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On Perturbed Symmetric Distributions Associated with the Truncated Bivariate Elliptical Models†

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Abstract

This paper proposes a class of perturbed symmetric distributions associated with the bivariate elliptically symmetric(or simply bivariate elliptical) distributions. The class is obtained from the nontruncated marginals of the truncated bivariate elliptical distributions. This family of distributions strictly includes some univariate symmetric distributions, but with extra parameters to regulate the perturbation of the symmetry. The moment generating function of a random variable with the distribution is obtained and some properties of the distribution are also studied. These developments are followed by practical examples.

Keywords: Perturbed symmetric distribution; truncated bivariate elliptical distribution; Skew-elliptical distribution.

1. Introduction

A distribution of $k \times 1$ random vector $\mathbf{X}$, written $\mathbf{X} \sim EC_k(\theta, \Sigma, g^{(k)})$, is said to have k-variate elliptically symmetric(or simply elliptical) distribution with location vector $\theta \in R^k$ and a $k \times k$ (positive definite) dispersion matrix Σ and the density generating function $g^{(k)}$. The density of $\mathbf{X}$ distribution is given by

$$f(\mathbf{x}\,|\,\theta, \Sigma) = |\Sigma|^{-\frac{1}{2}} g^{(k)}\left[(\mathbf{x}-\theta)'\Sigma^{-1}(\mathbf{x}-\theta)\right], \tag{1.1}$$

for some density generator function $g^{(k)}(u),\ u \geq 0$, such that

$$\int_0^\infty u^{\frac{k}{2}-1} g^{(k)}(u)du = \frac{\Gamma\left(\dfrac{k}{2}\right)}{\pi^{\frac{k}{2}}}. \tag{1.2}$$

By varying the function $g^{(k)}$, distributions with longer or shorter tails than the k-variate normal can be obtained. The density (1.1) has appeared at various places in the literature, sometimes in a somewhat casual manner. A systematic treatment of this distribution has been given by Fang *et al.* (1990) and Fang and Zhang (1990). Various univariate/multivariate extensions of the distribution is considered by Azzalini (1985),

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Branco and Dey (2001), Azzalini and Capitanio (2003), Ma and Genton (2004), Arellano-Valle *et al.* (2006) and Kim (2007, 2008) among others.

The major goal of this paper is to study a class of perturbed symmetric distributions as an extension of univariate elliptically symmetric distribution. The class is obtained from deriving the distribution of

$$Z \equiv [X \mid a < Y < b], \tag{1.3}$$

where (X, Y) is a bivariate elliptical variable with the location parameter (θ_1, θ_2) and 2×2 scale matrix Σ. Although cases of perturbed distributions(obtained from a conditioning mechanism) are well addressed in the literature, so far as we know, there are few results concerning the class of perturbed distributions associated with a doubly truncated bivariate elliptical distribution. This motivates the investigation in this article. The interest in studying the distribution (1.3) comes from both theoretical and applied directions. On the theoretical side, it provides a class of distributions that enjoys a number of formal properties which resemble those of the elliptical distributions as given in Fang and Zhang (1990) and produces a general class of distributions that strictly includes many symmetric distributions as well as the univariate skewed distributions considered by Arellano-Valle *et al.* (2006). From the applied viewpoint, the distribution is a unimodal empirical distribution with presence of skewness and possibly heavy(or light) tail (see, Figure 3.1). This implies that (1.3) is useful to modeling random phenomena which have heavier(or lighter) tails than the normal as well as having some skewness. Moreover, the class of distributions obtained from (1.3) provides yet other models that enable us to analyze a screened data in terms of the sum of truncated and untruncated observations.

2. Preliminaries

Prior to suggesting the class of perturbed symmetric distributions, we provide lemmas useful for studying property of the class.

If we set $\Psi = \{\psi_{ij}\}$ with $\psi_{11} = \psi_{22} = 1$ and $\psi_{12} = \psi_{21} = \rho$, the properties of $\mathrm{EC}_2(\mathbf{0}, \Psi, g^{(2)})$ distribution yield following theorems. From now on, we will use Ψ to denote this standard form of 2×2 dispersion matrix.

Lemma 2.1 Let $\mathbf{W} \sim \mathrm{EC}_2(\mathbf{0}, \Psi, g^{(2)})$, $\mathbf{W} = (W_1, W_2)'$. Then the conditional distribution of W_2 given that $W_1 = w$ is $\mathrm{EC}_1(\alpha, \beta, g_{q(w)})$ distribution. Here $\alpha = \rho w$, $\beta = 1 - \rho^2$, $g_{q(w)} = g^{(2)}(u + q(w))/g^{(1)}(q(w))$, where $g^{(1)}(u) = 2\int_0^\infty g^{(2)}(r^2 + u)dr$ and $q(w) = w^2$.

Proof: Straightforward application of the result of Branco and Dey (2001, pp. 102) yields the result. □

Lemma 2.2 Let $\mathbf{W} \sim \mathrm{EC}_2(\mathbf{0}, \Psi, g^{(2)})$, $\mathbf{W} = (W_1, W_2)'$. If Z is set to equal to W_1 conditionally on $a < W_2 < b$. Then the *pdf* of Z is

$$f_Z(z) = \frac{f_{g^{(1)}}(z)\left\{F_{g_{q(z)}}(\lambda_1 b - \lambda z) - F_{g_{q(z)}}(\lambda_1 a - \lambda z)\right\}}{F_{g^{(1)}}(b) - F_{g^{(1)}}(a)}, \quad \text{for } z \in \mathcal{R}, \tag{2.1}$$

where $\lambda = \rho/\sqrt{1-\rho^2}$, $\lambda_1 = (1 + \lambda^2)^{1/2} = 1/\sqrt{1-\rho^2}$ and $f_{g^{(1)}}(\cdot)$ and $F_{g^{(1)}}(\cdot)$ are the *pdf* and the *cdf* of $\mathrm{EC}_1(0, 1, g^{(1)})$, respectively. $F_{g_{q(z)}}$ is the *cdf* of $\mathrm{EC}_1(0, 1, g_{q(z)})$ with $q(z) = z^2$.

Proof: The *pdf* of Z can be expressed as

$$f_Z(z) = \frac{P(a < W_2 < b \,|\, z) f_{W_1}(z)}{P(a < W_2 < b)}.$$

Using the property of $\mathrm{EC}_2(\mathbf{0}, \Psi, g^{(2)})$ (see, Fang *et al.* 1990, p.43) we obtain the marginal distribution, $W_1 \sim \mathrm{EC}_1(0, 1, g^{(1)})$. Further Lemma 2.1 gives $W_2 \,|W_1 = z \sim \mathrm{EC}_1(\alpha, \beta, g_{q(z)})$ and hence

$$\begin{aligned} P(a < W_2 < b \,|z) &= P\left(\frac{a-\alpha}{\sqrt{\beta}} < \frac{W_2 - \alpha}{\sqrt{\beta}} < \frac{b-\alpha}{\sqrt{\beta}} \,\Big|\, z\right) \\ &= F_{g_{q(z)}}(\lambda_1 b - \lambda z) - F_{g_{q(z)}}(\lambda_1 a - \lambda z). \end{aligned}$$

Noticing that $f_{W_1}(z) = f_{g^{(1)}}(z)$ and $P(a < W_2 < b) = F_{g^{(1)}}(b) - F_{g^{(1)}}(a)$, we have the result. □

3. Perturbed Symmetric Distribution

A generalization of Lemma 2.2 yields the following result. Let $\mathbf{X} = (X, Y)'$ and let $\mathbf{X} \sim \mathrm{EC}_2(\theta, \Sigma, g^{(2)})$ with $\theta = (\theta_1, \theta_2)'$, $\Sigma = \{\sigma_{ij}\}$, $\sigma_{ii} = \sigma_i^2$ and $\sigma_{12} = \rho\sigma_1\sigma_2$. Then, the *pdf* of $Z \equiv [X \,|a < Y < b]$ variable is

$$f_Z^*(z) = \frac{f_{g^{(1)}}(u_1(z)) \left\{F_{g_{q_*(z)}}(\lambda_1 u(b) - \lambda u_1(z)) - F_{g_{q_*(z)}}(\lambda_1 u(a) - \lambda u_1(z))\right\}}{\sigma_1 \left\{F_{g_{(1)}}(u(b)) - F_{g_{(1)}}(u(a))\right\}}, \qquad (3.1)$$

for $z \in R$, where $u_1(z) = (z - \theta_1)/\sigma_1$, $u(a) = (a - \theta_2)/\sigma_2$, $u(b) = (b - \theta_2)/\sigma_2$, $F_{g_{q_*(z)}}$ is the *cdf* of $\mathrm{EC}_1(0, 1, g_{q_*(z)})$ with $q_*(z) = u_1(z)^2$. The density of Z variable is obtained from (2.1) using the transformation relations $X = \sigma_1 W_1 + \theta_1$ and $Y = \sigma_2 W_2 + \theta_2$. From now on, we use the same notations without redefining.

Definition 3.1 If a random variable Z has density function (3.1), then we say that Z is a perturbed symmetric random variable with parameters θ, Σ and a non-negative perturb function

$$p(z; \theta_2, \sigma_2, \rho) = \frac{\left\{F_{g_{q_*(z)}}(\lambda_1 u(b) - \lambda u_1(z)) - F_{g_{q_*(z)}}(\lambda_1 u(a) - \lambda u_1(z))\right\}}{\left\{F_{g_{(1)}}(u(b)) - F_{g_{(1)}}(u(a))\right\}}. \qquad (3.2)$$

For brevity we shall also say that Z is $\mathrm{PS}_{(a,b)}(\theta, \Sigma, g^{(2)})$.

We see, from (3.1), that the perturbed distribution arise when the symmetric density $f_{g^{(1)}}(u_1(z))/\sigma_1$ of potential observation z gets distorted so that it is multiplied by some non-negative perturb function $p(z; \theta_2, \sigma_2, \rho)$. From Lemma 2.1 and (3.1), we can get an alternative and convenient expression for the *pdf* of $\mathrm{PS}_{(a,b)}(\theta, \Sigma, g^{(2)})$ distribution as

$$f_Z^*(z) = \frac{\displaystyle\int_{u(a)}^{u(b)} g^{(2)}\left[\frac{\{r - \rho u_1(z)\}^2}{1-\rho^2} + u_1(z)^2\right] dr}{\sigma_1\sqrt{1-\rho^2}\left\{F_{g_{(1)}}(u(b)) - F_{g_{(1)}}(u(a))\right\}}, \quad z \in \mathcal{R}. \qquad (3.3)$$

There are many subclasses of bivariate elliptical distributions, such as bivariate normals, bivariate t-distributions, symmetric bivariate Pearson types VII, bivariate scale mixture normal distributions and some others (see, Fang *et al.* 1990, for the others). These are the most useful subclasses from both practical and theoretical aspects. In the following examples, we see that (3.3) yields some perturbed symmetric distributions from these subclasses.

Example 3.1 (A Perturbed Normal Distribution)
The generator function for a bivariate normal is

$$g_N^{(2)}(u) = (2\pi)^{-1} \exp\left(-\frac{u}{2}\right). \tag{3.4}$$

Thus the density $f_Z^*(z)$ of $Z \sim \mathrm{PS}_{(a,b)}(\theta, \Sigma, g_N^{(2)})$ variable is obtained from (3.3) and it is

$$f_Z^*(z) = \frac{\phi(u_1(z))\,\{\Phi(\lambda_1 u(b) - \lambda u_1(z)) - \Phi(\lambda_1 u(a) - \lambda u_1(z))\}}{\sigma_1\{\Phi(u(b)) - \Phi(u(a))\}}, \quad z \in \mathcal{R}. \tag{3.5}$$

This agrees with the density given in Kim (2007). When $a = 0$, $b = \infty$, $\theta = \mathbf{0}$ and $\Sigma = \Psi$, we see that (3.5) reduces to the density of skew-normal distribution ($\mathcal{SN}$) by Azzalini (1985).

Example 3.2 (A Perturbed t_ν Distribution)
The generator function for a bivariate t-distribution with the degrees of freedom ν is

$$g_\nu^{(2)}(u) = C_1(\nu + u)^{-\frac{\nu+2}{2}}, \tag{3.6}$$

where $C_1 = \nu^{\nu/2}\Gamma\{(\nu+2)/2\}/(\pi\ \Gamma\{\nu/2\})$. It follows from (3.3) that the *pdf* $f_Z^*(z)$ of $\mathrm{PS}_{(a,b)}(\theta, \Sigma, g_\nu^{(2)})$ distribution is

$$f_Z^*(z) = \frac{f_\nu(u_1(z))\{F_{\nu+1}(v_2(z)) - F_{\nu+1}(v_1(z))\}}{\sigma_1\{F_\nu(u(b)) - F_\nu(u(a))\}}, \quad z \in \mathcal{R}, \tag{3.7}$$

where $f_\nu(\cdot)$ and $F_\nu(\cdot)$ denote the *pdf* and the *cdf* of a univariate standard t_ν distribution, while $F_{\nu+1}(\cdot)$ is the *cdf* of a univariate standard $t_{\nu+1}$ distribution.

Example 3.3 (A Perturbed Pearson Type VII Distribution)
A 2×1 random vector $\mathbf{X}$ is said to have a symmetric bivariate Pearson Type VII distribution if it has a density generator function

$$g_{VII}^{(2)}(u) = C\left(1 + \frac{u}{m}\right)^{-N}, \quad N > 1,\ m > 0 \tag{3.8}$$

and $C = (\pi m)^{-1}\Gamma(N)/\Gamma(N-1)$. Some analytic algebra plugging (3.8) in (3.3) leads to the *pdf* of $\mathrm{PS}_{(a,b)}(\theta, \Sigma, g_{VII}^{(2)})$ distribution given by

$$f_Z^*(z) = (\delta\sigma_1)^{-1} f_{N-\frac{1}{2},m}(u_1(z))\{F_{N,m+1}(w_2(z)) - F_{N,m+1}(w_1(z))\}, \quad z \in \mathcal{R}, \tag{3.10}$$

where $\delta = F_{N-1/2,m}(u(b)) - F_{N-1/2,m}(u(a))$,

$$w_1(z) = \frac{\{\lambda_1 u(a) - \lambda u_1(z)\}\sqrt{m+1}}{\sqrt{m + u_1(z)^2}} \quad \text{and} \quad w_2(z) = \frac{\{\lambda_1 u(b) - \lambda u_1(z)\}\sqrt{m+1}}{\sqrt{m + u_1(z)^2}}.$$

Here $f_{N-1/2,m}(\cdot)$ and $F_{N-1/2,m}(\cdot)$ denote the *pdf* and the *cdf* of a univariate standard Pearson type VII distribution with parameters $N - 1/2$ and m, while $F_{N,m+1}(\cdot)$ is the *cdf* of a univariate standard Pearson type VII distribution with parameters N and $m+1$.

The perturbed Pearson type VII distribution includes a number of important distributions such as the perturbed t distribution(for N=m/2+1) and the perturbed Cauchy distribution(for m=1 and N=3/2).

Example 3.4 (A Perturbed Scale Mixture of Normal Distribution)

The generator function for a bivariate scale mixture of normal is

$$g^{(2)}_{H,K}(u) = \int_0^\infty \{2\pi K(\eta)\}^{-1} \exp\left\{-\frac{u}{2K(\eta)}\right\} dH(\eta), \tag{3.11}$$

where η is a mixing variable with the *cdf* $H(\eta)$ and $K(\eta)$ is a weight function. From (3.3), the *pdf* $f^*_Z(z)$ of the perturbed scale mixture of normal distribution, written $\mathrm{PS}_{(a,b)}(\theta, \Sigma, g^{(2)}_{H,K})$, is given as

$$\frac{1}{2\pi\delta_1 K(\eta)\sigma_1\sqrt{1-\rho^2}} \int_0^\infty \int_{u(a)}^{u(b)} \exp\left[-\frac{\{r-\rho u_1(z)\}^2}{2K(\eta)(1-\rho^2)} + \frac{u_1(z)^2}{2K(\eta)}\right] \partial r \; \partial H(\eta).$$

Transforming $\{r - \rho u_1(z)\}/\sqrt{K(\eta)(1-\rho^2)}$ to w yields

$$f^*_Z(z) = \frac{1}{\delta_1\sigma_1\sqrt{K(\eta)}} \int_0^\infty \phi\left(\frac{u_1(z)}{\sqrt{K(\eta)}}\right) \{\Phi(u^*_1(z)) - \Phi(u^*_2(z))\} dH(\eta), \tag{3.12}$$

for $z \in \mathcal{R}$, where

$$\delta_1 = \int_0^\infty \Phi\left(\frac{u(b)}{\sqrt{K(\eta)}}\right) - \Phi\left(\frac{u(a)}{\sqrt{K(\eta)}}\right) dH(\eta),$$

$$u^*_1(z) = \frac{\lambda_1 u(b) - \lambda u_1(z)}{\sqrt{K(\eta)}} \quad \text{and} \quad u^*_2(z) = \frac{\lambda_1 u(a) - \lambda u_1(z)}{\sqrt{K(\eta)}}.$$

Note that (3.12) reduces to (3.5) when $H(\eta)$ is degenerate with $K(\eta) = 1$. Also note that (3.12) is reduced to (3.7) when $K(\eta) = 1/\eta$ with $H(\eta)$ as the *cdf* of a gamma distribution, *i.e.*, $\eta \sim G(\nu/2, 2/\nu)$ so that $\nu\eta \sim \chi^2_\nu$. Other combinations of $H(\eta)$ and $K(\eta)$ functions for the scale mixture of normal distribution are considered by Chen and Dey (1998) and Branco and Dey (2001), among other. Those combinations can be applied to (3.12) to coming at various classes of perturbed symmetric distributions. For example, through the scale mixtures by Chen and Dey (1998), a class of perturbed stable densities, perturbed logistic densities and that of perturbed symmetric power densities can also be obtained from (3.12).

4. Property of the Distribution

4.1. Distributional properties

Now we will state some interesting properties for the $\mathrm{PS}_{(a,b)}(\theta, \Sigma, g^{(2)})$ distribution as well as the associate examples. Some properties are focused on those of $\mathrm{PS}_{(a,b)}(\theta, \Sigma, g^{(2)}_{K,H})$

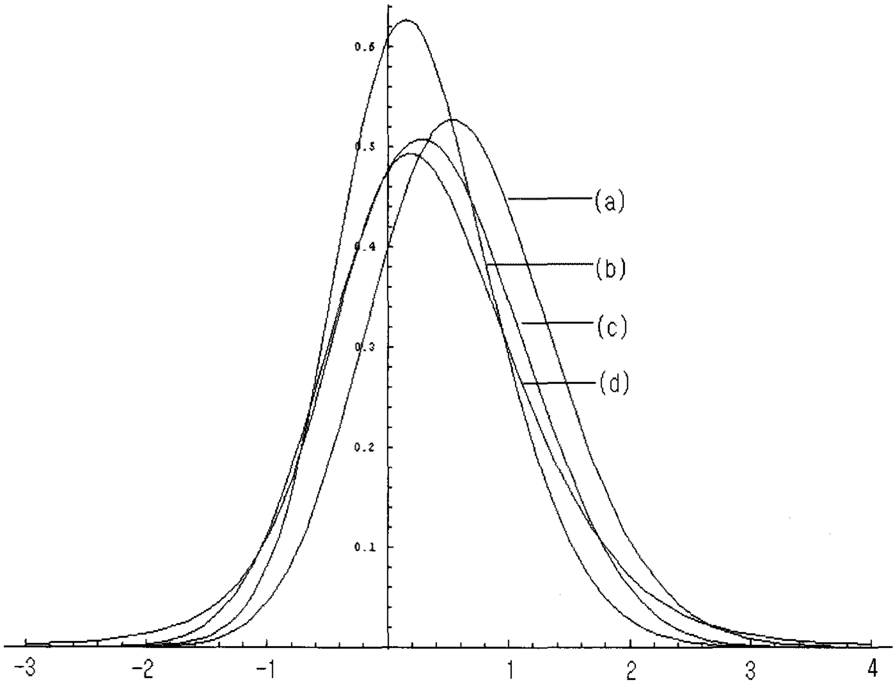

Figure 3.1: Various Shapes of perturbed symmetric about zero densities with $\rho = 0.8$ ((a) the *pdf* of $\mathcal{SN}(\lambda) \equiv \mathrm{PE}_{(0,\infty)}(\mathbf{0}, \Psi, g_N^{(2)})$ distribution, where $\lambda = \rho/\sqrt{1-\rho^2}$; (b) the *pdf* of $\mathrm{PE}_{(-.5,2)}(\mathbf{0}, \Psi, g_N^{(2)})$ distribution; (c) the *pdf* of $\mathrm{PE}_{(-.5,2)}(\mathbf{0}, \Psi, g_\nu^{(2)})$ distribution with $\nu = 3$; (d) the *pdf* of $\mathrm{PE}_{(-.5,2)}(\mathbf{0}, \Psi, g_{VII}^{(2)})$ distribution with $m = N = 7$)

which incudes several perturbed distributions obtained from well-known symmetric distributions.

Property 4.1. If $Z \sim \mathrm{PS}_{(a,b)}(\theta, \Sigma, g^{(2)})$, then $-Z \sim \mathrm{PS}_{(a,b)}(\theta^*, \Sigma^*, g^{(2)})$, where θ^* and Σ^* are obtained from θ and Σ by changing their elements θ_1 and ρ with $-\theta_1$ and $-\rho$, respectively.

Property 4.2. Let $U(Z) = (Z - \theta_1)/\sigma_1$, where $Z \sim \mathrm{PS}_{(a,b)}(\theta, \Sigma, g^{(2)})$. Then

$$U(Z) \sim \mathrm{PS}_{(u(a),u(b))}(\mathbf{0}, \Psi, g^{(2)}). \tag{4.1}$$

Property 4.3. If $Z \sim \mathrm{PS}_{(\theta_2,\infty)}(\theta, \Sigma, g^{(2)})$, its *pdf* is

$$f_Z(z) = \frac{2}{\sigma_1} f_{g^{(1)}}(u_1(z)) F_{g_{q^*(z)}}(\lambda u_1(z)), \quad \text{for } z \in R, \tag{4.2}$$

where $F_{g_{q^*(z)}}(\cdot)$ is the *cdf* of $\mathrm{EC}_1(0, 1, g_{q^*(z)})$ with $q^*(z) = u_1(z)^2$.

The density (4.2) does not depend on the values of θ_2 and σ_2 and it is equivalent to the univariate "skew-elliptical" distribution introduced by Branco and Dey (2001). In this sense, Definition 3.1 extends the result of Branco and Dey (2001) where they studied only for the case of $(\theta_2 = 0, \sigma_2 = 1)$ and $(a = 0, b = \infty)$. When the relationship between λ and ρ in (4.2) is considered, we see λ has the sign of ρ, ranges from $-\infty$ to ∞ and is equal to zero if and only if $\rho = 0$. This directly gives the following property.

Property 4.4. Suppose $\mathbf{X} \sim \mathrm{EC}_2(\theta, \Sigma, g^{(2)})$, where $\mathbf{X} = (X, Y)'$, $\theta = (\theta_1, \theta_2)'$, $\Sigma = \{\sigma_{ij}\}$ and $\sigma_{12} = \rho\sigma_1\sigma_2$. If $\rho = 0$ then

$$[X \mid Y > \theta_2] \equiv [X \mid Y \leq \theta_2] \sim \mathrm{EC}_1(\theta_1, \sigma_{11}, g^{(1)}). \tag{4.3}$$

One can give a probabilistic representation of the distribution $\mathrm{PS}_{(a,b)}(\theta, \Sigma, g^{(2)}_{H,K})$ in terms of a scale mixture of independent normal and truncated normal laws. The scale mixture scheme directly gives the following theorem.

Theorem 4.1 . Conditionally on η having the *cdf* $H(\eta)$, let $V \sim N(\mu_1, K(\eta)\tau_1^2)$ and $W \sim N(\mu_2, K(\eta)\tau_2^2)$ be independent random variables. Then, for any real values a_1 and a_2 with $a_1 \neq 0$,

$$Z = a_1 V + a_2 W_{I(a,b)} \sim \mathrm{PS}_{(a,b)}(\theta, \Sigma, g^{(2)}_{H,K}), \tag{4.4}$$

where $W_{I(a,b)}$ denotes a truncated $N(\mu_2, K(\eta)\tau_2^2)$ variable and a and b are lower and upper truncation points, respectively. $\theta = (\theta_1, \theta_2)$ with $\theta_1 = \sum_{i=1}^2 a_i\mu_i$ and $\theta_2 = \mu_2$; $\Sigma = \{\sigma_{ij}\}, \sigma_{ii} = \sigma_i^2$, with $\sigma_{11} = K(\eta)\sum_{i=1}^2 a_i^2\tau_i^2$, $\sigma_{22} = K(\eta)\tau_2^2$ and $\rho = a_2\sigma_2/\sigma_1$.

Proof: Let $U = a_1V + a_2W$. Then (U, W) is a bivariate normal variable with $N_2(\theta, \Sigma)$ distribution. Conditionally on η, the distribution of U conditionally on $a < W < b$ is $\mathrm{PS}_{(a,b)}(\theta, \Sigma, g^{(2)}_N)$, by Example 1. Given η, V and W in $U = a_1V + a_2W$ are independent normal variables and hence the distribution of U given that $a < W < b$ equals that of $a_1V + a_2W_{I(a,b)}$ figuring in the statement of Theorem 4.1. Thus the scale mixture distribution (4.4) is directly obtained by using Example 4. □

When $a_2 = 0$, Z is a scale mixture of normal distribution. Thus the representation in (4.4) reveals the structure of the class of $\mathrm{PS}_{(a,b)}(\theta, \Sigma, g^{(2)}_{H,K})$ distributions and indicates the kind of departure from the symmetric distribution. Furthermore, the representation provides one-for-one method of generating a random variable Z with density (3.12). For generating the truncated normal variable $W_{I(a,b)}$, the one-for-one method by Devroye (1986) may be used.

Corollary 4.1 Conditionally on η, let V and W be independent $N(0, K(\eta))$ random variables and let $(X, Y) \sim N_2(\mathbf{0}, K(\eta)\Psi)$, where η is a random variable with the *cdf* $H(\eta)$ and weight function $K(\eta)$. Then

$$\frac{1}{\sqrt{1+\lambda^2}}V + \frac{\lambda}{\sqrt{1+\lambda^2}}W_{I(a,b)} \equiv [X|a < Y < b] \sim \mathrm{PS}_{(a,b)}(\mathbf{0}, \Psi, g^{(2)}_{H,K}), \tag{4.5}$$

where $\lambda = \rho/\sqrt{1-\rho^2}$.

Proof: Setting $\mu_1 = \mu_2 = 0$ and $\tau_1 = \tau_2 = 1$, we have the result from Theorem 4.1. □

For $K(\eta)=1$, we see that (4.5) is equivalent to the probabilistic representation by Kim (2007). In the case $a=0$ and $b=\infty$, if we set $K(\eta)=1$ and $K(\eta)=1/\eta$, $\mathrm{PS}_{(a,b)}(\mathbf{0},\Psi,g^{(2)}_{H,K})$ distributions reduce to $\mathcal{SN}(\lambda)$ by Azzalini (1985) and the skew-t_ν distribution, respectively. Their probabilistic representations obtained from (4.5) agree with those of $\mathcal{SN}(\lambda)$ given in Henze (1986) and skew-t_ν derived by Kim (2002).

4.2. Moments of the perturbed distributions

In this section we derive a general expression for the moment generating function(mgf) for the perturbed scale mixture of normal distribution in (3.12). To compute the moments of $Z\sim\mathrm{PS}_{(a,b)}(\theta,\Sigma,g^{(2)}_{H,K})$, it suffices to compute the moments of $U(Z)=(Z-\theta_1)/\sigma_1$. We see from (3.12) that $U(Z)$ has the density

$$g(u(z))=\frac{1}{\delta_1\sqrt{K(\eta)}}\int_0^\infty\phi\left(\frac{u(z)}{\sqrt{K(\eta)}}\right)\{\Phi(u_1^*(z))-\Phi(u_2^*(z))\}dH(\eta),\tag{4.6}$$

for $z\in\mathcal{R}$.

Some algebra gives that the moment generating function of $U(Z)=(Z-\theta_1)/\sigma_1$ is

$$M_{U(Z)}(t)=\frac{1}{\delta_1}E_\eta\left[\{\Phi\left(u_t(b)\right)-\Phi\left(u_t(a)\right)\}e^{K(\eta)\frac{t^2}{2}}\right],\tag{4.7}$$

for $t\in\mathcal{R}$, where E_η denotes that the expectation is taken with respect to the distribution of η variate having the density $dH(\eta)/d\eta$ and

$$u_t(a)=\frac{u(a)}{\sqrt{K(\eta)}}-\rho\sqrt{K(\eta)}\,t,\quad u_t(b)=\frac{u(b)}{\sqrt{K(\eta)}}-\rho\sqrt{K(\eta)}\,t.$$

Naturally, the moments of $U(Z)=(Z-\theta_1)/\sigma_1$ can be obtained by using the moment generating function differentiation. For example:

$$E[U(Z)]=M'_{U(Z)}(t)|_{t=0}=-\frac{\rho}{\delta_1}E_\eta\left[\sqrt{K(\eta)}\{\phi(u^*(b))-\phi(u^*(a))\}\right],\tag{4.8}$$

where $u^*(a)=u(a)/\sqrt{K(\eta)}$ and $u^*(b)=u(b)/\sqrt{K(\eta)}$. Unfortunately, for higher moments this rapidly becomes tedious.

An alternative procedure makes use of the fact that

$$\frac{d}{dx}[x^{k+1}\phi(x)]=(k+1)x^k\phi(x)-x^{k+2}\phi(x),\tag{4.9}$$

for $k=-1,0,1,2,3,\ldots$, yields the following result.

Let $W=U(Z)/\sqrt{K(\eta)}$. Under the distribution (4.6), the relation (4.9) and integrating by parts gives $E[(k+1)W^k-W^{k+2}]$, that is

$$\frac{\rho}{\delta_1\lambda_1^{k+1}}\int_0^\infty\left[\phi(u^*(b))E\left[V+\lambda u^*(b)\right]^{k+1}-\phi(u^*(a))E\left[V+\lambda u^*(a)\right]^{k+1}\right]dH(\eta),\tag{4.10}$$

for $k=-1,0,1,\ldots$, where V is a $N(0,1)$ variate.

By setting $k = -1, 0, 1$, we obtain three expressions, which may be solved to yield the first three moments of $U(Z)$. Higher moments could be found similarly. One obtains,

$$\begin{aligned}
E[U(Z)] =& -\frac{\rho}{\delta_1}E_\eta\left[\sqrt{K(\eta)}\{\phi(u^*(b)) - \phi(u^*(a))\}\right],\\
E[U(Z)^2] =& E_\eta[K(\eta)] - \frac{\rho^2}{\delta_1}E_\eta\left[K(\eta)\{u^*(b)\phi(u^*(b)) - u^*(a)\phi(u^*(a))\}\right],\\
E[U(Z)^3] =& -\frac{\rho}{\delta_1}E_\eta\Big[K(\eta)^{\frac{3}{2}}\{(3-\rho^2+u^*(b)\rho^2)\phi(u^*(b))\\
& -(3-\rho^2+u^*(a)\rho^2)\phi(u^*(a))\}\Big].
\end{aligned}$$

By using the Binomial expansion, one can see that the general formula for the moments of $Z \sim \text{PE}_{(a,b)}(\theta, \Sigma, g^{(2)}_{H,K})$ is

$$E[Z^k] = \sum_{j=0}^{k}\binom{k}{j}\theta_1^{k-j}\sigma_1^j E[U(Z)^j]. \tag{4.11}$$

When $\theta_1 = 0$, $E[Z^k] = \sigma_1^k E[U(Z)^k]$. It is also noted that existence of the moments depends on the mixing distribution $H(\eta)$.

The class of perturbed scale mixture of normal distribution includes well-known skew-elliptical distributions as in the following two examples.

EXAMPLE 5. One member of the perturbed scale mixture normal distribution in (3.12) is the $\text{PS}_{(\theta_2,\infty)}(\theta, \Sigma, g^{(2)}_{H,K})$ distribution, for which H is degenerate with $K(\eta) = 1$. From (3.12), we obtain the *pdf* given by

$$f_Z^*(z) = \frac{2}{\sigma_1}\phi(u(z))\Phi(\lambda u_1(z)), \quad z \in \mathcal{R}. \tag{4.12}$$

This distribution is equivalent to the skew-normal distribution, $\mathcal{SN}(\theta_1, \sigma_1, \lambda)$, by Azzalini (1985). From (4.11), one finds the moments,

$$E[Z] = \theta_1 + \sigma_1\rho\sqrt{\frac{2}{\pi}}, \quad \text{Var}(Z) = \sigma_1^2\left(1 - \frac{2\rho^2}{\pi}\right)$$

and

$$\alpha_3 = \left(\frac{4}{\pi} - 1\right)\sqrt{\frac{2}{\pi}}\rho^3\left(1 - \frac{2\rho^2}{\pi}\right)^{-\frac{3}{2}}. \tag{4.13}$$

These values of Z agree with those given in Arnold *et al.* (1993). See Azzalini (1985) and Henze (1986) for the other properties of the distribution.

EXAMPLE 6. Another member of the perturbed scale mixture of normal distribution distribution is the skew-t_ν distribution. When $K(\eta) = 1/\eta$ and $\eta \sim G(\nu/2, 2/\nu)$, the *pdf* (3.12) of the $\text{PS}_{(\theta_2,\infty)}(\theta, \Sigma, g^{(2)}_{H,K})$ distribution reduces to

$$f_Z^*(z) = \frac{2}{\sigma_1}f_\nu(u_1(z))F_{\nu+1}\left(\frac{\lambda u_1(z)\sqrt{\nu+1}}{\sqrt{\nu + u_1(z)^2}}\right), \quad z \in \mathcal{R}, \tag{4.14}$$

where $f_\nu(\cdot)$ and $F_\nu(\cdot)$ denote the *pdf* and the *cdf* of a univariate standard t_ν distribution, while $F_{\nu+1}(\cdot)$ is the *cdf* of a univariate standard $t_{\nu+1}$ distribution. This distribution is equivalent to the skew-t_ν distribution, $\mathcal{ST}(\theta_1, \sigma_1, \nu, \lambda)$, by Kim (2002). From (4.11), we obtain,

$$\begin{aligned} E(Z) &= \theta_1 + \sigma_1 \rho \sqrt{\frac{\nu}{\pi}} \frac{\Gamma\{(\nu-1)/2\}}{\Gamma\{\nu/2\}}, && \text{for } \nu > 1, \\ \mathrm{Var}(Z) &= \sigma_1^2 \frac{\nu}{\nu-2} - E(Z)^2, && \text{for } \nu > 2, \\ \alpha_3 &= \frac{A}{\mathrm{Var}(Z)^{\frac{3}{2}}}, && \text{for } \nu > 3, \end{aligned} \tag{4.15}$$

where

$$\begin{aligned} A &= \sigma_1^3 \lambda \left[\left(\frac{\nu}{\pi}\right)^{\frac{1}{2}} \frac{3\nu\Gamma\{(\nu-1)/2\}}{(1+\lambda^2)^{\frac{3}{2}}(\nu-2)\Gamma\{\nu/2\}} \left(\frac{1}{\nu-3} + \lambda^2 B\right)\right], \\ B &= \left[\frac{2(\nu-2)}{3(\nu-3)} + \frac{2(\nu-2)\Gamma\{(\nu-1)/2\}^2}{3\pi\Gamma\{\nu/2\}^2} - 1\right] \quad \text{and} \quad \lambda = \frac{\rho}{\sqrt{1-\rho^2}}. \end{aligned}$$

It is easily seen that the distribution does not depend on θ_2 and σ_2 and leads to a parametric class of distributions that have strict inclusion of t_ν distribution (for the case $\theta_1 = 0$, $\sigma_1 = 1$ and $\rho = 0$) and perturbed Cauchy distribution(for $\nu = 1$). See Kim (2002) for the other properties and applications of the distribution (4.14). Note that $B > 0$ by using a numerical evaluation for $\nu > 3$. Thus (4.13) and (4.15) imply that the distributions in both examples are skewed to the right(the left) when $\rho > 0 (\rho < 0)$.

5. Applications

The aim of this section is to provide simple applications of the material presented so far, focusing on those of $\mathrm{PS}_{(a,b)}(\theta, \Sigma, g_N^{(2)})$ and $\mathrm{PS}_{(a,b)}(\theta, \Sigma, g_\nu^{(2)})$ distributions.

5.1. Comparing elasticities in regression

Suppose we have a model where quantity Q depends on price P, so that the demand function is

$$Q = \alpha P^\beta u. \tag{5.1}$$

When we take logs of (5.1), we obtain $\log Q = \log \alpha + \beta \log P + \log u$ which is of the standard form

$$Y = \alpha^* + \beta X + e. \tag{5.2}$$

In the econometric context, β is called the price elasticity of the demand Q. The price elasticity is formally defined as the relative change in quantity demanded divided by the relative change in price, *i.e.* β=(percent change in Q)/(percent change in P). See Rudy (2002) for the analysis of the price elasticity to diagnose consumer expenditure.

Let observational models of two quantities(Q_1 and Q_2) in (5.1) be

$$y_{ij} = \alpha_i^* + \beta_i x_{ij} + e_{ij}, \quad i = 1, 2;\ j = 1, \ldots, n_i, \tag{5.3}$$

where $e_{ij} \overset{iid}{\sim} N(0, \tau_i^2)$ and e_{1j} and e_{2j} are independent variables. Suppose, from the microeconomic theory, that the price elasticity of the demand for Q_1 is inelastic to price so that $\beta_1 \in (0, 1)$. Then the price elasticities of the demands for Q_1 and Q_2 can be compared by the distribution of β_2/β_1 obtained from the Bayesian approach. We shall take a reference prior that is independently uniform in α_i^*, β_i and $\log \tau_i^2$, so that $\pi(\alpha_i^*, \beta_i, \tau_i^2) \propto 1/\tau_i^2 I(0 < \beta_1 < 1)$. It is now clear that the marginal posterior distributions of β_i's $(i = 1, 2)$ are independent and

$$\beta_1|\text{Data} \sim TN\left(b_1, \frac{\tau_1^2}{S_{11}}\right) I(0 < \beta_1 < 1) \quad \text{and} \quad \beta_2|\text{Data} \sim N\left(b_2, \frac{\tau_2^2}{S_{22}}\right)$$

for known τ_i^2, $i = 1, 2$,

$$\beta_1|\text{Data} \sim Tt_{(n_i-2)}\left(b_1, \frac{s_1^2}{S_{11}}\right) I(0 < \beta_1 < 1) \quad \text{and} \quad \beta_2|\text{Data} \sim t_{(n_i-2)}\left(b_2, \frac{s_2^2}{S_{22}}\right)$$

for unknown τ_i^2, where $b_i = S_{0i}/S_{ii}$, $(n_i - 2)s_i^2 = S_{00} - S_{0y}^2/S_{ii}$, $S_{ii} = \sum_{j=1}^{n_i}(x_{ij} - \bar{x}_i)^2$, $S_{0i} = \sum_{j=1}^{n_i}(y_{ij} - \bar{y}_i)(x_{ij} - \bar{x}_i)$, $S_{00} = \sum_{j=1}^{n_i}(y_{ij} - \bar{y}_i)^2$, $\bar{x}_i = \sum_{j=1}^{n_i} x_{ij}/n_i$ and $\bar{y}_i = \sum_{j=1}^{n_i} y_{ij}/n_i$. Here $TN(a, b)I(0 < \beta_1 < 1)$ and $Tt_{(n_i-1)}(a, b)I(0 < \beta_1 < 1)$ denote the normal and $t_{(n_i-1)}$ distributions truncated below at 0 and above at 1, respectively. a and b are the location and scale parameters of each distribution.

For known τ_i^2, the posterior distribution function $F(c)$ of β_2/β_1 is the same as $P(Z \le 0)$ obtained from $Z \sim \text{PS}_{(0,1)}(\theta_c, \Sigma_c, g_N^{(2)})$ by Theorem 4.1, where $\theta_c = (\theta_1, \theta_2)$ with $\theta_1 = b_2 - cb_1$ and $\theta_2 = b_1$; $\Sigma_c = \{\sigma_{ij}\}$ with $\sigma_{11} = c^2\tau_1^2/S_{11} + \tau_2^2/S_{22}$, $\sigma_{22} = \tau_1^2/S_{11}$, $\sigma_{12} = -c\tau_1^2/S_{11}$ and $\rho = \sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}}$. Thus the distribution function $F_c^*(z)$ of $\text{PS}_{(0,1)}(\theta_c, \Sigma_c, g_N^{(2)})$ variable in (3.5) leads to the distribution function $F(c)$ of β_2/β_1. That is

$$F(c) = F_c^*(0) = 1 - \frac{L\left(-\frac{\theta_1}{\sqrt{\sigma_{11}}}, -\frac{\theta}{\sqrt{\sigma_{22}}}, \rho\right) - L\left(-\frac{\theta_1}{\sqrt{\sigma_{11}}}, \frac{1-\theta}{\sqrt{\sigma_{22}}}, \rho\right)}{\Phi\left(\frac{1-\theta_2}{\sqrt{\sigma_{22}}}\right) - \Phi\left(-\frac{\theta_2}{\sqrt{\sigma_{22}}}\right)},$$

where $L(\alpha, \beta, \rho)$ denotes the orthant probability function of the standard bivariate normal variable defined in Johnson and Kotz (1972). For the case τ_i^2 is unknown but with large n_i $(i = 1, 2)$, an approximate posterior distribution function of β_2/β_1 can be obtained by substituting $n_i s_i^2/(n_i - 2)$ for τ_i^2, $i = 1, 2$ in the scale matrix Σ_c of the $\text{PS}_{(0,1)}(\theta_c, \Sigma_c, g_N^{(2)})$ distribution.

5.2. Paired sample problem

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be paired random sample from bivariate normal distribution, where $\text{Corr}(X_i, Y_i) = \rho$, $X_i \overset{iid}{\sim} N(\mu_X, \sigma^2)$ and $Y_i \overset{iid}{\sim} N(\mu_Y, \sigma^2)$. Then sampling distributions of $n^{1/2}(\bar{X} - \mu_X)/s_p$ and $n^{1/2}(\bar{Y} - \mu_Y)/s_p$ are identically distributed as $N(0, \sigma^2/s_p^2)$ with $s_p^2/\sigma^2 \sim G(\nu/2, 2/\nu)$, where $\nu = 2n - 2$ and

$$\nu s_p^2 = (1 - \rho^2)^{-1}\left\{\sum_{i=1}^{n}(X_i - \bar{X})^2 - 2\rho\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y}) + \sum_{i=1}^{n}(Y_i - \bar{Y})^2\right\}.$$

Above result and the definition of the $\text{PS}_{(a,b)}(\theta, \Sigma, g_\nu^{(2)})$ distribution(also see Example 6) immediately gives

$$n^{\frac{1}{2}}\frac{\bar{X}-\mu_X}{s_p} \mid \bar{Y} > \mu_Y \sim \mathcal{ST}(0,1,\nu,\lambda),$$
$$n^{\frac{1}{2}}\frac{\bar{X}-\mu_X}{s_p} \mid \bar{Y} < \mu_Y \sim \mathcal{ST}(0,1,\nu,-\lambda),$$

where $\lambda = \rho\sqrt{1-\rho^2}$. From (4.14), we obtain the unconditional *pdf* of $Z_X = n^{1/2}(\bar{X} - \mu_X)/s_p$ that is

$$\begin{aligned} h(z_X) &= f_\nu(z_X)F_{\nu+1}\left(\frac{\lambda z_X\sqrt{\nu+1}}{\sqrt{\nu+z_X^2}}\right) + f_\nu(z_X)F_{\nu+1}\left(\frac{-\lambda z_X\sqrt{\nu+1}}{\sqrt{\nu+z_X^2}}\right) \\ &= f_\nu(z_X), \end{aligned}$$

where $f_\nu(z_X)$ is the *pdf* of a standard t_ν distribution. Thus, we see that irrespective of the value of ρ ($|\rho| \le 1$), $Z_X \sim t_{2n-2}$. Similar argument gives $Z_Y = n^{1/2}(\bar{Y}-\mu_Y)/s_p \sim t_{2n-2}$. These imply that, Z_X and Z_Y are pivotal quantities for μ_X and μ_Y for the bivariate normal population with common variance σ^2 and known ρ.

5.3. A Kalman filtering

A well-known property of the normal distribution is that, if Y is $N(Z,\sigma^2)$ where a priori Z is a normal variable, then the a posteriori distribution Z is still normal. Kim (2007) showed that analogous fact is true if a priori Z has probability density function in (3.5). Under this prior, some simple algebra shows that the a posteriori density function of Z given $Y=y$ is still of type (3.5) with $(\theta_1, \sigma_1, \lambda, u(a), u(b))$ replaced by

$$\frac{y/\tau^2+\theta_1/\sigma_1^2}{1/\tau^2+1/\sigma_1^2}, \quad \left(\frac{1}{\sigma_1^2}+\frac{1}{\tau^2}\right)^{-\frac{1}{2}}, \quad \lambda\left(1+\frac{\sigma_1^2}{\tau^2}\right)^{-\frac{1}{2}},$$
$$\left\{\lambda_1 u(a) + \frac{\lambda\sigma_1(y-\theta_1)}{\sigma_1^2+\tau^2}\right\}\left\{1+\frac{\lambda^2\tau^2}{\sigma_1^2+\tau^2}\right\}^{-\frac{1}{2}},$$
$$\left\{\lambda_1 u(b) + \frac{\lambda\sigma_1(y-\theta_1)}{\sigma_1^2+\tau^2}\right\}\left\{1+\frac{\lambda^2\tau^2}{\sigma_1^2+\tau^2}\right\}^{-\frac{1}{2}}.$$

Note that the parameter λ shrinks towards 0, independently of y and that the updating formulae of the first two parameters are the same as those of the normal case.

Consider now the Kalman filtering setting

$$\begin{aligned} Z_t &= \rho Z_{t-1} + \varepsilon_t, \\ Y_t &= Z_t + \eta_t, \quad t = 1,2,\ldots, \end{aligned}$$

where $\{\varepsilon_t\}$ is white noise $N(0,\sigma_\varepsilon^2)$ and $\{\eta_t\}$ is white noise $N(0,\sigma_\eta^2)$, with $\{\varepsilon_t\}$ independent of $\{\eta_t\}$. If the initial prior of Z_0 is normal, then all subsequent posterior distributions of

Z_t given $(Y_1, \ldots, Y_t)$ are still normal. An analogous property holds for the distribution (3.5): if the initial prior distributions of Z_0 is of type (3.5), then all subsequent posterior distributions of Z_t given $(Y_1, \ldots, Y_t)$ are again of type (3.5). This fact follows from the above conjugacy property and the application of Theorem 4.1. When Z is a random variable with density function (3.2) and ε is an independent $N(0, \sigma_\varepsilon^2)$ variable, $Z + \varepsilon$ has density function of type (3.5) by Theorem 4.1.

6. Conclusion

This paper has presented a new class of perturbed elliptical distributions. To form the class of distributions, we considered a conditioning method to the bivariate elliptical distributions. This introduces yet other perturb function, inducing perturbation of the symmetry with univariate elliptical distribution, that brings additional flexibility of modeling skewed and heavy tailed distribution. The results obtained in this paper extend many properties of the bivariate elliptical distributions in a nontrivial way. Several properties of the class are studied, especially for the perturbed normal and perturbed scale mixture of normal distributions. The study shows that we have at hand a class of distributions with following properties: (i) strict inclusion of the normal, skew-normal and their scale mixture densities, (ii) mathematical tractability and (iii) wide applicability in solving statistical problems as demonstrated by Section 5.

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