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GENERALIZED STABILITIES OF CAUCHY'S GAMMA-BETA FUNCTIONAL EQUATION

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Abstract. We obtain generalized super stability of Cauchy's gamma-beta functional equation

$$B(x,y)f(x+y) = f(x)f(y),$$

where $B(x,y)$ is the beta function and also generalize the stability in the sense of R. Ger of this equation in the following setting :

$$\left| \frac{B(x,y)f(x+y)}{f(x)f(y)} - 1 \right| < H(x,y),$$

where $H(x,y)$ is a homogeneous function of dgree p $(0 \le p < 1)$.

1. Introduction

In 1940, S. M. Ulam gave a wide ranging talk before the Mathematical Club of the University of Wisconsin in which he discussed a number of important unsolved problems (ref. [22]). Among those was the question concerning the stability of homomorphisms : Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all x, $y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In the next year, D. H. Hyers [5] answered the question of Ulam for the case where G_1 and G_2 are Banach spaces. Futhermore, the result of Hyers has been generalized by Th. M. Rassias [15]. Since then, the stability problems of various functional equations has been investigated by many authors (see [1-22]).

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Note that the gamma function $\Gamma(x)$ is defined by

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dx \quad (x > 0)$$

and the equation $f(x+1) = xf(x)$ is called the gamma functional equation. Also the beta function $B(x, y)$ is defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

and the equation

$$B(x+1, y+1) = \frac{xy}{(x+y)(x+y+1)}B(x, y)$$

is called the beta functional equation. S. -M. Jung [9] obtained stability theorems of the gamma functional equation and K. W. Jun, G. H. Kim and Y.W. Lee [8, 11] proved the stability of the beta functional equation.

In particular, Y.W. Lee and B. M. Choi [14] proved the super stability of Cauchy's gamma-beta functional equation

$$B(x, y)f(x+y) = f(x)f(y) \tag{1}$$

where $B(x, y)$ is the beta function and they obtain a stability in the sense of R. Ger of this equation in the following setting :

$$\left|\frac{B(x, y)f(x+y)}{f(x)f(y)} - 1\right| < \delta.$$

In this paper, we generalize these two results as follows; If a function f is unbounded and

$$|B(x, y)f(x+y) - f(x)f(y)| < \varphi(x, y)$$

where $B(x, y)$ is the beta function, then f satisfies the Cauchy's gamma-beta functional equation. And if a function f satisfies the condition

$$\left|\frac{B(x, y)f(x+y)}{f(x)f(y)} - 1\right| < H(x, y).$$

where $H(x, y)$ is a homogeneous function of dgree p $(0 < p < 1)$, that is, $H(tx, ty) = t^p H(x, y)$ for all $x, y \in (0, \infty)$, then f has a stability.

2. Generalized Super stability of the equation (1)

J. Baker, J. Lawrence and F. Zorzitto [2] proved the Hyers-Ulam stability of Cauchy's exponential equation

$$f(x+y)=f(x)f(y).$$

That is, if the Cauchy difference $f(x+y)-f(x)(y)$ of a real-valued function f defined on a real vector space is bounded for all x, y, then f is either bounded or exponential. Their result was generalized by J. Baker [1] : let S be a semi-group and let $f : S \to E$ be a mapping where E is a normed algebra in which the norm is multiplicative. If f satisfies the functional inequality

$$\| f(xy)-f(x)f(y) \| < \delta$$

for all x, $y \in S$, then f is either bounded or multiplicative. In particular, such a phenomenon for some functional equation is called super stability. First we discuss solutions of the functional equation (1). Secondly a generalized super stability problem for the equation (1) shall be investigated.

Proposition 2.1. *If f is a continuous solution with $f(1)=a>0$ of the equation (1), then $f(x)=a^x\Gamma(x)$ for all $x>0$. That is, $f(x)=a^x\Gamma(x)$ is the unique continuous solution of Cauchy's gamma-beta functional equation with $f(1)=a$ [See, 14].*

Proof. Clearly, $f(x)=a^x\Gamma(x)$ is a continuous solution with $f(1)=a$. Suppose that g is an another continuous solution of the gamma-beta functional equation (1) with $g(1)=a$. Since $\Gamma(1)=1$, $g(1)=a\Gamma(1)$. Then $g(2)=g(1)g(1)/B(1,1)=a^2\Gamma(2)$. If $g(n)=a^n\Gamma(n)$ for some positive integer n, then

$$g(n+1)=\frac{g(n)g(1)}{B(n+1,1)}=\frac{a^n\Gamma(n)a\Gamma(1)}{B(n+1,1)}=a^{n+1}\Gamma(n+1).$$

An induction argument implies that $g(n)=a^n\Gamma(n)$ for every positive integer n. Also we have

$$g(\frac{1}{2})^2=g(1)B(\frac{1}{2},\frac{1}{2})=a\Gamma(1)B(\frac{1}{2},\frac{1}{2})=a\Gamma(\frac{1}{2})^2.$$

Thus $g(\frac{1}{2})=a^{\frac{1}{2}}\Gamma(\frac{1}{2})$. If $g(\frac{1}{2^n})=a^{\frac{1}{2^n}}\Gamma(\frac{1}{2^n})$ for some positive integer n, then

$$\begin{aligned} g(\frac{1}{2^{n+1}})^2 &= g(\frac{1}{2^n})B(\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}}) \\ &= a^{\frac{1}{2^n}}\Gamma(\frac{1}{2^n})B(\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}}) \\ &= (a^{\frac{1}{2^{n+1}}})^2\Gamma(\frac{1}{2^{n+1}})^2. \end{aligned}$$

Also an induction argument implies that $g(\frac{1}{2^n}) = a^{\frac{1}{2^n}}\Gamma(\frac{1}{2^n})$ for all positive integer n. Note that for every positive integer m, $m = a_0 2^0 + a_1 2^1 + \cdots + a_k 2^k$ where $a_i = 0$ or 1 for each $i = 1, 2, \cdots, k$. We may assume that $a_i = 1$ for all $i = 1, 2, \cdots, k$. Then for any positive integer n,

$$\begin{aligned} g(\frac{m}{2^n}) &= g(\frac{1 + 2^1 + \cdots + 2^k}{2^n}) \\ &= \frac{g(\frac{1}{2^n})g(\frac{2^1+2^2+\cdots+2^k}{2^n})}{B(\frac{1}{2^n}, \frac{2^1+2^2+\cdots+2^k}{2^n})} \\ &= \frac{g(\frac{1}{2^n})g(\frac{2^1}{2^n})\cdots g(\frac{2^k}{2^n})}{\prod_{i=0}^{k-1} B(\frac{2^i}{2^n}, \frac{\sum_{j=i+1}^{k} 2^j}{2^n})} \\ &= a^{\frac{m}{2^n}}\Gamma(\frac{m}{2^n}). \end{aligned}$$

Now let r be any positive real number. For each $\epsilon > 0$ we can choose an integer n with $1/2\epsilon < 2^n$ and also choose an integer m such that $m < 2^n(r + \epsilon) \leq m + 1$. Then

$$r - \epsilon = r + \epsilon - 2\epsilon < \frac{m+1}{2^n} - \frac{1}{2^n} = \frac{m}{2^n} < r + \epsilon.$$

Thus, for any real number $r > 0$, there are positive integers n, m such that

$$\mid \frac{m}{2^n} - r \mid < \epsilon.$$

By the continuity of f we have

$$g(r) = a^r\Gamma(r)$$

for every positive real number r. Thus $f(x) = g(x)$ for all $x > 0$. □

Theorem 2.2. *Let a function $\varphi : (0,\infty) \times (0,\infty) \to (0,\infty)$ be given with $\varphi(x,y) = \varphi(y,x)$ for all $x, y \in (0,\infty)$ and $\varphi(x,y) \le \varphi(x,x)$ for $0 < x \le y$. Suppose that $f : (0,\infty) \to (0,\infty)$ be a function with $f(m) \ge max(2, 2\sqrt{\varphi(m,m)})$ for some positive integer m such that*

$$| B(x,y)f(x+y) - f(x)f(y) | < \varphi(x,y) \tag{2}$$

for all x, $y \in (0,\infty)$. Then

$$B(x,y)f(x+y) = f(x)f(y)$$

for all x, $y \in (0,\infty)$ (Compare, Theorem 2.2 in [14]).

Proof. If we replace x by m and also y by m in (2), respectively, we get

$$| B(m,m)f(2m) - f(m)^2 | < \varphi(m,m).$$

An induction argument implies that for all n

$$\begin{aligned} &| f(nm) \prod_{i=1}^{n-1} B(im,m) - f(m)^n | \\ &\le f(m)^{n-2}\varphi(m,m) + f(m)^{n-3}B(m,m)\varphi(2m,m) \\ &\quad + f(m)^{n-4}B(2m,m)B(m,m)\varphi(3m,m) + \cdots \\ &\quad + B((n-2)m,m)B((n-3)m,m)\cdots B(m,m)\varphi((n-1)m,m). \end{aligned} \tag{3}$$

Indeed, if the inequality (3) holds, we have

$$\begin{aligned} &| f((n+1)m) \prod_{i=1}^{n} B(im,m) - f(m)^{n+1} | \\ &\le | B(nm,m)f((n+1)m) - f(nm)f(m) | \prod_{i=1}^{n-1} B(im,m) \\ &\quad + f(m) | f(nm) \prod_{i=1}^{n-1} B(im,m) - f(m)^n | \\ &\le f(m)^{n-1}\varphi(m,m) + f(m)^{n-2}B(m,m)\varphi(2m,m) \\ &\quad + f(m)^{n-3}B(2m,m)B(m,m)\varphi(3m,m) \\ &\quad + \cdots + \varphi(nm,m) \prod_{i=1}^{n-1} B(im,m). \end{aligned}$$

Note that $\varphi(nm,m) = \varphi(m,m)$ for all $n \in N^+$ and

$$\begin{aligned}\prod_{i=1}^{n-1} B(im,m) &= \frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} \cdot \frac{\Gamma(2m)\Gamma(m)}{\Gamma(3m)} \cdot \cdots \cdot \frac{\Gamma((n-1)m)\Gamma(m)}{\Gamma(nm)} \\ &= \frac{\Gamma(m)^n}{\Gamma(nm)} \\ &= \frac{[(m-1)!]^n}{(nm-1)!} \to 0\end{aligned}$$

as $n \to \infty$. Also $B(im,m) = (im)!m!/((i+1)m)! < 1$ for all positive integer i. By (3), we get

$$\begin{aligned}&\left| \frac{f(nm)\prod_{i=1}^{n-1} B(im,m)}{f(m)^n} - 1 \right| \\ &\le \left(\frac{1}{f(m)^2} + \frac{1}{f(m)^3} + \cdots + \frac{1}{f(m)^n} \right) \delta \\ &< \frac{1}{f(m)^2} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \right) \varphi(m,m) = \frac{2\varphi(m,m)}{f(m)^2} \le \frac{1}{2}\end{aligned}$$

for all positive integer n. Thus we can easily show that

(4) $$f(nm) \to \infty \quad as \ \ n \to \infty$$

Since

$$\frac{B(x,y)B(nm,x+y)}{B(x,y+nm)B(y,nm)} = 1,$$

(5)

$$\begin{aligned}&|f(nm)| \mid B(x,y)f(x+y) - f(x)f(y) \mid \\ &\le |B(x,y)| \mid f(nm)f(x+y) - B(nm,x+y)f(nm+x+y) \mid \\ &\quad + B(y,nm) \mid B(x,y+nm)f(x+y+nm) - f(x)f(y+nm) \mid \\ &\quad + \mid B(y,nm)f(y+nm) - f(y)f(nm) \mid |f(x)| \\ &\le |B(x,y)|\varphi(nm,x+y) + |B(y,nm)|\varphi(x,y+nm) \\ &\quad + |f(x)|\varphi(y,nm) \\ &\le |B(x,y)|\varphi(x+y,x+y) + |B(y,y)|\varphi(x,x) + |f(x)|\varphi(y,y) < \infty\end{aligned}$$

for all sufficiently large n and $x,\ y \in (0,\infty)$. Since

$$\begin{aligned}B(y,nm) &= \frac{\Gamma(nm)\Gamma(y)}{\Gamma(y+nm)} \\ &= \frac{(nm-1)!}{(y+nm-1)(y+nm-2)\cdots(y+1)y} < \frac{1}{y},\end{aligned}$$

it follows from (4) and (5) that

$$B(x,y)f(x+y) = f(x)f(y)$$

for all $x,\ y \in (0,\infty)$. □

Corollary 2.3. *Let* $\varphi(x,y) = \delta, \frac{1}{xy}$, *or* $\frac{1}{a^{xy}}$ *for all* $x, y \in (0,\infty)$. *Suppose that* $f : (0,\infty) \to (0,\infty)$ *be a function with* $f(m) \geq max(2, 2\sqrt{\varphi(m,m)})$ *for some positive integer* m *such that*

$$| B(x,y)f(x+y) - f(x)f(y) | < \varphi(x,y) \tag{6}$$

for all $x,\ y \in (0,\infty)$. *Then*

$$B(x,y)f(x+y) = f(x)f(y)$$

for all $x,\ y \in (0,\infty)$.

3. Generalized stability in the sense of R. Ger of the equation (1)

R. Ger [4] suggested a new type of stability for the exponential equation

$$\left| \frac{f(x+y)}{f(x)f(y)} - 1 \right| \leq \delta.$$

Also Y.W.Lee and B.M. Choi [14] proved the stability of Cauchy's gamma-beta functional equation as following setting:

$$\left| \frac{B(x,y)f(x+y)}{f(x)f(y)} - 1 \right| \leq \delta.$$

In the following theorem, a generalized stability problem in the sense of R. Ger for the functional equation (1) shall be investigated. Throughout this section, we denote $H : (0,\infty) \times 0,\infty) \to (0,\infty)$ a homogeneous function with degree p $(0 \leq p < 1)$.

Theorem 3.1. *If a function* $f : (0,\infty) \to (0,\infty)$ *satisfies the inequality*

$$\left| \frac{B(x,y)f(x+y)}{f(x)f(y)} - 1 \right| \leq H(x,y) \tag{7}$$

for all $x,\ y \in (0,\infty)$, *then there exists a unique function* $F : (0,\infty) \to (0,\infty)$ *such that*

$$B(x,y)F(x+y) = F(x)F(y)$$

for all x, $y \in (0, \infty)$ *and*

$$e^{-(1+\frac{H(x,x)}{2-2^p})} \leq \frac{F(x)}{f(x)} \leq e^{(1+\frac{H(x,x)}{2-2^p})}$$

for all $x \in (0, \infty)$ *(Compare, Theorem 3.1 in [14]).*

Proof. If we define functions $G : D \to R$ by $G(x) = \ln f(x)$ for all $x \in D$, then the inequality (7) may be transformed into

$$|G(x+y) + \ln B(x,y) - G(x) - G(y)| \leq \ln(1 + H(x,y)). \tag{8}$$

Let $u(x) := \ln(1 + H(x,x))$. For $x = y$ the inequality (8) implies

$$|G(2x) + \ln B(x,x) - 2G(x)| \leq u(x) \tag{9}$$

for all $x \in D$. That is,

$$\left|\frac{G(2x)}{2} + \ln B(x,x)^{\frac{1}{2}} - G(x)\right| \leq \frac{1}{2}u(x) \tag{10}$$

for all $x \in D$. We use induction on n to prove

$$\begin{aligned} &\left|\frac{G(2^n x)}{2^n} + \ln \prod_{i=0}^{n-1} B(2^i x, 2^i x)^{\frac{1}{2^{i+1}}} - G(x)\right| \\ &\leq \sum_{i=0}^{n-1} \frac{1}{2^{i+1}} \ln(1 + H(2^i x, 2^i x)) \end{aligned} \tag{11}$$

for all $x \in D$. Indeed, on account of (10) the inequality (11) holds true for $n = 1$. Suppose that inequality (11) holds true for some $n > 1$. Then (10) and (11) imply

$$\begin{aligned} &\left|\frac{G(2^{n+1} x)}{2^{n+1}} + \ln \prod_{i=0}^{n} B(2^i x, 2^i x)^{\frac{1}{2^{i+1}}} - G(x)\right| \\ &\leq \left|\frac{G(2^{n+1} x)}{2^{n+1}} + \ln \prod_{i=1}^{n} B(2^i x, 2^i x)^{\frac{1}{2^{i+1}}} - \frac{G(2x)}{2}\right| \\ &+ \left|\frac{G(2x)}{2} + \ln B(x,x)^{\frac{1}{2}} - G(x)\right| \\ &\leq \sum_{i=0}^{n} \frac{\ln(1 + H(2^i x, 2^i x))}{2^{i+1}}, \end{aligned}$$

which ends the proof of (11). Note that

$$\begin{aligned}
&\sum_{i=0}^{\infty} \frac{\ln(1+H(2^i x, 2^i x))}{2^{i+1}} \\
&\leq \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} + \sum_{i=0}^{\infty} \frac{1}{2} 2^{i(p-1)} H(x,y) \\
&\leq 1 + \frac{H(x,y)}{2-2^p}.
\end{aligned}$$

For any $x \in D$ and for every positive integer n we define

$$P_n(x) := \frac{G(2^n x)}{2^n} + \ln \prod_{i=0}^{n-1} B(2^i x, 2^i x)^{\frac{1}{2^{i+1}}}.$$

Let $m, n > 0$ be integers with $n > m$. By (11), we have

$$\begin{aligned}
&|P_n(x) - P_m(x)| \\
&= \frac{1}{2^m} \left| \frac{G(2^{n-m}(2^m x))}{2^{n-m}} + \ln \prod_{i=m}^{n-1} B(2^i x, 2^i x)^{\frac{1}{2^{i-m+1}}} - G(2^m x) \right| \\
&= \frac{1}{2^m} \left| \frac{G(2^{n-m}(2^m x))}{2^{n-m}} + \ln \prod_{i=0}^{n-m-1} B(2^i(2^m x), 2^i(2^m x))^{\frac{1}{2^{i+1}}} - G(2^m x) \right| \\
&\leq \frac{1}{2^m} \sum_{i=0}^{n-m-1} \frac{u(2^i(2^m x))}{2^{i+1}} = \sum_{i=m}^{n-1} \frac{u(2^i x)}{2^{i+1}} \\
&\leq \sum_{i=m}^{\infty} \frac{\ln(1+H(2^i x, 2^i x))}{2^{i+1}}
\end{aligned}$$

for all $x \in D$. Taking the limit as $m \to \infty$, we get

$$\lim_{m\to\infty} |P_n(x) - P_m(x)| = 0$$

for all $x \in D$. Therefore, the sequence $\{P_n(x)\}$ is a Cauchy sequence, and we may define a function

$$L(x) := \lim_{n\to\infty} P_n(x).$$

Note that

$$B(x+y, x+y) = \frac{B(x,x+y)B(y,y+2x)}{B(x,y)} = \frac{B(x,x)B(y,y)B(2x,2y)}{B(x,y)^2}$$

for all $x, y \in D$. Thus we have

(12)
$$\begin{aligned}&\prod_{i=0}^{n-1}\left[\frac{B(2^i(x+y),2^i(x+y))}{B(2^ix,2^ix)B(2^iy,2^iy)}\right]^{\frac{1}{2^{i+1}}} = \prod_{i=0}^{n-1}\left[\frac{B(2^{i+1}x,2^{i+1}y)}{B(2^ix,2^iy)^2}\right]^{\frac{1}{2^{i+1}}}\\&= \left[\frac{B(2x,2y)}{B(x,y)^2}\right]^{\frac{1}{2}}\cdot\left[\frac{B(2^2x,2^2y)}{B(2x,2y)^2}\right]^{\frac{1}{2^2}}\cdots\left[\frac{B(2^nx,2^ny)}{B(2^{n-1}x,2^{n-1}y)^2}\right]^{\frac{1}{2^n}}\\&= \frac{B(2^nx,2^ny)^{\frac{1}{2^n}}}{B(x,y)}\end{aligned}$$

for all $x,\ y > 0$. Then we arrive at for all $(x,y) \in D \times D$,

$$\begin{aligned}&|L(x+y)+\ln B(x,y)-L(x)-L(y)|\\&= \lim_{n\to\infty}\Big|\frac{G(2^nx+2^ny)}{2^n}+\ln\prod_{i=0}^{n-1}B(2^ix+2^iy,2^ix+2^iy)^{\frac{1}{2^{i+1}}}\\&\qquad-\left(\frac{G(2^nx)}{2^n}+\ln\prod_{i=0}^{n-1}B(2^ix,2^ix)^{\frac{1}{2^{i+1}}}\right)\\&\qquad-\left(\frac{G(2^ny)}{2^n}+\ln\prod_{i=0}^{n-1}B(2^iy,2^iy)^{\frac{1}{2^{i+1}}}\right)\\&\qquad+\ln\prod_{i=0}^{n-1}\left[\frac{B(2^ix,2^ix)B(2^iy,2^iy)}{B(2^ix+2^iy,2^ix+2^iy)}\right]^{\frac{1}{2^{i+1}}}\cdot B(2^nx,2^ny)^{\frac{1}{2^n}}\Big|\\&= \lim_{n\to\infty}\left|\frac{G(2^nx+2^ny)}{2^n}+\ln B(2^nx,2^ny)^{\frac{1}{2^n}}-\frac{G(2^nx)}{2^n}-\frac{G(2^ny)}{2^n}\right|\\&\le \lim_{n\to\infty}\frac{\ln(1+H(2^nx,2^ny))}{2^n}=0.\end{aligned}\tag{13}$$

Let $F(x) := e^{L(x)}$ for all $x \in D$. Then by (13), we have

$$B(x,y)F(x+y) = F(x)F(y)$$

for all $x, y \in D$. Taking the limit in (13) as $n \to \infty$, we have

$$|L(x)-F(x)| \le \lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{\ln(1+H(2^ix,2^iy))}{2^{i+1}} \le 1+\frac{H(x,x)}{2-2^p}$$

for all $x \in D$. That is,

$$\left|\ln\frac{F(x)}{f(x)}\right| \le \varepsilon(x,x)$$

for all $x \in D$. Thus we have

$$e^{-(1+\frac{H(x,x)}{2-2^p})} \le \frac{F(x)}{f(x)} \le e^{(1+\frac{H(x,x)}{2-2^p})}$$

for all $x \in D$. It remains to show that F is unique. Suppose that $W : D \to (0, \infty)$ is another such function with

$$B(x,y)W(x+y) = W(x)W(y)$$

and

$$e^{-(1+\frac{H(x,x)}{2-2^p})} \le \frac{W(x)}{f(x)} \le e^{(1+\frac{H(x,x)}{2-2^p})}$$

for all $x, y \in D$. Note that for all $x \in D$

$$\frac{F(2x)}{W(2x)} = \frac{F(x)^2}{W(x)^2}, \cdots, \frac{F(2^n x)}{W(2^n x)} = \frac{F(x)^{2^n}}{W(x)^{2^n}}$$

and

$$\begin{aligned}\frac{1+\frac{H(x,y)}{2-2^p}}{2^{n-1}} &= \frac{1}{2^{n-1}} \sum_{i=0}^{\infty} \frac{(1+H(2^{n+i}x, 2^{n+i}x))}{2^{i+1}} \\ &= \sum_{i=n}^{\infty} \frac{(1+H(2^i x, 2^i x))}{2^{i+1}} \to 0\end{aligned}$$

as $n \to \infty$. Thus we have

$$e^{\frac{-(1+H(2^n x, 2^n x))}{2^{n-1}}} \le \frac{F(x)}{W(x)} = \left(\frac{F(2^n x)}{f(2^n x)} \frac{f(2^n x)}{W(2^n x)}\right)^{\frac{1}{2^n}} \le e^{\frac{1+H(2^n x, 2^n x)}{2^{n-1}}}$$

for all $n \ge 1$, and so $F(x) = W(x)$ for all $x \in D$.

□

Corollary 3.2. *If a function $f : (0, \infty) \to (0, \infty)$ satisfies the inequality*

$$\left| \frac{B(x,y)f(x+y)}{f(x)f(y)} - 1 \right| \le ||x||^p + ||y||^p \tag{14}$$

for all x, $y \in (0, \infty)$, then there exists a unique function $F : (0, \infty) \to (0, \infty)$ such that

$$B(x,y)F(x+y) = F(x)F(y)$$

for all x, $y \in (0, \infty)$ and

$$e^{-(1+\frac{||x||^p}{2-2^p})} \le \frac{F(x)}{f(x)} \le e^{(1+\frac{||x||^p}{2-2^p})}$$

for all $x \in (0, \infty)$.

Corollary 3.3. *If a function $f : (0,\infty) \to (0,\infty)$ satisfies the inequality*

$$\left|\frac{B(x,y)f(x+y)}{f(x)f(y)} - 1\right| \leq \delta \tag{15}$$

for all x, $y \in (0,\infty)$, then there exists a unique function $F : (0,\infty) \to (0,\infty)$ such that

$$B(x,y)F(x+y) = F(x)F(y)$$

for all x, $y \in (0,\infty)$ and

$$e^{-(1+\delta)} \leq \frac{F(x)}{f(x)} \leq e^{(1+\delta)}$$

for all $x \in (0,\infty)$.

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