Honam Mathematical J. **30** (2008), No. 3, pp. 581–587

A STUDY OF AVERAGE ERROR BOUND OF TRAPEZOIDAL RULE

Meehyea Yang[†] and Bum Il Hong[‡]

Abstract. In this paper, to have a better *a posteriori* error bound of the average case error between the true value of $I(f)$ and the Trapezoidal rule on subintervals using zero mean-Gaussian, we prove that a new average error between the difference of the true value of $I(f)$ from the composite Trapezoidal rule and that of the composite Trapezoidal rule from the simple Trapezoidal rule is bounded by $c_r h^{2r+3}$ through direct computation of constants c_r for $r \leq 2$ under the assumption that we have subintervals (for simplicity equal length h) partitioning $[0,1]$.

1. Introduction

Many computational problems in science and engineering can only be solved approximately since the available information is partial. For instance, for problems defined on a space of functions, information about f is typically provided by few function values, $N(f) = [f(x_0), f(x_1), \ldots, f(x_n)]$. Knowing $N(f)$, the solution is approximated by an algorithm. Therefore we have the error between the true and the approximate solutions. The error between the true and the approximate solutions can be reduced by acquiring more information.

The error between the true solution and the approximation depends on a problem setting. In the worst case setting, the error of a numerical scheme is defined by its worst performance with respect to the given class of functions. In this paper, we concentrate on another setting, the average case setting. In this setting, we assume that the class F of input functions is equipped with a probability measure. The general references are [1], [6] and [8].

[‡]Corresponding author.

Received August 8, 2008. Revised September 1, 2008.
2000 Mathematics Subject Classification : 65D30, 65G10, 28C20.
Key words and phrases : Trapezoidal rule, Error analysis, Wiener measure.
[†] This research was supported by the University of Incheon Research Grant in 2007.

2. Definitions

It is well known that the average case setting requires the space of functions to be equipped with a probability measure. In this paper, we choose a probability measure μ_r which is a variant of an r-fold Wiener measure ω_r. The probability measure ω_r is a Gaussian measure with zero mean and correlation function given by $M_{\omega_r}(f(x)\,f(y)) = \int_F f(x)f(y)\,\omega_r(df) = \int_0^1 \frac{(x-t)_+^r}{r!}\frac{(y-t)_+^r}{r!}\,dt$, where $(z-t)_+^r = [\max\{0,(z-t)\}]^r$. Equivalently, f distributed according to ω_r can be viewed as a Gaussian stochastic process with zero mean and autocorrelation given above. However, since ω_r is concentrated on functions with boundary conditions $f(0) = f'(0) = \cdots = f^{(r)}(0) = 0$, we choose to study a slightly modified measure μ_r that preserves basic properties of ω_r, yet does not require any boundary conditions. More precisely, we assume that a function f, as a stochastic process, is given by

$$f(x) \;=\; f_1(x) + f_2(1-x), \quad x \in [0,1],$$

where f_1 and f_2 are independent and distributed according to ω_r. Then by [4-7], the corresponding probability measure μ_r is a zero mean Gaussian with the correlation function given by

$$\begin{aligned} M_{\mu_r}(f(x)\,f(y)) &= \int_0^1 \frac{(x-t)_+^r(y-t)_+^r \;+\; (1-x-t)_+^r(1-y-t)_+^r}{r!\;r!}\,dt \\ &= \int_0^1 \frac{(x-t)_+^r(y-t)_+^r \;+\; (t-x)_+^r(t-y)_+^r}{r!\;r!}\,dt. \end{aligned}$$

We study the problem of approximating an integral $I(f) = \int_0^1 f(x)\,dx$ for $f \in F = C^r[0,1]$, assuming that the class of integrands is equipped with the probability measure μ_r.

Assume that we have n subintervals (not necessarily equal length) partitioning $[0,1]$. Let $x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$. But for simplicity, we let $x_i = ih$, for $i = 0,\ldots,n$ where $h = \frac{1}{n}$. With this indexing, we get

$$I_i(f) \equiv \int_{x_{i-1}}^{x_{i+1}} f(x)\,dx \quad \text{and} \quad T_i(f) \;=\; h\,\{f(x_{i-1}) + f(x_{i+1})\}$$

while T_i is the basic Trapezoidal rule using $f(x_{i-1})$ and $f(x_{i+1})$. Let $\overline{T_i}$ be the composite Trapezoidal rule that uses $f(x_{i-1})$, $f(x_i)$ and $f(x_{i+1})$, i.e.,

$$\overline{T_i}(f) \;=\; \frac{h}{2}\,\{f(x_{i-1}) + 2f(x_i) + f(x_{i+1})\}\,.$$

Also let

$$X_i(f) = I_i(f) - \overline{T_i}(f) \quad \text{and} \quad Y_i(f) = \overline{T_i}(f) - T_i(f).$$

3. Error bounds for a subinterval using distribution

Since the true value of $I(f)$ is unknown, we do not know the actual error of $X(f) = |I(f) - T(f)|$. Therefore $|Y_i(f)|$ is often used as an *a posteriori* error bound for $|X_i(f)|$. However, it is well known that $|Y_i(f)|$ is not a good bound if $f \notin C^3[0,1]$. As a result, in order to find a new and better *a posteriori* error bound in the sense of probability, we need to know the distributions not only X_i and Y_i, but also $X_i - Y_i$. In fact, the distributions of X_i and Y_i, and error bounds for $M_{\mu_r}(X_iX_j)$ and $M_{\mu_r}(Y_iY_j)$ were already studied [2,3].

In this section, we consider two consecutive subintervals. In order to find a new error bound for the subintervals, we compute the distribution of $X_i - Y_i$. it is also Gaussian with zero-mean and covariances like X_i and Y_i. A new error bound for $M_{\mu_r}([X_i - Y_i][X_j - Y_j])$ is given in next theorem that is the main result of this paper.

THEOREM 3.1. *For a non-negative integer $r \leq 2$,*

$$M_{\mu_r}([X_i - Y_i][X_j - Y_j]) = \delta_{ij} \cdot c_r \cdot h^{2r+3}, \quad \text{for} \quad i \leq j,$$

where δ_{ij} is the Kronecker delta and the constants c_r are independent of h's and equal $c_0 = \frac{16}{3}$, $c_1 = \frac{38}{15}$ and $c_2 = \frac{4}{7}$ respectively.

Proof. Let $[X_{i1} - Y_{i1}] = [X_i - Y_i](f_1)$ and $[X_{i2} - Y_{i2}] = [X_i - Y_i](f_2)$. Then $[X_i - Y_i](f) = [X_{i1} - Y_{i1}] + [X_{i2} - Y_{i2}]$, and due to the independence of f_1 and f_2, we have

$$\begin{aligned} &M_{\mu_r}([X_i - Y_i][X_j - Y_j]) \\ =&M_{\omega_r}([X_{i1} - Y_{i1}][X_{j1} - Y_{j1}]) + M_{\omega_r}([X_{i2} - Y_{i2}][X_{j2} - Y_{j2}]). \end{aligned} \tag{1}$$

It is easy to see that $Y_i(f) = -\frac{h}{2}\nabla_i f = -\frac{h}{2}\nabla_i f_1 - \frac{h}{2}\nabla_i f_2$ where $h = (x_{i+1} - x_{i-1})/2$ and $\nabla_i f$ is the backward difference f, i.e.,

$$\nabla_i f = f(x_{i-1}) - 2f(x_i) + f(x_{i+1}).$$

Now,

$$M_{\omega_r}\left([X_{i1}-Y_{i1}][X_{j1}-Y_{j1}]\right) \;=\; \int_0^{x_{i+1}} L_{i1}(t)\cdot L_{j1}(t)\,dt,$$

where

$$L_{i1}(t) \;=\; \int_{x_{i-1}}^{x_{i+1}} \frac{(x-t)_+^r}{r!}dx - \overline{T}_i\left(\frac{(\cdot-t)_+^r}{r!}\right) + \frac{h}{2}\,\nabla_i\left(\frac{(\cdot-t)_+^r}{r!}\right),$$

and

$$L_{j1}(t) \;=\; \int_{x_{j-1}}^{x_{j+1}} \frac{(y-t)_+^r}{r!}\,dy - \overline{T}_j\left(\frac{(\cdot-t)_+^r}{r!}\right) + \frac{h}{2}\,\nabla_j\left(\frac{(\cdot-t)_+^r}{r!}\right).$$

Similarly,

$$M_{\omega_r}\left([X_{i2}-Y_{i2}][X_{j2}-Y_{j2}]\right) \;=\; \int_{x_{j-1}}^{1} L_{i2}(t)\cdot L_{j2}(t)\,dt.$$

Since the operator ∇_i is exact for polynomials of degree ≤ 2, $L_{j1}(t)=0$ for $t\le x_{i+1}$ and $L_{i2}(t)=0$ for $t\ge x_{j-1}$. Therefore $M_{\mu_r}([X_i-Y_i][X_j-Y_j])=0$ when $i<j$. Therefore it is enough to compute the case of $i=j$. If $i=j$, then

$$\begin{aligned}
&M_{\omega_r}\left([X_{i1}-Y_{i1}][X_{j1}-Y_{j1}]\right)\\
&=M_{\omega_r}\left([X_{i1}-Y_{i1}]^2\right)\\
(2)\quad &=\int_0^{x_{i+1}} L_{i1}(t)\cdot L_{i1}(t)\,dt\\
&=\int_{x_{i-1}}^{x_{i+1}}\left[\int_{x_{i-1}}^{x_{i+1}} \frac{(x-t)_+^r}{r!}dx - \overline{T}_i\left(\frac{(\cdot-t)_+^r}{r!}\right) + \frac{h}{2}\nabla_i\left(\frac{(\cdot-t)_+^r}{r!}\right)\right]^2 dt.
\end{aligned}$$

We now compute $\frac{h}{2}\nabla_i\left(\frac{(\cdot-t)_+^r}{r!}\right)$ on $[x_{i-1},x_{i+1}]$. By the definition of ∇_i,

$$\frac{h}{2}\nabla_i\left(\frac{(\cdot-t)_+^r}{r!}\right) = \frac{h}{2}\left(\frac{(x_{i-1}-t)_+^r}{r!} - 2\frac{(x_i-t)_+^r}{r!} + \frac{(x_{i+1}-t)_+^r}{r!}\right).$$

If we use the change of variable, $u=\frac{t-x_{i-1}}{2h}$, $\quad t\in[x_{i-1},x_{i+1}]$, then we have

(3)

$$\begin{aligned}
\overline{T}_i\left(\frac{(\cdot-t)_+^r}{r!}\right) &= \frac{h}{2}\left(\frac{(2h)^r(0-u)_+^r}{r!} + 2\frac{(2h)^r(\frac{1}{2}-u)_+^r}{r!} + \frac{(2h)^r(1-u)_+^r}{r!}\right)\\
\frac{h}{2}\nabla_i\left(\frac{(\cdot-t)_+^r}{r!}\right) &= \frac{h}{2}\left(\frac{(2h)^r(0-u)_+^r}{r!} - 2\frac{(2h)^r(\frac{1}{2}-u)_+^r}{r!} + \frac{(2h)^r(1-u)_+^r}{r!}\right)
\end{aligned}$$

where $u \in [0,1]$. Therefore, from (3), we have

$$-\overline{T}_i\left(\frac{(\cdot - t)_+^r}{r!}\right) + \frac{h}{2}\nabla_i\left(\frac{(\cdot - t)_+^r}{r!}\right) = -2h\frac{(2h)^r(\frac{1}{2}-u)_+^r}{r!}. \tag{4}$$

Moreover, if we apply $z = \frac{x - x_{i-1}}{2h}$ similarly to $\int_{x_{i-1}}^{x_{i+1}} \frac{(x-t)_+^r}{r!}dx$ in (1), then we have

$$\int_{x_{i-1}}^{x_{i+1}} \frac{(x-t)_+^r}{r!}dx = \int_0^1 \frac{(2h)^r(z-u)_+^r}{r!}\,2hdz. \tag{5}$$

So, by (4) and (5), (2) becomes

$$\begin{aligned}
&M_{\omega_r}\left([X_{i1} - Y_{i1}][X_{j1} - Y_{j1}]\right) \\
=&\int_0^1 \left[\int_0^1 \frac{(2h)^r(z-u)_+^r}{r!}\,2hdz - 2h\frac{(2h)^r(\frac{1}{2}-u)_+^r}{r!}\right]^2 2hdu \\
=&c_{r1}h^{2r+3}
\end{aligned}$$

where

$$c_{r1} = 2^{2r+3}\int_0^1\left[\int_0^1 \frac{(z-u)_+^r}{r!}\,dz - \frac{(\frac{1}{2}-u)_+^r}{r!}\right]^2 du.$$

Similarly, we can easily show that

$$M_{\omega_r}\left([X_{i2} - Y_{i2}][X_{j2} - Y_{j2}]\right) = M_{\omega_r}\left([X_{i2} - Y_{i2}]^2\right) = c_{r2}h^{2r+3},$$

where

$$c_{r2} = 2^{2r+3}\int_0^1\left[\int_0^1 \frac{(u-z)_+^r}{r!}\,dz - \frac{(u-\frac{1}{2})_+^r}{r!}\right]^2 du.$$

Since

$$\int_0^1\left[\int_0^1 \frac{(z-u)_+^r}{r!}\,dz - \frac{(\frac{1}{2}-u)_+^r}{r!}\right]^2 du = \int_0^1\left[\int_0^1 \frac{(u-z)_+^r}{r!}\,dz - \frac{(u-\frac{1}{2})_+^r}{r!}\right]^2 du,$$

$c_{r1} = c_{r2}$ for any non-negative integer $r \le 2$. Therefore, by (1), $c_r = c_{r1} + c_{r2} = 2c_{r1}$. We now calculate c_{r1}.

$$\begin{aligned}
c_{r1} =&2^{2r+3}\int_0^1\left[\int_0^1 \frac{(z-u)_+^r}{r!}\,dz - \frac{(\frac{1}{2}-u)_+^r}{r!}\right]^2 du \\
=&2^{2r+3}\int_0^1\left[\int_u^1 \frac{(z-u)^r}{r!}\,dz - \frac{(\frac{1}{2}-u)_+^r}{r!}\right]^2 du.
\end{aligned}$$

For $r=0$,

$$c_{01}=2^3\int_0^1\left[\int_u^1 dz-(\frac{1}{2}-u)_+^0\right]^2 du=2^3\int_0^1[1-u-1]^2\,du=\frac{8}{3}.$$

For $r=1$,

$$\begin{aligned}c_{11}=&2^{2+3}\int_0^1\left[\int_u^1(z-u)\,dz-(\frac{1}{2}-u)_+\right]^2 du\\=&2^5\int_0^1\left[\frac{1}{2}(1-u)^2-(\frac{1}{2}-u)_+\right]^2 du\\=&2^5\Big(\int_0^{\frac{1}{2}}\left[\frac{1}{2}(1-u)^2-(\frac{1}{2}-u)\right]^2 du+\int_{\frac{1}{2}}^1\left[\frac{1}{2}(1-u)^2-(u-\frac{1}{2})\right]^2 du\Big)\\=&\frac{19}{15}.\end{aligned}$$

for $r=2$,

$$\begin{aligned}&c_{21}\\=&2^{2\cdot 2+3}\int_0^1\left[\int_u^1\frac{(z-u)^2}{2}\,dz-\frac{(\frac{1}{2}-u)_+^2}{2}\right]^2 du\\=&2^7\int_0^1\left[\frac{1}{6}(1-u)^3-\frac{(\frac{1}{2}-u)_+^2}{2}\right]^2 du\\=&2^7\Big(\int_0^{\frac{1}{2}}\left[\frac{1}{6}(1-u)^3-\frac{(\frac{1}{2}-u)^2}{2}\right]^2 du+\int_{\frac{1}{2}}^1\left[\frac{1}{6}(1-u)^2-\frac{(u-\frac{1}{2})^2}{2}\right]^2 du\Big)\\=&2^7\int_0^1\left[\frac{1}{6}(1-u)^3-\frac{(\frac{1}{2}-u)^2}{2}\right]^2 du\\=&\frac{2}{7}.\end{aligned}$$

Therefore $c_0=2c_{01}=\frac{16}{3}$, $c_1=2c_{11}=\frac{38}{15}$ and $c_2=2c_{21}=\frac{4}{7}$. This completes the proof. □

Remark 1. Even though we used equally spaced knots x_i for simplicity, unequally knots x_i do give us the same result. In fact, we are easily able to show that $M_{\mu_r}([X_i-Y_i][X_j-Y_j])=\delta_{ij}\cdot c_r\cdot h_i^{2r+3}, \quad \text{for} \quad i\le j$

Remark 2. In the case of $r\ge 3$, we expect that the error bound is still bounded by ch^8 at most and a proof is much more complicated than the case of $r\le 2$. We will explore this in the later paper.

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Meehyea Yang
Department of Mathematics,
The University of Incheon,
Incheon 402-749, Korea
E-mail: mhyang@incheon.ac.kr

Bum Il Hong
Department of Mathematics,
Kyung Hee University,
Yongin, Kyunggi 446-701, Korea
E-mail: bihong@khu.ac.kr