overline{C_{n+2}}$ then $\mathcal{K}_n(j_n\, \mu_{n,n+1}\, j_{n+1}^{-1}\, y) = F_n(j_n\, \mu_{n,n+1}\, j_{n+1}^{-1}\, y) \;\cup\; j_n\, \mu_{n,n+1}\, j_{n+1}^{-1}\, F_{n+1}(y)$ and so on.

Proof. For each $n \in N$ there exists (Theorem 1.2) $y_n \in \overline{C_n}$ with $y_n \in F_n\, y_n$ in E_n. Essentially the same reasoning as in Theorem 2.1 establishes the result. □

Remark 2.6. Note we could replace $\overline{C_n} \subseteq Y_n$ above with $\overline{C_n}$ a subset of the closure of Y_n in E_n if Y is a closed subset of E (so in this case we can take $Y = C$ if C is a closed subset of E).

For our next definitions E and E_n are as described in the beginning of Section 2, C is a convex subset of E, V a bounded pseudo–open subset of E and $F : Y \to 2^E$ with $Y \subseteq E$. Also assume either $\overline{U_n} = \overline{V_n \cap \overline{C_n}} \subseteq Y_n$ for each $n \in N$ (here $U_n = V_n \cap \overline{C_n}$) or $\overline{U_n}$ is a subset of the closure of Y_n in E_n for each $n \in N$ (with Y a closed subset of E). In addition assume for each $n \in N$ that $F_n : \overline{U_n} \to 2^{E_n}$.

Definition 2.1. $F \in K(Y,E)$ if for each $n \in N$ we have $F_n \in K(\overline{U_n}, E_n)$ (i.e. for each $n \in N$, $F_n : \overline{U_n} \to CK(E_n)$ is a upper semicontinuous condensing map with $F_n(x) \cap \overline{I_{\overline{C_n}}(x)} \neq \emptyset$ for $x \in \overline{U_n}$); here $\overline{U_n}$ denotes the closure of U_n in $\overline{C_n}$.

Definition 2.2. $F \in K_\partial(Y,E)$ if $F \in K(Y,E)$ and for each $n \in N$ we have $x \notin F_n\,(x)$ for $x \in \partial U_n$; here ∂U_n denotes the boundary of U_n in $\overline{C_n}$.

Definition 2.3. A map $F \in K_\partial(Y,E)$ is *essential* in $K_\partial(Y,E)$ if for each $n \in N$ we have that $F_n \in K_{\partial U_n}(\overline{U_n}, E_n)$ is essential in $K_{\partial U_n}(\overline{U_n}, E_n)$ (i.e. for each $n \in N$, every map $G \in K_{\partial U_n}(\overline{U_n}, E_n)$ with $G|_{\partial U_n} = F_n|_{\partial U_n}$ has a fixed point in $\overline{U_n} \setminus \partial U_n$).

Remark 2.7. Note if $j_n\, \mu_n(0) \in U_n$ for each $n \in N$ then $0 \in K_\partial(Y,E)$ is essential in $K_\partial(Y,E)$ by Remark 1.3.

Definition 2.4. (We assume $j_n\, \mu_n(0) \in U_n$ for each $n \in N$). $F, 0 \in K_\partial(Y,E)$ are *homotopic* in $K_\partial(Y,E)$, written $F \cong 0$ in $K_\partial(Y,E)$, if for each $n \in N$ we have $F_n \cong j_n\, \mu_n(0)$ in $K_{\partial U_n}(\overline{U_n}, E_n)$.

Theorem 2.3. *Let E and E_n be as described in the beginning of Section 2, C a convex subset in E, V a bounded pseudo–open subset of E and $F : Y \to 2^E$ with $Y \subseteq E$. Also assume either $\overline{U_n} = \overline{V_n \cap \overline{C_n}} \subseteq Y_n$ for each $n \in N$ (here $U_n = V_n \cap \overline{C_n}$) or $\overline{U_n}$ is a subset of the closure of Y_n in E_n for each $n \in N$ (with Y a closed subset of E). Suppose $0 \in V \cap C$ and for each $n \in N$ assume $F_n : \overline{U_n} \to 2^{E_n}$ and also suppose $F \in K_\partial(Y,E)$ with* (2.5) *and the following condition satisfied:*

$$F \cong 0 \quad \text{in} \quad K_\partial(Y,E). \tag{2.19}$$

Also assume (2.9) *and* (2.10) *hold. Then* F *has a fixed point in* E.

Proof. Fix $n \in N$. Now Remark 2.7 guarantees that the zero map (i.e. $G(x) = j_n\,\mu_n\,(0)$) is essential in $K_{\partial U_n}(\overline{U_n}, E_n)$ for each $n \in N$. Now Theorem 1.3 guarantees that F_n is essential in $K_{\partial U_n}(\overline{U_n}, E_n)$ so in particular there exists $y_n \in U_n$ with $y_n \in F_n\,y_n$. Essentially the same reasoning as in Theorem 2.1 (with Remark 2.3) establishes the result. □

Remark 2.8. If for each $n \in N$ the map $F_n : \overline{U_n} \to CK(E_n)$ is countably condensing instead of condensing in Definition 2.1 (and throughout) then we assume $F_n(x) \cap I_{\overline{C_n}}(x) \neq \emptyset$ for $x \in \overline{U_n}$ instead of $F_n(x) \cap \overline{I_{\overline{C_n}}(x)} \neq \emptyset$ for $x \in \overline{U_n}$ in Definition 2.1 (and throughout).

Remark 2.9. Notice $0 \in V \cap C$ and (2.19) could be replaced by $F \cong G$ in $K_\partial(Y, E)$ (of course we assume $G \in K_\partial(Y, E)$ and we must specify G_n for $n \in N$ here).

Remark 2.10. Note Remark 2.4 holds in this situation also.

Theorem 2.4. *Let* E *and* E_n *be as described in the beginning of Section 2,* C *a convex subset in* E, V *a bounded pseudo–open subset of* E *and* $F : Y \to 2^E$ *with* $Y \subseteq E$. *Also assume either* $\overline{U_n} = \overline{V_n \cap \overline{C_n}} \subseteq Y_n$ *for each* $n \in N$ *(here* $U_n = V_n \cap \overline{C_n}$*) or* $\overline{U_n}$ *is a subset of the closure of* Y_n *in* E_n *for each* $n \in N$ *(with* Y *a closed subset of* E*). Suppose* $0 \in V \cap C$ *and for each* $n \in N$ *assume* $F_n : \overline{U_n} \to 2^{E_n}$ *and also suppose* $F \in K_\partial(Y, E)$ *with* (2.5), (2.9), (2.10) *and the following condition satisfied:*

$$\text{(2.20)} \qquad \begin{cases} \text{for each } n \in N,\ y \notin \lambda\,F_n\,y \ \text{ in } E_n \text{ for all} \\ \lambda \in (0,1] \ \text{ and } \ y \in \partial\,U_n. \end{cases}$$

Then F *has a fixed point in* E.

Proof. Now (2.19) is immediate if we take for each $n \in N$, $H_n(x, \lambda) = \lambda\,F(x)$ for $(x, \lambda) \in \overline{U_n} \times [0,1]$. Our result follows from Theorem 2.3. □

Next we present a Krasnoselskii type result for weakly inward maps in the Fréchet space setting.

Theorem 2.5. *Let* E *and* E_n *be as described in the beginning of Section 2,* C *a convex subset in* E, U *and* V *are bounded pseudo–open subsets of* E *with* $0 \in U \subseteq \overline{U} \subseteq V$ *and* $F : Y \to 2^E$ *with* $Y \subseteq E$. *Also assume either* $\overline{W_n} = \overline{V_n \cap \overline{C_n}} \subseteq Y_n$ *for each* $n \in N$ *(here* $W_n = V_n \cap \overline{C_n}$*) or* $\overline{W_n}$ *is a subset of the closure of* Y_n *in* E_n *for each* $n \in N$ *(with* Y *a closed subset of* E*). Also for each* $n \in N$ *assume* $F_n : \overline{W_n} \to 2^{E_n}$ *and suppose the following conditions are satisfied:*

$$\text{(2.21)} \qquad \overline{W_1} \supseteq \overline{W_2} \supseteq \ldots\ldots\ldots\ldots$$

$$(2.22)\quad \begin{cases} \textit{for each } n \in N,\ F_n : \overline{W_n} \to CK(E_n) \textit{ is a upper} \\ \textit{semicontinuous condensing map with } F_n(x) \cap \overline{I_{\overline{C_n}}(x)} \neq \emptyset \\ \textit{for } x \in \overline{W_n};\ \textit{ here } \overline{W_n} \textit{ denotes the closure of } W_n \textit{ in } \overline{C_n} \end{cases}$$

$$(2.23)\quad \begin{cases} \textit{for each } n \in N,\ y \notin \lambda F_n y \textit{ in } E_n \textit{ for all} \\ \lambda \in [0,1] \textit{ and } y \in \partial W_n \end{cases}$$

$$(2.24)\quad \begin{cases} \textit{for each } n \in N,\ \exists v_n \in \overline{C_n} \setminus \{0\} \textit{ with } x \notin F_n x + \delta v_n \\ \textit{for } \delta \geq 0 \textit{ and } x \in \partial \Omega_n;\ \textit{ here } \Omega_n = U_n \cap \overline{C_n} \end{cases}$$

$$(2.25)\quad \begin{cases} \textit{for each } n \in N,\ F_n(\,.\,) + \mu v_n : \overline{\Omega_n} \to CK(E_n) \textit{ is} \\ \textit{weakly inward with respect to } \overline{C_n} \textit{ for all } \mu \geq 0 \\ \textit{(i.e. } [F_n(x) + \mu v_n] \cap \overline{I_{\overline{C_n}}(x)} \neq \emptyset \textit{ for } x \in \overline{\Omega_n}) \end{cases}$$

$$(2.26)\quad \begin{cases} \textit{for each } n \in N,\ \textit{ the map } \mathcal{K}_n : \overline{W_n} \to 2^{E_n} \textit{ given in} \\ \textit{Remark 2.11 is hemicompact} \end{cases}$$

$$(2.27)\quad \begin{cases} \textit{for every } k \in N \textit{ and any subsequence } A \subseteq \{k, k+1, ...\} \\ \textit{if } x \in \overline{C_n} \textit{ is such that } x \in \overline{W_n} \setminus \Omega_n \textit{ for some } n \in A \\ \textit{then there exists a } \gamma > 0 \textit{ with } |j_k\, \mu_{k,n}\, j_n^{-1}\, x|_k \geq \gamma \end{cases}$$

and

$$(2.28)\quad \begin{cases} \textit{if there exists a } w \in Y \textit{ and a sequence } \{y_n\}_{n \in N} \\ \textit{with } y_n \in \overline{W_n} \setminus \Omega_n \textit{ and } y_n \in F_n\, y_n \textit{ in } E_n \textit{ such that} \\ \textit{for every } k \in N \textit{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, ...\} \textit{ of } N \textit{ with } j_k\, \mu_{k,n}\, j_n^{-1}\,(y_n) \to w \\ \textit{in } E_k \textit{ as } n \to \infty \textit{ in } S,\ \textit{ then } w \in F\, w \textit{ in } E. \end{cases}$$

Then F has a fixed point in E.

Remark 2.11. The definition of $\mathcal{K}_n$ is as follows. If $y \in \overline{W_n}$ and $y \notin \overline{W_{n+1}}$ then $\mathcal{K}_n(y) = F_n(y)$ and so on.

Proof. Fix $n \in N$. Now $\overline{C_n}$ is convex and U_n, V_n are open bounded subsets of E_n with $j_n\, \mu_n(0) \in U_n \subseteq V_n$. It just remains to show $U_n \subseteq \overline{U_n} \subseteq V_n$. Of course since $U \subseteq \overline{U} \subseteq V$ we have

$$U_n = j_n \mu_n(U) \subseteq j_n \mu_n(\overline{U}) \subseteq j_n \mu_n(V) = V_n$$

and since $j_n \mu_n$ is continuous $U_n \subseteq j_n \mu_n(\overline{U}) \subseteq \overline{j_n\, \mu_n(U)} = \overline{U_n}$. Also we see $\overline{\mu_n(U)} \subseteq \mu_n(V)$ (note $\overline{U} \subseteq V$) so since j_n is an isometry

$$\overline{U_n} = \overline{j_n\, \mu_n(U)} = j_n\, \overline{\mu_n(U)} \subseteq j_n\, \mu_n(V) = V_n.$$

Theorem 1.4 guarantees there exists $y_n \in \overline{W_n} \setminus \Omega_n$ with $y_n \in F_n\, y_n$ in E_n. As in Theorem 2.1 there exists is a subsequence $N_1^\star$ of N and a $z_1 \in \overline{W_1}$ with $j_1\, \mu_{1,n}\, j_n^{-1}\,(y_n) \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$. Also $y_n \in \overline{W_n} \setminus \Omega_n$ together with (2.22) yields $|j_1\,\mu_{1,n}\,j_n^{-1}\,(y_n)|_1 \geq \gamma$ for $n \in N$ and so $|z_1|_1 \geq \gamma$. Let $N_1 = N_1^\star \setminus \{1\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq ......, \quad N_k^\star \subseteq \{k, k+1, ...\}$$

and $z_k \in \overline{W_k}$ with $j_k\,\mu_{k,n}\,j_n^{-1}\,(y_n) \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$. Note $j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,z_{k+1} = z_k$ in E_k for $k \in \{1, 2, ...\}$ and $|z_k|_k \geq \gamma$. Also let $N_k = N_k^\star \setminus \{k\}$. Now essentially the same reasoning as in Theorem 2.1 (with Remark 2.3) guarantees the result. □

Remark 2.12. Note (2.27) is only needed to guarantee that the fixed point y satisfies $|j_k\,\mu_k\,(y)|_k \geq \gamma$ for $k \in N$. If we assume all the conditions in Theorem 2.5 except (2.27) then again F has a fixed point in E but the above property is not guaranteed.

We next present a Mönch type result using Theorem 1.5.

Theorem 2.6. *Let E and E_n be as described in the beginning of Section 2, $X \subseteq E$ and $F : Y \to 2^E$ with $\overline{int\,X_n} \subseteq Y_n$ for each $n \in N$ or $\overline{int\,X_n}$ is a subset of the closure of Y_n in E_n for each $n \in N$ (with Y a closed subset of E). Also for each $n \in N$ assume $F_n : \overline{int\,X_n} \to 2^{E_n}$ and suppose the following conditions are satisfied:*

$$\overline{int\,X_1} \supseteq \overline{int\,X_2} \supseteq .... \tag{2.29}$$

$$x_0 \in pseudo - int\,(X) \tag{2.30}$$

$$\begin{cases} \text{for each } n \in N,\ F_n : \overline{int\,X_n} \to CK(E_n) \text{ is a upper} \\ \text{semicontinuous map} \end{cases} \tag{2.31}$$

$$\begin{cases} \text{for each } n \in N,\ M \subseteq \overline{int\,X_n} \text{ with} \\ M \subseteq co\,(\{j_n\,\mu_n(x_0)\} \cup F_n(M)) \text{ with } \overline{M} = \overline{C} \\ \text{and } C \subseteq M \text{ countable, implies } \overline{M} \text{ is compact} \end{cases} \tag{2.32}$$

$$\begin{cases} \text{for each } n \in N,\ y \notin (1-\lambda)\,j_n\,\mu_n(x_0) + \lambda\,F_n\,y \text{ in } E_n \\ \text{for all } \lambda \in (0,1] \text{ and } y \in \partial\,int\,X_n \end{cases} \tag{2.33}$$

$$\begin{cases} \text{for each } n \in N, \text{ the map } \mathcal{K}_n : \overline{int\,X_n} \to 2^{E_n} \text{ given in} \\ \text{Remark 2.13 is hemicompact} \end{cases} \tag{2.34}$$

and

$$\begin{cases} \text{if there exists a } w \in Y \text{ and a sequence } \{y_n\}_{n \in N} \\ \text{with } y_n \in int\,X_n \text{ and } y_n \in F_n\,y_n \text{ in } E_n \text{ such that} \\ \text{for every } k \in N \text{ there exists a subsequence} \\ S \subseteq \{k+1, k+2, ...\} \text{ of } N \text{ with } j_k\,\mu_{k,n}\,j_n^{-1}\,(y_n) \to w \\ \text{in } E_k \text{ as } n \to \infty \text{ in } S, \text{ then } w \in F\,w \text{ in } E. \end{cases} \tag{2.35}$$

Then F has a fixed point in E.

Remark 2.13. The definition of $\mathcal{K}_n$ is as follows. If $y \in \overline{int\, X_n}$ and $y \notin \overline{int\, X_{n+1}}$ then $\mathcal{K}_n(y) = F_n(y)$ and so on.

Remark 2.14. Suppose in Theorem 2.6 we have

$$(2.29)^\star \qquad int\, X_1 \supseteq int\, X_2 \supseteq ....$$

and

$$(2.34)^\star \qquad \text{for each } n \in N, \text{ the map } \mathcal{K}_n : int\, X_n \to 2^{E_n} \text{ is hemicompact}$$

instead of (2.29) and (2.34); here if $y \in int\, X_n$ and $y \notin int\, X_{n+1}$ then $\mathcal{K}_n(y) = F_n(y)$ and so on. In addition we assume $F : Y \to 2^E$ with $\overline{int\, X_n} \subseteq Y_n$ (or $\overline{int\, X_n}$ is a subset of the closure of Y_n in E_n if Y is a closed subset of E) for each $n \in N$ is replaced by $F : X \to 2^E$ and suppose (2.35) is true with $w \in Y$ replaced by $w \in X$. Then the result in Theorem 2.6 is again true.

Also we have the following result for Mönch inward type maps (just apply Theorem 1.6 in this case).

Theorem 2.7. *Let E and E_n be as described in the beginning of Section 2, C a convex subset in E, V a pseudo-open bounded subset of E, $0 \in V \cap C$, and $F : Y \to 2^E$ with $Y \subseteq E$, and $\overline{U_n} = \overline{V_n \cap \overline{C_n}} \subseteq Y_n$ for each $n \in N$ (here $U_n = V_n \cap \overline{C_n}$) or $\overline{U_n}$ is a subset of the closure of Y_n in E_n (with Y a closed subset of E). Also for each $n \in N$ assume $F_n : \overline{U_n} \to 2^{E_n}$ and suppose* (2.5)*,* (2.7)*,* (2.8) *and the following conditions hold:*

$$(2.36) \qquad \begin{cases} \text{for each } n \in N,\ F_n : \overline{U_n} \to CK(E_n) \text{ is} \\ \text{upper semicontinuous and } F_n(\overline{U_n}) \text{ is bounded;} \\ \text{here } \overline{U_n} \text{ denotes the closure of } U_n \text{ in } \overline{C_n} \end{cases}$$

and

$$(2.37) \qquad \begin{cases} \text{for each } n \in N,\ D \subseteq E_n \text{ with} \\ D \subseteq co\,(\{j_n\, \mu_n(0)\} \cup F_n(D \cap U_n)) \text{ and } \overline{D} = \overline{B} \\ \text{with } B \subseteq D \text{ countable, implies } \overline{D \cap U_n} \text{ is compact.} \end{cases}$$

In addition assume (2.9) *and* (2.10) *hold. Then F has a fixed point in E.*

Remark 2.15. Note Remark 2.4 holds in this situation also.

Finally in this section we consider contractive type maps. First we consider single valued maps (just apply Theorem 1.7).

Theorem 2.8. *Let E and E_n be as described in the beginning of Section 2, $X \subseteq E$ and $F : Y \to E$ with $\overline{int\, X_n} \subseteq Y_n$ for each $n \in N$ or $\overline{int\, X_n}$ is a subset of the closure of Y_n in E_n for each $n \in N$ (with Y a closed subset of E). Also for each $n \in N$ assume $F_n : \overline{int\, X_n} \to E_n$ and suppose* (2.29) *and the following conditions are satisfied:*

$$(2.38) \qquad 0 \in pseudo - int\,(X)$$

$$(2.39) \qquad \text{for each } n \in N,\ F_n(\overline{int\, X_n}) \text{ is bounded}$$

$$
\left\{
\begin{array}{l}
\textit{for each } n \in N, \textit{ there exists a continuous} \\
\textit{nondecreasing function } \phi_n : [0,\infty) \to [0,\infty) \\
\textit{satisfying } \phi_n(z) < z \textit{ for } z > 0 \textit{ such that} \\
|F_n\, x - F_n\, y|_n \le \phi_n(|x-y|_n) \textit{ for all } x, y \in \overline{int\, X_n}
\end{array}
\right.
\tag{2.40}
$$

and

$$
\left\{
\begin{array}{l}
\textit{for each } n \in N,\ y \neq \lambda\, F_n\, y \textit{ in } E_n \textit{ for all} \\
\lambda \in (0,1] \textit{ and } y \in \partial\, int\, X_n.
\end{array}
\right.
\tag{2.41}
$$

Also assume (2.34) *and* (2.35) *(with* $y_n \in F_n\, y_n$ *and* $w \in F\, w$ *replaced by* $y_n = F_n\, y_n$ *and* $w = F\, w$*) hold. Then* F *has a fixed point in* E.

Remark 2.16. Note there is an analogue of Remark 2.14 in this situation and in the next also.

Theorem 2.9. *Let* E *and* E_n *be as described in the beginning of Section 2,* $X \subseteq E$ *and* $F : Y \to 2^E$ *with* $\overline{int\, X_n} \subseteq Y_n$ *for each* $n \in N$ *or* $\overline{int\, X_n}$ *is a subset of the closure of* Y_n *in* E_n *for each* $n \in N$ *(with* Y *a closed subset of* E*). Also for each* $n \in N$ *assume* $F_n : \overline{int\, X_n} \to 2^{E_n}$ *and suppose* (2.29), (2.38) *and the following conditions are satisfied:*

$$
\textit{for each } n \in N,\ F_n(\overline{int\, X_n}) \textit{ is bounded}
\tag{2.42}
$$

$$
\left\{
\begin{array}{l}
\textit{for each } n \in N,\ F_n : \overline{int\, X_n} \to C(E_n), \textit{ and there} \\
\textit{exists a continuous strictly increasing function} \\
\phi_n : [0,\infty) \to [0,\infty) \textit{ satisfying } \phi_n(z) < z \textit{ for } z > 0 \\
\textit{such that } H_n(F_n\, x, F_n\, y) \le \phi_n(|x-y|_n) \\
\textit{for all } x, y \in \overline{int\, X_n}
\end{array}
\right.
\tag{2.43}
$$

$$
\left\{
\begin{array}{l}
\textit{for each } n \in N, \textit{ the map } \Phi_n : [0,\infty) \to [0,\infty), \\
\textit{given by } \Phi_n(x) = x - \phi_n(x), \textit{ is strictly increasing,} \\
\Phi_n^{-1}(a) + \Phi_n^{-1}(b) \le \Phi_n^{-1}(a+b) \textit{ for } a, b \ge 0, \\
\textit{with } \sum_{i=0}^{\infty} \phi_n^i(t) < \infty \textit{ for } t > 0 \textit{ and} \\
\sum_{i=1}^{\infty} \phi_n^i(x - \phi(x)) \le \phi_n(x) \textit{ for } x > 0
\end{array}
\right.
\tag{2.44}
$$

and

$$
\left\{
\begin{array}{l}
\textit{for each } n \in N,\ y \notin \lambda\, F_n\, y \textit{ in } E_n \textit{ for all} \\
\lambda \in (0,1] \textit{ and } y \in \partial\, int\, X_n.
\end{array}
\right.
\tag{2.45}
$$

Also assume (2.34) *and* (2.35) *hold. Then* F *has a fixed point.*

References

[1] R. P. Agarwal, J. H. Dshalalow and D. O'Regan, *Fixed point theory for Mönch type maps defined on closed subsets of Fréchet spaces: the projective limit approach*, Int. Jour. Math. Math. Sciences **17** (2005), 2775–2782.

[2] R. P. Agarwal, J. H. Dshalalow and D. O'Regan, *Leray-Schauder principles for inward Kakutani Mönch type maps*, Nonlinear Functional Analysis and Applications **10** (2005), 325–330.

[3] R. P. Agarwal, J. Dshalalow and D. O'Regan, *Fixed point and homotopy results for generalized contractive maps of Reich type*, Applicable Analysis **82** (2003), 329–350.

[4] R. P. Agarwal, M. Frigon and D. O'Regan, *A survey of recent fixed point theory in Fréchet spaces*, Nonlinear analysis and applications: to V. Lakshmikantham on his 80th birthday, **Vol 1**, 75–88, Kluwer Acad. Publ., Dordrecht, 2003.

[5] R. P. Agarwal and D. O'Regan, *Fixed point theory for weakly inward Kakutani maps: the projective limit approach*, Proc. Amer. Math. Soc. **135** (2007), 417–426.

[6] K. Deimling, *Multivalued differential equations*, Walter de Gruyter, Berlin, 1992.

[7] L. V. Kantorovich and G. P. Akilov, *Functional analysis in normed spaces*, Pergamon Press, Oxford, 1964.

[8] D. O'Regan, *Fixed point theorems for nonlinear operators*, Jour. Math. Anal. Appl. **202** (1996), 413–432.

[9] D. O'Regan, *A continuation theory for weakly inward maps*, Glasgow Math. J. **40** (1998), 311–321.

[10] D. O'Regan, *Homotopy and Leray-Schauder type results for admissible inward multimaps*, Jour. Concrete Appl. Math. **2** (2004), 67–76.

[11] D. O'Regan, *An essential map approach for multimaps defined on closed subsets of Fréchet spaces*, Applicable Analysis **85** (2006), 503–513.

[12] D. O'Regan, *Leray-Schauder results for inward acyclic and approximable maps defined on Fréchet space*, Applied Math. Letters **19** (2006), 976–982.

[13] D. O'Regan and R. Precup, *Fixed point theory for set valued maps and existence principles for integral inclusions*, Jour. Math. Anal. Appl. **245** (2000), 594–612.

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