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TIETZE EXTENSION THEOREM FOR ORDERED FUZZY PRE-EXTREMALLY DISCONNECTED SPACES

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Abstract. In this paper, a new class of fuzzy topological spaces called ordered fuzzy pre-extremally disconnected spaces is introduced. Tietze extension theorem for ordered fuzzy pre-extremally disconnected spaces has been discussed as in [9] besides proving several other propositions and lemmas.

1. Intorduction and Preliminaries

The fuzzy concept has invaded almost all branches of Mathematics since the introduction of the concept by Zadeh [10]. Fuzzy sets have applications in many fields such as information [7] and control [8]. The theory of fuzzy topological spaces was introduced and developed by Chang [5] and since then various notions in classical topology have been extended to fuzzy topological space [2,3]. A new class of fuzzy topological spaces called ordered fuzzy pre-extremally disconnected spaces is introduced in this paper by using the concepts of fuzzy extremally disconnected spaces [1], fuzzy pre-open sets [4] and ordered fuzzy topology [6]. Some interesting properties and characterizations are studied. Tietze extension theorem for ordered fuzzy pre-extremally disconnected spaces has been discussed as in [9] besides proving several other propositions and lemmas.

Definition 1. Let (X, T) be a fuzzy topological space and let λ be any fuzzy set in X. λ is called *fuzzy pre-open* [4] if $\lambda \leq \text{int cl } \lambda$. The complement of fuzzy pre-open set is *fuzzy pre-closed* set.

Definition 2 ([6])**.** A fuzzy set λ in (X, T) is called *increasing* (resp. *decreasing*) if $\lambda(x) \leq \lambda(y)$ (resp. $\lambda(x) \geq \lambda(y)$) whenever $x \leq y$ in (X, T) and $x, y \in X$.

Definition 3 ([6])**.** An ordered set on which there is given a fuzzy topology is called an *ordered fuzzy topological space.*

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2. Ordered fuzzy pre-extremally disconnected spaces

In this section, the concept of ordered fuzzy pre-extremally disconnected space is introduced. Characterizations and properties are studied.

Notation. $\mathrm{I}^0(\lambda)$ denotes increasing fuzzy interior of λ. $\mathrm{I}(\lambda)$ denotes increasing fuzzy closure of λ.

Definition 4. Let $(X, T, \leq)$ be an ordered fuzzy topological space and let λ be any fuzzy set in $(X, T, \leq)$, λ is called *increasing fuzzy pre-open* if $\lambda \leq \mathrm{I}^o(\mathrm{I}(\lambda))$. The complement of increasing fuzzy pre open set is called *decreasing fuzzy pre-closed* set.

Definition 5. Let λ be any fuzzy set in the ordered fuzzy topological space $(X, T, \leq)$. Then we define

$$\begin{aligned}
\mathrm{I}^{\mathrm{fp}}(\lambda) &= \text{increasing fuzzy pre-closure of } \lambda \\
&= \wedge\{\mu/\mu \text{ is a fuzzy pre-closed increasing set and } \mu \geq \lambda\}, \\
\mathrm{D}^{\mathrm{fp}}(\lambda) &= \text{decreasing fuzzy pre-closure of } \lambda \\
&= \wedge\{\mu/\mu \text{ is a fuzzy pre-closed decreasing set and } \mu \geq \lambda\ \}, \\
\mathrm{I}^{0\mathrm{fp}}(\lambda) &= \text{increasing fuzzy pre-interior of } \lambda \\
&= \vee\{\mu/\mu \text{ is a fuzzy pre-open increasing set and } \mu \leq \lambda\ \}, \\
\mathrm{D}^{0\mathrm{fp}}(\lambda) &= \text{decreasing fuzzy pre-interior of } \lambda \\
&= \vee\{\mu/\mu \text{ is a fuzzy pre-open decreasing set and } \mu \leq \lambda\ \}.
\end{aligned}$$

Clearly, $\mathrm{I}^{\mathrm{fp}}(\lambda)$ (resp. $\mathrm{D}^{\mathrm{fp}}(\lambda)$) is the smallest fuzzy pre-closed increasing (resp. decreasing) set containing λ and $\mathrm{I}^{0\mathrm{fp}}(\lambda)$ (resp. $\mathrm{D}^{0\mathrm{fp}}(\lambda)$) is the largest fuzzy pre-open increasing (resp. decreasing) set contained in λ.

Proposition 1. *For any fuzzy set λ of an ordered fuzzy topological space $(X, T, \leq)$, the following hold.*

(a) $1 - \mathrm{I}^{\mathrm{fp}}(\lambda) = \mathrm{D}^{0\mathrm{fp}}(1 - \lambda)$.
(b) $1 - \mathrm{D}^{\mathrm{fp}}(\lambda) = \mathrm{I}^{0\mathrm{fp}}(1 - \lambda)$.
(c) $1 - \mathrm{I}^{0\mathrm{fp}}(\lambda) = \mathrm{D}^{\mathrm{fp}}(1 - \lambda)$.
(d) $1 - \mathrm{D}^{0\mathrm{fp}}(\lambda) = \mathrm{I}^{\mathrm{fp}}(1 - \lambda)$.

Proof. We shall prove (a) only, (b), (c) and (d) can be proved in a similar manner.

(a) Since $\mathrm{I}^{\mathrm{fp}}(\lambda)$ is a fuzzy pre-closed increasing set containing λ, $1-\mathrm{I}^{\mathrm{fp}}(\lambda)$ is a fuzzy pre-open decreasing set such that $1 - \mathrm{I}^{\mathrm{fp}}(\lambda) \leq 1 - \lambda$. Let μ be another fuzzy pre-open decreasing set such that $\mu \leq 1 - \lambda$. Then $1 - \mu$ is a fuzzy pre-closed increasing set such that $1 - \mu \geq \lambda$. It follows that $\mathrm{I}^{\mathrm{fp}}(\lambda) \leq 1 - \mu$. That is, $\mu \leq 1-\mathrm{I}^{\mathrm{fp}}(\lambda)$. Thus, $1-\mathrm{I}^{\mathrm{fp}}(\lambda)$ is the largest fuzzy pre-open decreasing set such that $1 - \mathrm{I}^{\mathrm{fp}}(\lambda) \leq 1 - \lambda$. That is, $1 - \mathrm{I}^{\mathrm{fp}}(\lambda) = \mathrm{D}^{0\mathrm{fp}}(1 - \lambda)$. □

Definition 6. Let $(X, T, \leq)$ be an ordered fuzzy topological space. Let λ be any fuzzy pre-open increasing (resp. decreasing) set in $(X, T, \leq)$. If $\mathrm{I}^{\mathrm{fp}}(\lambda)$ (resp. $\mathrm{D}^{\mathrm{fp}}(\lambda)$) is fuzzy pre-open increasing (resp. decreasing) in $(X, T, \leq)$, then $(X, T, \leq)$ is said to be *upper* (resp. *lower*) *fuzzy pre-extremally disconnected.* A fuzzy topological space $(X, T, \leq)$ is said to be *ordered fuzzy pre-extremally disconnected* if it is both upper and lower fuzzy pre-extremally disconnected.

Proposition 2. *For an ordered fuzzy topological space $(X, T, \leq)$, the following are equivalent.*

(a) $(X, T, \leq)$ *is upper fuzzy pre-extremally disconnected.*
(b) *For each fuzzy pre-closed decreasing set λ, $\mathrm{D}^{0\mathrm{fp}}(\lambda)$ is fuzzy pre-closed decreasing.*
(c) *For each fuzzy pre-open increasing set λ, we have $\mathrm{I}^{\mathrm{fp}}(\lambda) + \mathrm{D}^{\mathrm{fp}}(1 - \mathrm{I}^{\mathrm{fp}}(\lambda)) = 1$.*
(d) *For each pair of fuzzy pre-open increasing set λ, pre-open decreasing set μ in $(X, T, \leq)$ with $\mathrm{I}^{\mathrm{fp}}(\lambda) + \mu = 1$, we have $\mathrm{I}^{\mathrm{fp}}(\lambda) + \mathrm{D}^{\mathrm{fp}}(\mu) = 1$.*

Proof. **(a) ⇒ (b).** Let λ be any fuzzy pre-closed decreasing set. We claim $\mathrm{D}^{0\mathrm{fp}}(\lambda)$ is a fuzzy pre-closed decreasing set. Now, $1 - \lambda$ is fuzzy pre-open increasing and so by assumption (a), $\mathrm{I}^{\mathrm{fp}}(1 - \lambda)$ is fuzzy pre-open increasing. That is, $\mathrm{D}^{0\mathrm{fp}}(\lambda)$ is fuzzy pre-closed decreasing. **(b) ⇒ (c).** Let λ be any fuzzy pre-open increasing set. Then,

$$1 - \mathrm{I}^{\mathrm{fp}}(\lambda) = \mathrm{D}^{0\mathrm{fp}}(1 - \lambda). \tag{1}$$

Consider $\mathrm{I}^{\mathrm{fp}}(\lambda) + \mathrm{D}^{\mathrm{fp}}(1 - \mathrm{I}^{\mathrm{fp}}(\lambda)) = \mathrm{I}^{\mathrm{fp}}(\lambda) + \mathrm{D}^{\mathrm{fp}}(\mathrm{D}^{0\mathrm{fp}}(1 - \lambda))$. As λ is any fuzzy pre-open increasing, $1 - \lambda$ is fuzzy pre-closed decreasing and by assumption (b), $\mathrm{D}^{0\mathrm{fp}}(1 - \lambda)$ is fuzzy pre-closed decreasing. Therefore, $\mathrm{D}^{\mathrm{fp}}(\mathrm{D}^{0\mathrm{fp}}(1 - \lambda)) = \mathrm{D}^{0\mathrm{fp}}(1 - \lambda)$. Now,

$$\begin{aligned} \mathrm{I}^{\mathrm{fp}}(\lambda) + \mathrm{D}^{\mathrm{fp}}(\mathrm{D}^{0\mathrm{fp}}(1 - \lambda)) &= \mathrm{I}^{\mathrm{fp}}(\lambda) + \mathrm{D}^{0\mathrm{fp}}(1 - \lambda) \\ &= \mathrm{I}^{\mathrm{fp}}(\lambda) + 1 - \mathrm{I}^{\mathrm{fp}}(\lambda) \\ &= 1. \end{aligned}$$

That is, $\mathrm{I}^{\mathrm{fp}}(\lambda) + \mathrm{D}^{\mathrm{fp}}(1 - \mathrm{I}^{\mathrm{fp}}(\lambda)) = 1$.

(c) ⇒ (d). Let λ be any fuzzy pre-open increasing set and μ be any fuzzy pre-open decreasing set such that

$$\mathrm{I}^{\mathrm{fp}}(\lambda) + \mu = 1. \tag{2}$$

By assumption (c),

$$\mathrm{I}^{\mathrm{fp}}(\lambda) + \mathrm{D}^{\mathrm{fp}}(1 - \mathrm{I}^{\mathrm{fp}}(\lambda)) = 1 = \mathrm{I}^{\mathrm{fp}}(\lambda) + \mu. \tag{3}$$

That is, $\mu = \mathrm{D}^{\mathrm{fp}}(1 - \mathrm{I}^{\mathrm{fp}}(\lambda))$. Since $\mu = 1 - \mathrm{I}^{\mathrm{fp}}(\lambda)$,

$$\mathrm{D}^{\mathrm{fp}}(\mu) = \mathrm{D}^{\mathrm{fp}}(1 - \mathrm{I}^{\mathrm{fp}}(\lambda)). \tag{4}$$

From (3) and (4) we have $\mathrm{I}^{\mathrm{fp}}(\lambda) + \mathrm{D}^{\mathrm{fp}}(\mu) = 1$.

(d) $\Rightarrow$ (a). Let λ be any fuzzy pre-open increasing set. Put $\mu = 1 - \mathrm{I}^{\mathrm{fp}}(\lambda)$. Clearly, μ is fuzzy pre-open decreasing and from the construction of μ it follows that $\mathrm{I}^{\mathrm{fp}}(\lambda) + \mu = 1$. By assumption (d), we have $\mathrm{I}^{\mathrm{fp}}(\lambda) + \mathrm{D}^{\mathrm{fp}}(\mu) = 1$ and so $\mathrm{I}^{\mathrm{fp}}(\lambda) = 1 - \mathrm{D}^{\mathrm{fp}}(\mu)$ is fuzzy pre-open increasing. Therefore $(X, T, \leq)$ is upper fuzzy pre-extremally disconnected. □

Proposition 3. *Let $(X, T, \leq)$ be an ordered fuzzy topological space. Then $(X, T, \leq)$ is an upper fuzzy pre-extremally disconnected space if and only if for fuzzy decreasing pre-open set λ and fuzzy decreasing pre-closed set μ such that $\lambda \leq \mu$, we have $\mathrm{D}^{\mathrm{fp}}(\lambda) \leq \mathrm{D}^{0\mathrm{fp}}(\mu)$.*

Proof. Suppose $(X, T, \leq)$ is an upper fuzzy pre-extremally disconnected space. Let λ be any fuzzy pre-open decreasing set, μ be any fuzzy pre-closed decreasing set such that $\lambda \leq \mu$. Then by (b) of Proposition 2, $\mathrm{D}^{0\mathrm{fp}}(\mu)$ is fuzzy pre-closed decreasing. Also, since λ is fuzzy pre-open decreasing and $\lambda \leq \mu$, it follows that $\lambda \leq \mathrm{D}^{0\mathrm{fp}}(\mu)$. Again, since $\mathrm{D}^{0\mathrm{fp}}(\mu)$ is fuzzy pre-closed decreasing, it follows that $D(\lambda) \leq \mathrm{D}^{0\mathrm{fp}}(\mu)$. To prove the converse, let μ be any fuzzy pre-closed decreasing set. By Definition 5, $\mathrm{D}^{0\mathrm{fp}}(\mu)$ is fuzzy pre-open decreasing and it is also clear that $\mathrm{D}^{0\mathrm{fp}}(\mu) \leq \mu$. Therefore by assumption, it follows that $\mathrm{D}^{\mathrm{fp}}(\mathrm{D}^{0\mathrm{fp}}(\mu)) \leq \mathrm{D}^{0\mathrm{fp}}(\mu)$. This implies that $\mathrm{D}^{0\mathrm{fp}}(\mu)$ is fuzzy pre-closed decreasing. Hence, by (b) of Proposition 2, it follows that $(X, T, \leq)$ is upper fuzzy pre-extremally disconnected. □

Notation. An ordered fuzzy set which is both fuzzy decreasing (resp. increasing) pre-open and pre-closed is called fuzzy decreasing (resp. increasing) pre clopen set.

Remark 1. Let $(X, T, \leq)$ be an upper fuzzy pre-extremally disconnected space. Let $\{\lambda_i, 1 - \mu_i \mid i \in N\}$ be a collection such that λ_i's are fuzzy pre-open decreasing sets, μ_i's are fuzzy pre-closed decreasing sets and let λ, $1 - \mu$ be fuzzy pre-open decreasing and pre-open increasing sets respectively. If $\lambda_i \leq \lambda \leq \mu_j$ and $\lambda_i \leq \mu \leq \mu_j$ for all $i, j \in N$, then there exists a fuzzy pre-clopen decreasing set γ such that $\mathrm{D}^{\mathrm{fp}}(\lambda_i) \leq \gamma \leq \mathrm{D}^{0\mathrm{fp}}(\mu_j)$ for all $i, j \in N$. By Proposition 3, $\mathrm{D}^{\mathrm{fp}}(\lambda_i) \leq \mathrm{D}^{\mathrm{fp}}(\lambda) \wedge \mathrm{D}^{0\mathrm{fp}}(\mu) \leq \mathrm{D}^{0\mathrm{fp}}(\mu_j)$ $(i, j \in N)$. Put $\gamma = \mathrm{D}^{\mathrm{fp}}(\lambda) \wedge \mathrm{D}^{0\mathrm{fp}}(\mu)$. Now, γ satisfies our required condition.

Proposition 4. *Let $(X, T, \leq)$ be an upper fuzzy pre-extremally disconnected space. Let $\{\lambda_q\}_{q \in Q}$ and $\{\mu_q\}_{q \in Q}$ be monotone increasing collections of fuzzy pre-open decreasing sets and fuzzy pre-closed decreasing sets of $(X, T, \leq)$ respectively and suppose that $\lambda_{q_1} \leq \mu_{q_2}$ whenever $q_1 < q_2$ (Q is the set of rational numbers). Then there exists a monotone increasing collection $\{\gamma_q\}_{q \in Q}$ of fuzzy pre-clopen decreasing sets of $(X, T, \leq)$ such that $\mathrm{D}^{\mathrm{fp}}(\lambda_{q_1}) \leq \gamma_{q_2}$ and $\gamma_{q_1} \leq \mathrm{D}^{0\mathrm{fp}}(\mu_{q_2})$ whenever $q_1 < q_2$.*

Proof. Let us arrange into a sequence $\{q_n\}$ of rational numbers without repetitions. For every $n \geq 2$, we shall define inductively a collection $\{\gamma_{q_i} \mid 1 \leq i <$

$n\} \subset I^X$ such that

$$(S_n) \qquad \begin{cases} \mathrm{D}^{\mathrm{fp}}(\lambda_q) \le \gamma_{q_i} & \text{if } q < q_i, \\ \gamma_{q_i} \le \mathrm{D}^{0\mathrm{fp}}(\mu_q) & \text{if } q_i < q, \end{cases}$$

for all $i < n$.

By Proposition 3, the countable collections $\{\mathrm{D}^{\mathrm{fp}}(\lambda_q)\}$ and $\{\mathrm{D}^{0\mathrm{fp}}(\mu_q)\}$ satisfying $\mathrm{D}^{\mathrm{fp}}(\lambda_{q_1}) \le \mathrm{D}^{0\mathrm{fp}}(\mu_{q_2})$ if $q_1 < q_2$. By Remark 1, there exists fuzzy pre-clopen decreasing set δ_1 such that $\mathrm{D}^{\mathrm{fp}}(\lambda_{q_1}) \le \delta_1 \le \mathrm{D}^{0\mathrm{fp}}(\mu_{q_2})$. Setting $\gamma_{q_1} = \delta_1$ we get (S_2). Assume that fuzzy sets γ_{q_i} are already defined for $i < n$ and satisfy (S_n). Define $\Sigma = \vee\{\gamma_{q_i} \,|\, i < n, q_i < q_n\} \vee \lambda_{q_n}$ and $\Phi = \wedge\{\gamma_{q_j} \,|\, j < n, q_j > q_n\} \wedge \mu_{q_n}$. Then we have that $\mathrm{D}^{\mathrm{fp}}(\gamma_{q_i}) \le \mathrm{D}^{\mathrm{fp}}(\Sigma) \le \mathrm{D}^{0\mathrm{fp}}(\gamma_{q_j})$ and $\mathrm{D}^{\mathrm{fp}}(\gamma_{q_i}) \le \mathrm{D}^{0\mathrm{fp}}(\Phi) \le \mathrm{D}^{0\mathrm{fp}}(\gamma_{q_j})$ whenever $q_i < q_n < q_j$ $(i, j < n)$ as well as $\lambda_q \le \mathrm{D}^{\mathrm{fp}}(\Sigma) \le \mu_{q'}$ and $\lambda_q \le \mathrm{D}^{0\mathrm{fp}}(\Phi) \le \mu_{q'}$ whenever $q < q_n < q'$. This shows that the countable collections $\{\gamma_{q_i} \,|\, i < n, q_i < q_n\} \cup \{\lambda_q \,|\, q < q_n\}$ and $\{\gamma_{q_j} \,|\, j < n, q_j > q_n\} \cup \{\mu_q \,|\, q > q_n\}$ together with Σ and Φ fulfil all conditions of Remark 7.3.1. Hence, there exists a fuzzy pre-clopen decreasing set δ_n such that $\mathrm{D}^{\mathrm{fp}}(\delta_n) \le \mu_q$ if $q_n < q$, $\lambda_q \le \mathrm{D}^{0\mathrm{fp}}(\delta_n)$ if $q < q_n$, $\mathrm{D}^{\mathrm{fp}}(\gamma_{q_i}) \le \mathrm{D}^{0\mathrm{fp}}(\delta_n)$ if $q_i < q_n$, $\mathrm{D}^{\mathrm{fp}}(\delta_n) \le \mathrm{D}^{0\mathrm{fp}}(\gamma_{q_j})$ if $q_n < q_j$, where $1 \le i, j \le n - 1$. Now setting $\gamma_{q_n} = \delta_n$ we obtain the fuzzy sets $\gamma_{q_1}, \gamma_{q_2}, \ldots, \gamma_{q_n}$ that satisfy (S_{n+1}). Therefore the collection $\{\gamma_{q_i} \,|\, i = 1, 2, \ldots\}$ has required property. This completes the proof. □

Definition 7. Let $(X, T, \le)$ and $(Y, S, \le)$ be ordered fuzzy topological spaces. A mapping $f\colon (X, T, \le) \to (Y, S, \le)$ is called an *increasing* (resp. *decreasing*) *fuzzy pre-continuous* if $f^{-1}(\lambda)$ is fuzzy pre-open increasing (resp. decreasing) set of $(X, T, \le)$ for every fuzzy pre-open set λ of $(Y, S, \le)$. If f is both increasing and decreasing fuzzy pre-continuous, then it is called ordered fuzzy pre-continuous.

Definition 8. Let $(X, T, \le)$ be an ordered fuzzy topological space. A function $f : X \to R(I)$ is called *lower* (resp. *upper*) *fuzzy pre-continuous*, if $f^{-1}(R_t)$ (resp. $f^{-1}(L_t)$) is an increasing or decreasing fuzzy pre-open set for each $t \in R$.

Lemma 1. *Let $(X, T, \le)$ be an ordered fuzzy topological space, let $\lambda \in I^X$, and let $f\colon X \to R(I)$ be such that*

$$f(x)(t) = \begin{cases} 1 & \text{if } t < 0, \\ \lambda(x) & \text{if } 0 \le t \le 1, \\ 0 & \text{if } t > 1, \end{cases}$$

for all $x \in X$. Then f is lower (resp. upper) fuzzy pre-continuous iff λ is fuzzy pre-open (resp. pre-closed) increasing or decreasing set.

Definition 9. The characteristic function of $\lambda \in I^X$ is the map $\chi_\lambda : X \to [0, 1](I)$ defined by $\chi_\lambda(x) = (\lambda(x))$, $x \in X$.

Proposition 5. *Let $(X, T, \leq)$ be an ordered fuzzy topological space, and let $\lambda \in I^X$. Then χ_λ is lower (resp. upper) fuzzy pre-continuous iff λ is fuzzy pre-open (resp. pre-closed) increasing or decreasing set.*

Proof. The proof follows from Lemma 1. □

Proposition 6. *Let $(X, T, \leq)$ be an ordered fuzzy topological space. Then the following statements are equivalent.*

(a) *$(X, T, \leq)$ is upper fuzzy pre-extremally disconnected.*
(b) *If $g, h : X \to R(I)$, g is lower fuzzy pre-continuous, h is upper fuzzy pre-continuous and $g \leq h$, then there exists an increasing fuzzy pre-continuous function $f : (X, T, \leq) \to R(I)$ such that $g \leq f \leq h$.*
(c) *If $1 - \lambda$ is fuzzy pre-open increasing and μ is fuzzy pre-open decreasing such that $\mu \leq \lambda$, then there exists an increasing fuzzy pre-continuous function $f : (X, T, \leq) \to [0, 1](I)$ such that $\mu \leq (1 - L_1)f \leq R_0 f \leq \lambda$.*

Proof. **(a) ⇒ (b).** Define $H_r = L_r h$ and $G_r = (1 - R_r)g$, $r \in Q$. Thus we have two monotone increasing families of respectively fuzzy pre-open decreasing and fuzzy pre-closed decreasing sets of $(X, T, \leq)$. Moreover $H_r \leq G_s$ if $r < s$. By Proposition 4, there exists a monotone increasing family $\{F_r\}_{r \in Q}$ of fuzzy pre-clopen decreasing sets of $(X, T, \leq)$ such that $\mathrm{D}^{\mathrm{fp}}(H_r) \leq F_s$ and $F_r \leq \mathrm{D}^{0\mathrm{fp}}(G_s)$ whenever $r < s$. Letting $V_t = \wedge_{r<t}(1 - F_r)$ for all $t \in R$, we define a monotone decreasing family $\{V_t \,|\, t \in R\} \subset I^X$. Moreover we have $\mathrm{I}^{\mathrm{fp}}(V_t) \leq \mathrm{I}^{0\mathrm{fp}}(V_s)$, whenever $s < t$. We have

$$\begin{aligned} \vee_{t \in R} V_t &= \vee_{t \in R} \wedge_{r<t} (1 - F_r) \\ &\geq \vee_{t \in R} \wedge_{r<t} (1 - G_r) \\ &= \vee_{t \in R} \wedge_{r<t} g^{-1}(R_r) \\ &= \vee_{t \in R} g^{-1}(R_t) \\ &= g^{-1}(\vee_{t \in R}(R_t)) = 1. \end{aligned}$$

Similarly, $\wedge_{t \in R} V_t = 0$.

We now define a function $f : (X, T, \leq) \to R(I)$ satisfying the required properties. Let $f(x)(t) = V_t(x)$ for all $x \in X$ and $t \in R$. By the above discussion, it follows that f is well defined. To prove f is fuzzy increasing pre-continuous, we observe that

$$\vee_{s>t} V_s = \vee_{s>t} \mathrm{I}^{0\mathrm{fp}}(V_s) \quad \text{and} \quad \wedge_{s<t} V_s = \wedge_{s<t} \mathrm{I}^{\mathrm{fp}}(V_s).$$

Then $f^{-1}(R_t) = \vee_{s>t} V_s = \mathrm{I}^{0\mathrm{fp}}(V_s)$ is fuzzy pre-open increasing. Now,

$$f^{-1}(1 - L_t) = \wedge_{s<t} V_s = \wedge_{s<t} \mathrm{I}^{\mathrm{fp}}(V_s)$$

is fuzzy pre-closed increasing, so that f is fuzzy increasing pre-continuous. To conclude the proof it remains to show that $g \leq f \leq h$, that is $g^{-1}(1 - L_t) \leq f^{-1}(1 - L_t) \leq h^{-1}(1 - L_t)$ and $g^{-1}(R_t) \leq f^{-1}(R_t) \leq h^{-1}(R_t)$ for each $t \in R$.

We have

$$\begin{aligned} g^{-1}(1-L_t) &= \wedge_{s<t} g^{-1}(1-L_s) \\ &= \wedge_{s<t} \wedge_{r<s} g^{-1}(R_r) \\ &= \wedge_{s<t} \wedge_{r<s} (1-G_r) \\ &\leq \wedge_{s<t} \wedge_{r<s} (1-F_r) \\ &= \wedge_{s<t} V_s = f^{-1}(1-L_t) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(1-L_t) &= \wedge_{s<t} V_s \\ &= \wedge_{s<t} \wedge_{r<s} (1-F_r) \\ &\leq \wedge_{s<t} \wedge_{r<s} (1-H_r) \\ &= \wedge_{s<t} \wedge_{r<s} h^{-1}(1-L_r) \\ &= \wedge_{s<t} h^{-1}(1-L_s) \\ &= h^{-1}(1-L_t). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} g^{-1}(R_t) &= \vee_{s>t} g^{-1}(R_s) \\ &= \vee_{s>t} \vee_{r>s} g^{-1}(R_r) \\ &= \vee_{s>t} \vee_{r>s} (1-G_r) \\ &\leq \vee_{s>t} \vee_{r<s} (1-F_r) \\ &= \vee_{s>t} V_s = f^{-1}(R_t) \end{aligned}$$

and

$$\begin{aligned} f^{-1}(R_t) &= \vee_{s>t} V_s \\ &= \vee_{s>t} \vee_{r<s} (1-F_r) \\ &\leq \vee_{s>t} \vee_{r>s} (1-H_r) \\ &= \vee_{s>t} \vee_{r>s} h^{-1}(1-L_r) \\ &= h^{-1}(R_s) = h^{-1}(R_t). \end{aligned}$$

Thus, (b) is proved.

(b) ⇒ (c). Suppose $1-\lambda$ is a fuzzy pre-open increasing set and μ is a fuzzy pre-open decreasing set, $\mu \leq \lambda$. Then $\chi_\mu \leq \chi_\lambda$ and χ_μ, χ_λ are lower and upper fuzzy pre-continuous functions respectively. Hence by (b), there exists an increasing fuzzy pre-continuous function $f : (X, T, \leq) \to R(I)$ such that $\chi_\mu \leq f \leq \chi_\lambda$. Clearly, $f(x) \in [0,1](I)$ for all $x \in X$ and $\mu = (1-L_1)\chi_\mu \leq (1-L_1)f \leq R_0 f \leq R_0 \chi_\lambda = \lambda$.

(c) ⇒ (a). This follows from Proposition 3 and the fact that $(1-L_1)f$ and $R_0 f$ are fuzzy pre-closed decreasing and pre-open decreasing sets respectively. Hence, the result. □

Note 1. The Propositions 2 to 5, and Remark 1 can be discussed for other cases also.

3. Tietze extension theorem for ordered fuzzy pre-extremally disconnected spaces

In this section, Tietze extension theorem for ordered fuzzy pre-extremally disconnected spaces is studied.

Proposition 7 (Tietze Extension Theorem)**.** *Let* $(X, T, \leq)$ *be an upper fuzzy pre-extremally disconnected space and let* $A \subset X$ *be such that* χ_A *is fuzzy pre-open increasing set in* $(X, T, \leq)$*. Let* $f\colon (A, T/A) \to [0,1](I)[6]$ *be an increasing fuzzy pre-continuous function. Then* f *has an increasing fuzzy pre-continuous extension over* $(X, T, \leq)$*.*

Proof. Let $g, h\colon X \to [0,1](I)$ be such that $g = f = h$ on A, and

$$g(x) = \langle 0 \rangle,\ h(x) = \langle 1 \rangle \text{ if } x \notin A.$$

We now have

$$R_t g = \begin{cases} \mu_t \wedge \chi_A & \text{if } t \geq 0, \\ 1 & \text{if } t < 0, \end{cases}$$

where μ_t is fuzzy pre-open increasing such that

$$\mu_t / A = R_t f$$

and

$$L_t h = \begin{cases} \lambda_t \wedge \chi_A & \text{if } t \leq 1, \\ 1 & \text{if } t > 1, \end{cases}$$

where λ_t is fuzzy pre-open increasing such that

$$\lambda_t / A = L_t f.$$

Thus, g is lower fuzzy pre-continuous, h is upper fuzzy pre-continuous and $g \leq h$. By Proposition 6, there is an increasing fuzzy pre-continuous function $F\colon (X, T, \leq) \to [0,1](I)$ such that $g \leq F \leq h$; hence $F \equiv f$ on A. □

Note 2. The above proposition can be discussed for other cases also.

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