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PRODUCT PROPERTIES OF SINGULAR KOBAYASHI METRICS

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Abstract. In this paper, we prove the product properties of the singular Kobayashi metrics relative to the higher order Kobayashi metric.

1. Introduction

S. Kobayashi initiated studying his pseudodistance([6]) and H. L. Royden published the infinitesimal form in [9]. The infinitesimal form that is called as the Kobayashi or Kobayashi-Royden metric is studied in [1], [7], [8] etc. The higher order Kobayashi metric was introduced by J. Yu in [10] and N. Nikolov also investigated the higher order Kobayashi metric in [8].

The higher order Kobayashi metric is the generalization of the Kobayashi metric. Many properties for higher order Kobayashi metric are obtained as counterparts for the properties of Kobayashi metric.

In this article, we will deal with product products of singular Kobayashi metric. To do this, we introduce some notations which are dealt in this article. By $\mathbb{N}$ and $\mathbb{C}$ we denote the set of natural numbers and the set of complex numbers, respectively. We use the usual inner product $\langle \cdot, \cdot \rangle$ and the usual norm $|| \cdot ||$ on $\mathbb{C}^n$.

Further, by F^c_Ω and K_Ω we denote the Carathéodory metric and the Kobayashi metric for some domain Ω.

2. The higher order Kobayashi metrics

The higher order Kobayashi metric is introduced in [10] by J. Yu as the generalization of the Kobayashi metric.

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Let $D \subset \mathbb{C}^n$ be a domain and denote by $\mathcal{O}(\Delta, D)$ the space of all holomorphic mappings from the unit disk $\Delta \subset \mathbb{C}$ into D.

For each $p \in \mathbb{N}$ and $(z, X) \in D \times \mathbb{C}^n$, the *p-th order Kobayashi metric* is defined by

(2.1)

$$K_D^p(z, X) :=$$

$$\inf\{|r|^{-1} \mid \exists \psi \in \mathcal{O}(\Delta, D) \text{ s.t. } \psi(0) = z, \nu(\psi) \geq p, \psi^{(p)}(0) = p!rX\},$$

where $\nu(\psi)$ stands for the order of vanishing of $\psi - \psi(0)$ at 0. Clearly $K_D^1(z, X)$ is the usual Kobayashi metric. The *singular Kobayashi metric* is defined as follows ;

$$K_D^\infty(z, X) := \inf_{m \in \mathbb{N}} K_D^m(z, X). \tag{2.2}$$

We refer to [3] and [10] for proofs of Proposition 2.1 and Proposition 2.3.

Proposition 2.1. *Let D be a domain in $\mathbb{C}^n$ and $k \in \mathbb{N}$. Then K_D^k is upper semicontinuous on $D \times \mathbb{C}^n$.*

Since the infimum of any collection of upper semicontinuous functions is upper semicontinuous, we have the following

Corollary 2.2. *Let $D \subset \mathbb{C}^n$ be a domain. Then the singular Kobayashi metric K_D^∞ is also upper semicontinuous on $D \times \mathbb{C}^n$.*

Proposition 2.3. *Let $D \subset \mathbb{C}^n$ be a domain. Then for each $m \geq 1$, we have*

1. *K_D^m has the length decreasing property. In particular, K_D^m is biholomorphically invariant.*
2. *$K_D^m \equiv K_\Delta$, the usual Kobayashi metric for the unit disc Δ.*
3. *$F_D^c(z, X) \leq K_D^\infty(z, X) \leq K_D^m(z, X) \leq K_D(z, X)$.*
4. *$K_D^m(z, \mu X) = |\mu| K_D^m(z, X)$, $K_D^\infty(z, \mu X) = |\mu| K_D^\infty(z, X)$, for all $\mu \in \mathbb{C}$.*

For a domain $D \subset \mathbb{C}^n$, we define a map $\tilde{K}_D : D \times \mathbb{C}^n \longrightarrow \mathbb{R}$ by

$$\tilde{K}_D(z, X) := \left(\limsup_{m \to \infty} K_D^m(z, X) \right)^*, \tag{2.3}$$

where $*$ stands for the upper semicontinuous regularization. [1]

[1]If $v : D \longrightarrow [-\infty, +\infty)$ is locally bounded from above, then we define a map v^* for each $z \in D$ by

$$v^*(z) := \limsup_{z' \to z} v(z') = \inf\{\phi(z) : \phi \in \mathcal{C}(D, \mathbb{R}), v \leq \phi\}$$

Lemma 2.4. *Let D be a domain in $\mathbb{C}^n$. Then we have*

$$K_D^{pq}(z,X) \leq \min\left\{K_D^p(z,X), K_D^q(z,X)\right\}$$

for all $p,q \in \mathbb{N}$ and for all $(z,X) \in D \times \mathbb{C}^n$.

Proof. Assume, without loss of generality, that $K_D^p(z,X) \leq K_D^q(z,X)$. Let ϕ be an element in the defining family for $K_D^p(z,X)$. That is, $\phi \in \mathcal{O}(\Delta, D), \phi(0) = z, \nu(\phi) \geq p$ and $\phi^{(p)}(0) = p!rX$. Define a map $g : \Delta \longrightarrow \Delta$ by $g(\zeta) = \zeta^q$ for all $\zeta \in \Delta$ and put $\psi = \phi \circ g$. Then $\psi \in \mathcal{O}(\Delta, D), \psi(0) = z$ and $\nu(\psi) \geq pq$. We claim that ψ is an element of the defining family for $K_D^{pq}(z,X)$. To do this, we need to show that $\psi^{(pq)}(0) = (pq)!rX$. Applying Maclaurin's seris expansion to the coordinate functions of ϕ and ψ, respectively, we have

$$\phi(\zeta) = \phi(0) + r\zeta^p X + o(|\zeta|^p)\mathbb{E}$$

and

$$\psi(\zeta) = \phi(\zeta^q) = \phi(0) + r\zeta^{pq} X + o(|\zeta|^{pq})\mathbb{E}$$

as $\zeta \longrightarrow 0$. Here o stands for Landau's notation and $\mathbb{E} = (1, 1, \cdots, 1) \in \mathbb{C}^n$. Hence we have $\psi^{(pq)}(0) = (pq)!rX$. It follows from this that $K_D^{pq}(z,X) \leq K_D^p(z,X)$ holds. $\square$

Note that by Lempert's Theorem([2],[7]) with Proposition 2.3, if $D \subset \mathbb{C}^n$ is a convex domain, then

$$K_D^{pq}(z,X) = K_D^p(z,X) = K_D^q(z,X) = K_D(z,X)$$

for all $(z,X) \in D \times \mathbb{C}^n$ and for all $p,q \in \mathbb{N}$. Hence we have the following

Corollary 2.5. *Let $D \subset \mathbb{C}^n$ be a convex domain. Then*

$$K_D^{\infty}(z,X) = K_D(z,X) = \tilde{K}_D(z,X)$$

for all $(z,X) \in D \times \mathbb{C}^n$.

3. Product properties of singular metrics

Theorem 3.1. ([3]) *Let $D \subset \mathbb{C}^n$ and $G \subset \mathbb{C}^l$ be domains. Then the following formula holds :*

$$K_{D\times G}^m((z,w),(X,Y)) = \max\{K_D^m(z,X), K_G^m(w,Y)\}.$$

for all $(z,w) \in D \times G$, $(X,Y) \in \mathbb{C}^n \times \mathbb{C}^l$ and each $m \in \mathbb{N}$.

Theorem 3.2. *Let $D \subset \mathbb{C}^n$ and $G \subset \mathbb{C}^l$ be domains. Then the following formula holds :*

$$K^{\infty}_{D\times G}((z,w),(X,Y)) \;=\; \max\{K^{\infty}_D(z,X), K^{\infty}_G(w,Y)\}$$

for all $(z,w) \in D \times G$ and $(X,Y) \in \mathbb{C}^n \times \mathbb{C}^l$.

Proof. First we know that for all $m \in \mathbb{N}$

$$K^{\infty}_D(z,X) \;\le\; K^m_D(z,X),\; K^{\infty}_G(w,Y) \;\le\; K^m_G(w,Y).$$

Hence we have the following relationship, by Theorem 3.1,

$$\begin{aligned} \max\{K^{\infty}_D(z,X), K^{\infty}_G(w,Y)\} &\le \max\{K^m_D(z,X), K^m_G(w,Y)\} \\ &= K^m_{D\times G}((z,w),(X,Y)). \end{aligned}$$

Taking infimum above inequality we get

$$\max\{K^{\infty}_D(z,X), K^{\infty}_G(w,Y)\} \;\le\; K^{\infty}_{D\times G}((z,w),(X,Y)).$$

To prove the reverse inequality, suppose that

$$K^{\infty}_{D\times G}((z,w),(X,Y)) \;>\; A \;>\; \max\{K^{\infty}_D(z,X), K^{\infty}_G(w,Y)\}.$$

Then, by definition, we have for all $m \in \mathbb{N}$

$$K^m_{D\times G}((z,w),(X,Y)) \;>\; A \;>\; \max\{K^{\infty}_D(z,X), K^{\infty}_G(w,Y)\}.$$

Hence there are two natural number $p, q \in \mathbb{N}$ such that, for all $m \in \mathbb{N}$,

$$\begin{aligned} K^m_{D\times G}((z,w),(X,Y)) \;>\; A &> \max\{K^p_D(z,X), K^q_G(w,Y)\} \\ &\ge \max\{K^{pq}_D(z,X), K^{pq}_G(w,Y)\} \\ &= K^{pq}_{D\times G}((z,w),(X,Y)). \end{aligned}$$

It is absurd and so we get

$$K^{\infty}_{D\times G}((z,w),(X,Y)) = \max\{K^{\infty}_D(z,X), K^{\infty}_G(w,Y)\}. \quad \square$$

Theorem 3.3. *Let $D \subset \mathbb{C}^n$ and $G \subset \mathbb{C}^l$ be domains. Then the following formula holds :*

$$\tilde{K}_{D\times G}((z,w),(X,Y)) \;=\; \max\{\tilde{K}_D(z,X), \tilde{K}_G(w,Y)\}$$

for all $(z,w) \in D \times G$ and $(X,Y) \in \mathbb{C}^n \times \mathbb{C}^l$.

Proof. First we know that, by Proposition 2.3,

$$\tilde{K}_{D\times G}((z,w),(X,Y)) \;\ge\; \max\{\tilde{K}_D(z,X), \tilde{K}_G(w,Y)\}.$$

To prove the reverse inequality, suppose that

$$\tilde{K}_{D\times G}((z,w),(X,Y)) \;>\; A \;>\; \max\{\tilde{K}_D(z,X), \tilde{K}_G(w,Y)\}.$$

Then, by definition, we have

$$\left(\limsup_{m\to\infty} K_D^m(z,X)\right)^* < A \quad \text{and} \quad \left(\limsup_{m\to\infty} K_G^m(w,Y)\right)^* < A.$$

It follows that there is a natural number $m_0 \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ with $m_0 \leq m$

$$K_D^m(z,X) < A \quad \text{and} \quad K_G^m(w,Y) < A.$$

But since $K_{D\times G}^m((z,w),(X,Y)) = \max\{K_D^m(z,X), K_G^m(w,Y)\}$ for all $m \in \mathbb{N}$, we get

$$K_{D\times G}^m((z,w),(X,Y)) < A, \quad \text{for all } m \in \mathbb{N} \quad \text{with } m_0 \leq m.$$

From this we have the following formula

$$\limsup_{m\to\infty} K_{D\times G}^m((z,w),(X,Y)) < A.$$

Hence $\tilde{K}_{D\times G}((z,w),(X,Y)) \leq A$, which is contradiction to our assumption. □

By applying Corollary 2.5, Theorem 3.2 and Theorem 3.3, we obtain

Corollary 3.4. *Let $D \subset \mathbb{C}^n, G \subset \mathbb{C}^l$ be two convex domains. Then we have*

$$K_{D\times G}^\infty((z,w),(X,Y)) = K_{D\times G}((z,w),(X,Y)) = \tilde{K}_{D\times G}((z,w),(X,Y)),$$

for all $(z,w) \in D \times G$ and $(X,Y) \in \mathbb{C}^n \times \mathbb{C}^l$.

Corollary 3.5. *Let $D \subset \mathbb{C}^n, G \subset \mathbb{C}^l$ be two open unit balls with center at origin, respectively. Then we have*

$$\begin{aligned} & K_{D\times G}^\infty((z,w),(X,Y)) \\ = \ & \max\left\{\sqrt{\frac{||X||^2}{1-||z||^2} + \frac{|\langle z,X\rangle|^2}{(1-||z||^2)^2}}, \sqrt{\frac{||Y||^2}{1-||w||^2} + \frac{|\langle w,Y\rangle|^2}{(1-||w||^2)^2}}\right\} \\ = \ & \tilde{K}_{D\times G}((z,w),(X,Y)) \end{aligned}$$

for all $(z,w) \in D \times G$ and $(X,Y) \in \mathbb{C}^n \times \mathbb{C}^l$.

In particular, we also have the following.

Corollary 3.6. *Let $D := \Delta^n \subset \mathbb{C}^n$ be a unit polydisc with center at origin. Then we have*

$$K_D^\infty(z,X) = \max\left\{\frac{|X_1|}{1-|z_1|^2}, \cdots, \frac{|X_n|}{1-|z_n|^2}\right\} = \tilde{K}_D(z,X),$$

for all $(z,X) \in D \times \mathbb{C}^n$.

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