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# ON THE HAJECK-RENYI-TYPE INEQUALITY FOR $\tilde{\rho}$-MIXING SEQUENCES

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**Abstract.** Let $\{\Omega,\ \mathcal{F},\ P\}$ be a probability space and $\{X_n \mid n \geq 1\}$ be a sequence of random variables defined on it. We study the Hajeck-Renyi-type inequality for $\tilde{\rho}$-mixing random variable sequences and obtain the strong law of large numbers by using this inequality. We also consider the strong law of large numbers for weighted sums of $\tilde{\rho}$-mixing sequences.

## 1. Introduction and Preliminaries

Let $\{\Omega,\ \mathcal{F},\ P\}$ be a probability space and $\{X_n \mid n \geq 1\}$ be a sequence of random variables defined on it. Let $\{X_n \mid n \geq 1\}$ be a sequence of random variables. Write $F_S = \sigma(X_i,\ i \in S \subset N)$. Since $\sigma$-algebras $F,\ R$ in $\mathcal{F}$,

$$\rho(X,Y) = \sup\{Corr(X,Y)|X \in L_2(F),\ Y \in L_2(R)\},$$

where $Corr(X,Y) = EXY - EXEY/\sqrt{VarX\ VarY}$.

Bradley([1],[2]) introduced the following coefficients of dependence:

$$\tilde{\rho}(k) = \sup\{\rho(F_S,\ F_T)\},\ k \geq 0,$$

where the supremum is taken over all finite subsets $S, T \subset N$ such that $\text{dist}(S,\ T) \geq k$. Obviously, $0 \leq \tilde{\rho}(k+1) \leq \tilde{\rho}(k) \leq 1,\ k \geq 0$ and $\tilde{\rho}(0) = 1$.

**Definition 1.1** (Bradley [1] ). *A sequence of random variables $\{X_n \mid n \geq 1\}$ is said to be a $\tilde{\rho}$-mixing if there exists $k \in N$ such that $\tilde{\rho}(1) < 1$.*

Without loss of generality we may assume that $\{X_n|n \geq 1\}$ is such that $\tilde{\rho}(1) < 1$(See Bryc and Smolenski [3]). $\tilde{\rho}$-mixing sequences are similar to $\rho$-mixing sequences, but they are quite different from each

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other. So, the notion of $\tilde{\rho}$-mixing has been received considerable attention recently. We refer to Bradley([1],[2]) for the central limit theorem, Bryc and Smolenski([3]) for moment inequalities and almost sure convergence, Yang Shanchao([13]) for moment inequalities and strong law of large numbers, Wu Qunying([11],[12]) and Gan Shixin([7]) for almost sure convergence.

Haject-Renyi([8]) proved that if $\{X_n|n \geq 1\}$ is a sequence of independent random variables with $EX_n = 0$ and $EX_n^2 < \infty,\ n \geq 1$ and $\{b_n|n \geq 1\}$ is a positive nondecreasing real sequence, then for any $\varepsilon > 0$ and for any positive integer $m < n$,

$$P\left(\max_{m \leq j \leq n} |\frac{\sum_{i=1}^{j} X_i}{b_j}| \geq \varepsilon\right) \leq \varepsilon^{-2}\left(\sum_{j=m+1}^{n} \frac{EX_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{EX_j^2}{b_m^2}\right).$$

Since then, this inequality has been the concern of many authors(e.g., for martingales([4], [6]), for negatively associated random variables([9]), for associated random variables([5], [10]).

In this paper we study the Hajeck-Renyi-type inequality for $\tilde{\rho}$-mixing sequences and use this inequality to prove some strong laws of large numbers. Finally, we consider almost sure convergence for weighted sums of $\tilde{\rho}$-mixing sequences. For this goal we need the following lemma of Yang([13]).

**Lemma 1.1** (Yang [13]). *Let $\{X_n|n \geq 1\}$ be a $\tilde{\rho}$-mixing sequence with $\tilde{\rho}(1) < 1$, $EX_n = 0$ and $E|X_n|^p < \infty,\ n \geq 1$ and $p > 1$. Then there exists positive constants $C$ such that*

(1) $E(\max_{1 \leq k \leq n} |\sum_{i=1}^{k} X_i|^p) \leq C(\log 4n)^p \sum_{k=1}^{n} E|X_k|^p,$ *for any* $n \geq 1,$ $1 < p \leq 2,$

(2) $E(\max_{1 \leq k \leq n} |\sum_{i=1}^{k} X_i|^p) \leq C(\log 4n)^p(\sum_{k=1}^{n} E|X_k|^p + (\sum_{k=1}^{n} EX_k^2)^{p/2}),$ *for any* $n \geq 1$, $p > 2$.

## 2. The Hajeck-Renyi-type inequality for $\tilde{\rho}$-mixing sequences

**Theorem 2.1.** *Let $\{X_n|n \geq 1\}$ be a $\tilde{\rho}$-mixing sequence with $\tilde{\rho}(1) < 1$, $EX_n = 0$ and $E|X_n|^p < \infty,\ n \geq 1,\ p > 1$ and let $\{b_n|n \geq 1\}$ be a sequence of positive nondecreasing real numbers. Then for any $\varepsilon > 0$,*

(1) $P(\max_{1\le k\le n} |\frac{S_k}{b_k}| > \varepsilon) \le C(\log 4n)^p \sum_{k=1}^{n} \frac{E|X_k|^p}{b_k^p}$, *for any* $n \ge 1$, $1 < p \le 2$,

(2) $P(\max_{1\le k\le n} |\frac{S_k}{b_k}| > \varepsilon) \le C(\log 4n)^p \left( \sum_{k=1}^{n} \frac{E|X_k|^p}{b_k^p} + \left( \sum_{k=1}^{n} \frac{EX_k^2}{b_k^2} \right)^{p/2} \right)$, *for any* $n \ge 1$, $p > 2$.

**Proof of (1).**

Applying the proof of Theorem 2.1 in Liu et al.(1999), Markov's inequality and Lemma 1.2 we get that

$$
\begin{aligned}
P(\max_{1\le k\le n} |\frac{S_k}{b_k}| \ge \varepsilon) &= P(\max_{1\le k\le n} |\frac{\sum_{j=1}^k X_j}{b_k}| \ge \varepsilon) \\
&\le CE\left( \max_{1\le k\le n} |\frac{\sum_{j=1}^k X_j}{b_k}|^p \right) \\
&\le C(\log 4n)^p \sum_{k=1}^{n} \frac{E|X_k|^p}{b_k^p}. \quad \square
\end{aligned}
$$

From Theorem 2.1 we get the following more generalized Hajeck-Renyi-type inequality.

**Theorem 2.2.** *Let* $\{X_n | n \ge 1\}$ *be a* $\tilde{\rho}$*-mixing sequence with* $\tilde{\rho}(1) < 1$, $EX_n = 0$ *and* $E|X_n|^p < \infty$, $n \ge 1$, $p > 1$ *and let* $\{b_n | n \ge 1\}$ *be a sequence of positive nonnegative real numbers. Then for any positive integer* $m < n$, *and for* $\varepsilon > 0$,

$$
\begin{aligned}
(1)\ P(\max_{m\le k\le n} |\frac{1}{b_m}\sum_{i=1}^{k} X_i| \ge \varepsilon) &\le C(\log 4m)^p \sum_{j=1}^{m} \frac{E|X_j|^p}{{b_m}^p} \\
&+ C(\log 4(n-m))^p \sum_{j=m+1}^{n} \frac{E|X_j|^p}{b_j^p}, \quad \text{for } 1 < p \le 2,
\end{aligned}
$$

$$
\begin{aligned}
(2)\ &P(\max_{m\le k\le n} |\frac{1}{b_m}\sum_{i=1}^{k} X_i| \ge \varepsilon) \\
&\le C(\log 4m)^p \left( \sum_{k=1}^{m} \frac{E|X_k|^p}{b_m^p} + \left( \sum_{k=1}^{m} \frac{EX_k^2}{b_m^2} \right)^{p/2} \right)
\end{aligned}
$$

$$+ C(\log 4(n-m))^p \left( \sum_{j=m+1}^{n} \frac{E|X_j|^p}{b_j^p} + \left( \sum_{j=m+1}^{n} \frac{EX_j^2}{b_j^2} \right)^{p/2} \right), \quad \textit{for}$$

$p > 2$.

**Proof of (1).** By Theorem 2.1 and $\{b_n|n \geq 1\}$ is a sequence of positive nondecreasing real numbers, we get that

$$\begin{aligned} & P(\max_{m \leq k \leq n} |\frac{\sum_{i=1}^{k} X_i}{b_m}| \geq \varepsilon) \\ \leq \;& P(|\frac{\sum_{j=1}^{m} X_j}{b_m}| \geq \varepsilon/2) + P(\max_{m+1 \leq k \leq n} |\frac{\sum_{j=m+1}^{k} X_j}{b_j}| \geq \varepsilon/2) \\ \leq \;& P(\frac{1}{b_m} \max_{1 \leq k \leq m} |\sum_{j=1}^{k} X_j| \geq \varepsilon/2) + P(\max_{m+1 \leq k \leq n} |\frac{\sum_{j=m+1}^{k} X_j}{b_j}| \geq \varepsilon/2) \\ \leq \;& C(\log 4m)^p \sum_{j=1}^{m} \frac{E|X_j|^p}{b_m^p} + C(\log 4(n-m))^p \sum_{j=m+1}^{n} \frac{E|X_j|^p}{b_j^p}. \quad \square \end{aligned}$$

## 3. Strong laws of large numbers for $\tilde{\rho}$-mixing sequences

**Theorem 3.1.** *Let $\{X_n|n \geq 1\}$ be a $\tilde{\rho}$-mixing sequence with $\tilde{\rho}(1) < 1$, $EX_n = 0$ and $\sum_{n=1}^{\infty} (\log n)^p \frac{E|X_n|^p}{b_n^p} < \infty$, $n \geq 1$, $1 < p \leq 2$ and let $\{b_n|n \geq 1\}$ be a sequence of positive nondecreasing real numbers. Then for any $0 < r < 1$,*

(a) $E \sup_{n \geq 1} \left( \frac{|S_n|}{b_n} \right)^r < \infty$,

(b) *If $0 < b_n \uparrow \infty$, then $\frac{S_n}{b_n} \longrightarrow 0$ almost surely as $n \to \infty$.*

**Proof of (a).**
Note that

$$E \sup_{n \geq 1} \left( \frac{|S_n|}{b_n} \right)^r < \infty \Leftrightarrow \int_1^{\infty} P(\sup_{n \geq 1} \frac{|S_n|}{b_n} > t^{1/r}) \, dt \; < \infty.$$

By Theorem 2.1 we get that

$$\begin{aligned}
&\int_1^\infty P(\sup_{n\geq 1}\frac{|S_n|}{b_n}\geq t^{1/r})\ dt \\
&\leq \ C(\log 4n)^p\int_1^\infty t^{-p/r}\sum_{n=1}^\infty \frac{E|X_n|^p}{b_n^p}dt \\
&\leq \ C\sum_{n=1}^\infty (\log n)^p\frac{E|X_n|^p}{b_n^p}\int_1^\infty t^{-p/r}dt<\infty. \quad \square
\end{aligned}$$

**Proof of (b).**
By Theorem 2.2 we have

$$\begin{aligned}
&P(\max_{m\leq k\leq n}\frac{|S_k|}{b_k}\geq \varepsilon) \\
&\leq \ C(\log 4m)^p\sum_{j=1}^m \frac{E|X_j|^p}{b_m^p}+C(\log 4(n-m))^p\sum_{j=m+1}^n \frac{E|X_j|^p}{b_j^p}.
\end{aligned}$$

But

$$\begin{aligned}
&P(\sup_{k\geq m}\frac{|S_k|}{b_k}\geq \varepsilon) \\
&= \ \lim_{n\to\infty}P(\max_{m\leq k\leq n}\frac{|S_k|}{b_k}\geq \varepsilon) \\
&\leq \ C(\log 4m)^p\sum_{j=1}^m \frac{E|X_j|^p}{b_m^p}+C(\log 4(n-m))^p\sum_{j=m+1}^\infty \frac{E|X_j|^p}{b_j^p} \\
(3.1)\quad &\leq \ C\sum_{j=1}^m (\log j)^p\frac{E|X_j|^p}{b_m^p}+C\sum_{j=m+1}^\infty (\log j)^p\frac{E|X_j|^p}{b_j^p}.
\end{aligned}$$

By the Kronecker Lemma and $\sum_{n=1}^\infty (\log n)^p\frac{E|X_n|^p}{b_n^p}<\infty$, we get

$$\sum_{j=1}^m (\log j)^p\frac{E|X_j|^p}{b_m^p}\longrightarrow 0 \text{ as } m\longrightarrow\infty. \tag{3.2}$$

Hence, by (3.1) and (3.2), we get

$$\lim_{n\to\infty}P(\sup_{k\geq n}\frac{|S_k|}{b_k}\geq \varepsilon) \ = \ 0,$$

i.e. $\frac{S_n}{b_n}\longrightarrow 0$ almost surely as $n\longrightarrow\infty$. $\square$

**Remark 3.1.** *Taking $b_n = 1$, we can obtain the result of* $\sum_{n=1}^{\infty} S_n \longrightarrow 0$ *almost surely as* $n \longrightarrow \infty$.

**Corollary 3.1.** *Let $\{X_n|n \geq 1\}$ be a $\tilde{\rho}$-mixing sequence with $\tilde{\rho}(1) < 1$, $EX_n = 0$ and $\sum_{n=1}^{\infty}(\log n)^p \sup_{n\geq 1} E|X_n|^p < \infty$, $n \geq 1$, $1 < p \leq 2$. Then for $0 < t < 1$,*

(1) $\left(\frac{S_n}{n^{1/t}}\right) \longrightarrow 0$ *almost surely as* $n \longrightarrow \infty$.

(2) $E \sup_{n\geq 1}\left(\frac{|S_n|}{n^{1/t}}\right)^r < \infty$, *for any* $0 < r < 2$.

Taking $b_n = n^{1/t}$ in Theorem 3.1, we obtain the Marcinkiewicz strong laws of large numbers for the $\tilde{\rho}$-mixing sequences.

**Corollary 3.2.** *Let $\{X_n|n \geq 1\}$ be a identically distributed $\tilde{\rho}$-mixing sequence with $\tilde{\rho}(1) < 1$, $EX_1 = 0$ and $\sum_{n=1}^{\infty}(\log n)^p E|X_1|^p < \infty$, $1 < p \leq 2$. Then for $0 < t < 1$,*

$$\frac{\sum_{i=1}^{n} X_i}{n^{1/t}} \longrightarrow 0 \text{ almost surely as } n \longrightarrow \infty.$$

In addition, we get the almost sure convergence of weighted sums of $\tilde{\rho}$-mixing sequence as applications of Theorem 3.1.

**Theorem 3.2.** *Let $\{X_n|n \geq 1\}$ be a sequence of real numbers with $a_{ni} = 0$, $i > n$, $\sup_{n\geq 1}\sum_{i=1}^{n}|a_{ni}| < \infty$ and $\{b_n|n \geq 1\}$ be a sequence of positive nondecreasing real numbers such that $0 < b_n \uparrow \infty$. Then*

$$\sum_{i=1}^{n}\frac{a_{ni}X_i}{b_n} \longrightarrow 0 \text{ almost surely as } n \longrightarrow \infty.$$

**Proof.** Let $S_k = \sum_{j=1}^{k}\frac{X_j}{b_k}$, $c_{nj} = \frac{b_j}{b_n}(a_{nj} - a_{nj+1})$, for $1 \leq j \leq n-1$, and $c_{nn} = a_{nn}$. Then

$$\sum_{j=1}^{n}\frac{a_{nj}X_j}{b_n} = \sum_{j=1}^{n} c_{nj}S_j, \qquad \sum_{j=1}^{n}|c_{nj}| \leq 2\sup_{n\geq 1}\sum_{j=1}^{n}|a_{nj}| \tag{3.3}$$

and

$$\lim_{n\to\infty}|c_{nj}| = 0, \qquad \text{for every fixed } j. \tag{3.4}$$

So, it follows from (3.3) and (3.4) that we get for every sequence real numbers $d_n$ with $d_n \longrightarrow 0$ as $n \longrightarrow \infty$,

$$\sum_{j=1}^{n} c_{nj} d_j \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Hence, by Theorem 3.1(b), (3.3) and (3.4), we get the result of Theorem 3.2. □

**Theorem 3.3.** *Let* $\{X_n | n \geq 1\}$ *be a* $\tilde{\rho}$*-mixing sequence with* $\tilde{\rho}(1) < 1$, $EX_n = 0$ *and* $\sum_{n=1}^{\infty} (\log n)^p \sup_{n \geq 1} E|X_n|^p < \infty$, $n \geq 1$, $1 < p \leq 2$ *and* $\{a_{ni} | 1 \leq i \leq n, \ n \geq 1\}$ *be a sequence of real numbers with* $a_{ni} = 0$, $i > n$, $\sup_{n \geq 1} \sum_{i=1}^{n} |a_{ni}| < \infty$. *Then for* $0 < t < 1$,

$$\sum_{i=1}^{n} \frac{a_{ni} X_i}{n^{1/t}} \longrightarrow 0 \text{ almost surely as } n \longrightarrow \infty.$$

**Proof.** By taking $b_n = n^{1/t}$, from Corollary 3.1 and Theorem 3.2 we can obtain the result of Theorem 3.3 and the proof is omitted. □

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