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BOUNDARY CONTROL OF CHEMOTAXIS REACTION DIFFUSION SYSTEM

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Abstract. This paper is concerned with the boundary control of the chemotaxis reaction diffusion system. That is, we show the existence of the solution for the chemotaxis system with the boundary control and the existence of the optimal boundary control.

1. Introduction

In this paper we consider the following boundary control problem

$$(\mathbf{P}) \qquad \text{minimize} \;\; J(u)$$

with the cost functional $J(u)$ of the form

$$J(u) = \int_0^T \|y(u) - y_d\|^2_{H^1(0,L)} dt + \gamma \|u\|^2_{H^2(0,T)}, \quad u \in H^2(0,T),$$

where $y = y(u)$ is governed by the chemotaxis reaction diffusion system

$$(1.1) \quad \begin{aligned} &\frac{\partial y}{\partial t} = a\frac{\partial^2 y}{\partial x^2} - b\frac{\partial}{\partial x}\Big(y\frac{\partial \rho}{\partial x}\Big) \quad \text{in } (0,L)\times(0,T], \\ &\frac{\partial \rho}{\partial t} = d\frac{\partial^2 \rho}{\partial x^2} + fy - h\rho \quad \text{in } (0,L)\times(0,T], \\ &\frac{\partial y}{\partial x}(0,t) = \frac{\partial y}{\partial x}(L,t) = \frac{\partial \rho}{\partial x}(0,t) = 0, \;\; \frac{\partial \rho}{\partial x}(L,t) = u(t) \quad \text{on } (0,T], \\ &y(x,0) = y_0(x), \quad \rho(x,0) = \rho_0(x) \quad \text{in } (0,L). \end{aligned}$$

Here, $(0,L)$ is a bounded interval in $\mathbf{R}$. a, b, d, f, h and γ are given positive numbers. $y = y(x,t)$ describes the cell concentration in $(0,L)$ at time t, and $\rho = \rho(x,t)$ the chemoattractant concentration in $(0,L)$

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at time t. $u(t)$ is the control function. We refer to [2, 3, 6] and the references for (1.1).

Many papers have already been published to study the control problems for nonlinear parabolic equations. In [6], Ryu and Yagi studied the distributed optimal control problems for (1.1) with homogeneous boundary conditions regarding the forcing term as a control function. However, this paper handles optimal boundary control problems for (1.1) when the control is given by the boundary conditions. In this paper, we reduce (1.1) to the equations with homogeneous boundary by introducing an adequate lifting function for the boundary conditions. And then we show the existence of the weak solution and the existence of the optimal boundary control by using the method introduced in [6].

2. Preliminary

We shall state some results on the Sobolev spaces ([1, 7]) and the semilinear abstract equations ([6]). One uses the same constant C in all estimates without any confusion. I denotes an interval $(0,l)$.

When $s > \frac{1}{2}$, $H^s(I) \subset \mathcal{C}(\bar{I})$ with the estimate

$$\|\cdot\|_{\mathcal{C}} \leq C_s\|\cdot\|_{H^s}. \tag{2.1}$$

If $y \in H^1(I)$ and $\rho \in H^2(I)$ with $\frac{d\rho}{dx}(0) = \frac{d\rho}{dx}(l) = 0$, then

$$\left\langle \frac{d}{dx}\Big(y\frac{d\rho}{dx}\Big), \psi \right\rangle_{(H^1)', H^1} = \Big(y\frac{d\rho}{dx}, \frac{d\psi}{dx}\Big)_{L^2}, \quad \psi \in H^1(I).$$

By (2.1), we observe that

$$\Big\|\frac{d}{dx}\Big(y\frac{d\rho}{dx}\Big)\Big\|_{(H^1)'} \leq \begin{cases} C\|y\|_{L^\infty}\|\rho\|_{H^1} \\ C\|y\|_{L^2}\|\frac{\partial\rho}{\partial x}\|_{L^\infty} \end{cases}, \tag{2.2}$$

where $y \in H^1(I)$, $\rho \in H_n^2(I) = \{\rho \in H^2(I); \frac{\partial\rho}{\partial x}(0) = \frac{\partial\rho}{\partial x}(l) = 0\}$.

Furthermore, if $m_1 < m_2$, then

$$H^{m_2}(I) \text{ is compactly embedded in } H^{m_1}(I). \tag{2.3}$$

Let $\mathcal{V}$ and $\mathcal{H}$ be two separable real Hilbert spaces with dense and compact embedding $\mathcal{V} \hookrightarrow \mathcal{H}$. Identifying $\mathcal{H}$ and its dual $\mathcal{H}'$ and denoting the dual space of $\mathcal{V}$ by $\mathcal{V}'$, we have $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$.

We consider the following Cauchy problem

$$\text{(E)} \qquad \begin{aligned} &\frac{dY}{dt} + AY = F(Y) + G(t), \quad 0 < t \leq T, \\ &Y(0) = Y_0 \end{aligned}$$

in the space $\mathcal{V}'$. Here, A is the positive definite self-adjoint operator on $\mathcal{H}$ defined by a symmetric sesquilinear form $a(Y, \widetilde{Y})$ on $\mathcal{V}$, $\langle AY, \widetilde{Y}\rangle_{\mathcal{V},\mathcal{V}'} = a(Y, \widetilde{Y})$, which satisfies

$$\text{(a.i)} \qquad |a(Y, \widetilde{Y})| \leq M\|Y\|_{\mathcal{V}}\|\widetilde{Y}\|_{\mathcal{V}}, \quad Y, \widetilde{Y} \in \mathcal{V},$$

$$\text{(a.ii)} \qquad a(Y, Y) \geq \delta\|Y\|_{\mathcal{V}}^2, \quad Y \in \mathcal{V}$$

with some δ and $M > 0$. A is also a bounded operator from $\mathcal{V}$ to $\mathcal{V}'$.

$F(\cdot)$ is a given continuous function from $\mathcal{V}$ to $\mathcal{V}'$ satisfying:

For each $\eta > 0$, there exists an increasing continuous function $\mu_\eta : [0, \infty) \to [0, \infty)$ such that

$$\text{(f.i)} \qquad \|F(Y)\|_{\mathcal{V}'} \leq \eta\|Y\|_{\mathcal{V}} + \mu_\eta(\|Y\|_{\mathcal{H}}), \quad Y \in \mathcal{V};$$

$$\text{(f.ii)} \quad \|F(\widetilde{Y}) - F(Y)\|_{\mathcal{V}'} \leq \eta\|\widetilde{Y} - Y\|_{\mathcal{V}} + (\|\widetilde{Y}\|_{\mathcal{V}} + \|Y\|_{\mathcal{V}} + 1)\mu_\eta(\|\widetilde{Y}\|_{\mathcal{H}} + \|Y\|_{\mathcal{H}})\|\widetilde{Y} - Y\|_{\mathcal{H}}, \quad \widetilde{Y}, Y \in \mathcal{V}.$$

$G(\cdot) \in L^2(0, T; \mathcal{V}')$ is a given function and $Y_0 \in \mathcal{H}$ is an initial value.

We then obtain the following result (For the proof, see Ryu and Yagi [6]).

Theorem 2.1. Let (a.i), (a.ii), (f.i), and (f.ii) be satisfied. Then, for any $Y_0 \in \mathcal{H}$ and $G \in L^2(0, T; \mathcal{V}')$, there exists a unique weak solution

$$Y \in H^1(0, T(Y_0, G); \mathcal{V}') \cap \mathcal{C}([0, T(Y_0, G)]; \mathcal{H}) \cap L^2(0, T(Y_0, G); \mathcal{V})$$

to (E), the number $T(Y_0, G) > 0$ is determined by the norms $\|Y_0\|_{\mathcal{H}}$ and $\|G\|_{L^2(0,T;\mathcal{V}')}$.

3. Mathematical setting

In this section, we show the existence and uniqueness of a local solution for (1.1). To derive the existence of solutions for (1.1), we reduce (1.1) to a homogeneous problem. First we construct a lifting function for the boundary conditions,

$$\phi(x, t) = u(t)\frac{x^2}{2L}.$$

Here, $u \in H^2_\Gamma(0, T) = \{u \in H^2(0, T) : u(0) = 0\}$. Obviously

$$\text{(3.1)} \qquad \left|\frac{\partial^i \phi(x, t)}{\partial x^i}\right| \leq C|u(t)|, \quad \forall x \in (0, L), \ \forall t \in [0, T] \ (i = 0, 1, 2).$$

Let us set $w(x,t) = \rho(x,t) - \phi(x,t)$; then the system (1.1) is equivalent to the one:

$$
(3.2)\quad
\begin{aligned}
&\frac{\partial y}{\partial t} = a\frac{\partial^2 y}{\partial x^2} - b\frac{\partial}{\partial x}\Big(y\frac{\partial(w+\phi)}{\partial x}\Big) \qquad \text{in } (0,L)\times(0,T],\\
&\frac{\partial w}{\partial t} = d\frac{\partial^2 w}{\partial x^2} + fy - hw + g(x,t) \qquad \text{in } (0,L)\times(0,T],\\
&\frac{\partial y}{\partial x}(0,t) = \frac{\partial y}{\partial x}(L,t) = \frac{\partial w}{\partial x}(0,t) = \frac{\partial w}{\partial x}(L,t) = 0 \qquad \text{on } (0,T],\\
&y(x,0) = y_0(x), \quad w(x,0) = w_0 \qquad \text{in } (0,L).
\end{aligned}
$$

Here, $w_0 = \rho_0(x)$ and $g(x,t) = d\frac{\partial^2\phi}{\partial x^2} - h\phi - \frac{\partial\phi}{\partial t}$.

Now, we show the existence and uniqueness of a local solution for (3.2) as in [5, 6]. Let $A_1 = -a\frac{\partial^2}{\partial x^2} + a$ and $A_2 = -d\frac{\partial^2}{\partial x^2} + h$ with the same domain $\mathcal{D}(A_i) = H_n^2(0,L) = \{z \in H^2(0,L); \frac{\partial z}{\partial x}(0) = \frac{\partial z}{\partial x}(L) = 0\}$ $(i = 1,2)$. Then, A_i are two positive definite self-adjoint operators in $L^2(0,L)$. We set two product Hilbert spaces $\mathcal{V} \subset \mathcal{H}$ as

$$\mathcal{V} = H^1(0,L)\times H_n^2(0,L), \quad \mathcal{H} = L^2(0,L)\times H^1(0,L).$$

By identifying $\mathcal{H}$ with its dual space, we consider $\mathcal{V} \subset \mathcal{H} = \mathcal{H}' \subset \mathcal{V}'$. It is then seen that

$$\mathcal{V}' = (H^1(0,L))' \times L^2(0,L)$$

with the duality product

$$\langle \Phi, Y\rangle_{\mathcal{V}',\mathcal{V}} = \langle \zeta, y\rangle_{(H^1)',H^1} + (\varphi, A_2 w)_{L^2}, \quad \Phi = \begin{pmatrix}\zeta\\ \varphi\end{pmatrix},\ Y = \begin{pmatrix} y\\ w\end{pmatrix}.$$

We set also a symmetric bilinear form on $\mathcal{V}\times\mathcal{V}$:

$$a(Y,\widetilde{Y}) = a\int_0^L \frac{dy}{dx}\frac{d\tilde{y}}{dx}dx + a\int_0^L y\tilde{y}dx + \big(A_2^{1/2}w, A_2^{1/2}\tilde{w}\big)_{L^2},$$
$$Y = \begin{pmatrix} y\\ w\end{pmatrix}, \widetilde{Y} = \begin{pmatrix}\tilde{y}\\ \tilde{w}\end{pmatrix} \in \mathcal{V}.$$

Obviously, the form satisfies (a.i) and (a.ii). This form then defines a linear isomorphism $A = \begin{pmatrix} A_1 & 0\\ 0 & A_2\end{pmatrix}$ from $\mathcal{V}$ to $\mathcal{V}'$, and the part of A in $\mathcal{H}$ is a positive definite self-adjoint operator in $\mathcal{H}$. Let U_{ad} be a closed bounded convex subset in $H_\Gamma^2(0,T) = \{u \in H^2(0,T) : u(0) = 0\}$.

(3.2) is, then, formulated as an abstract equation

$$
(3.3)\quad
\begin{aligned}
&\frac{dY}{dt} + AY = F_u(Y) + G_u(t), \quad 0 < t \le T,\\
&Y(0) = Y_0
\end{aligned}
$$

in the space $\mathcal{V}'$. Here, $F(\cdot) : \mathcal{V} \to \mathcal{V}'$ is the mapping

$$F_u(Y) = \begin{pmatrix} ay - b\frac{\partial}{\partial x}\Big(y\frac{\partial(w+\phi)}{\partial x}\Big) \\ fy \end{pmatrix} \quad \text{and} \quad G_u(t) = \begin{pmatrix} 0 \\ g(x,t) \end{pmatrix}.$$

Y_0 is defined by $Y_0 = \binom{y_0}{w_0}$.

Moreover, for $u \in U_{ad}$, $F_u(\cdot)$ satisfies (f.i) and (f.ii)([5]). Indeed, by (2.2) and (3.1), it is seen that

$$\Big\|\frac{\partial}{\partial x}\Big(y\frac{\partial(w+\phi)}{\partial x}\Big)\Big\|_{(H^1)'} \le C\|y\|_{L^\infty}\|w+\phi\|_{H^1}$$
$$\le C\|y\|_{H^1}^{1/2}\|y\|_{L^2}^{1/2}(\|w\|_{H^1} + \|\phi\|_{H^1}) \le \varepsilon\|y\|_{H^1} + C_\varepsilon\|y\|_{L^2}(\|w\|_{H^1}+1)^2$$

with an arbitrary $\varepsilon > 0$. Therefore, (f.i) holds. On the other hand, by (2.1) and (2.2),

$$\Big\|\frac{\partial}{\partial x}\Big(y\frac{\partial(w+\phi)}{\partial x}\Big) - \frac{\partial}{\partial x}\Big(\tilde{y}\frac{\partial(\tilde{w}+\phi)}{\partial x}\Big)\Big\|_{(H^1)'}$$
$$\le C\Big[\|y-\tilde{y}\|_{L^2}(\|w\|_{H^2}+1) + \|w-\tilde{w}\|_{H^1}\|\tilde{y}\|_{H^1}\Big]$$
$$\le C(\|Y\|_{\mathcal{V}} + \|\widetilde{Y}\|_{\mathcal{V}} + 1)\|Y - \widetilde{Y}\|_{\mathcal{H}} \quad Y, \widetilde{Y} \in \mathcal{V},$$

where $Y = \binom{y}{w}, \widetilde{Y} = \binom{\tilde{y}}{\tilde{w}}$. Therefore, (f.ii) holds also.

Since $u \in U_{ad}$, we see that $G_u(\cdot) \in L^2(0,T;\mathcal{V}')$. From Theorem 2.1, we obtain the following result.

Theorem 3.1. *If $Y_0 \in \mathcal{H}$, there exists a unique weak solution*

$$Y \in H^1(0,S;\mathcal{V}') \cap \mathcal{C}([0,S];\mathcal{H}) \cap L^2(0,S;\mathcal{V})$$

to (3.3), the number $S \in (0,T]$ is determined by the norm $\|Y_0\|_{\mathcal{H}}$ and $\|G_u\|_{L^2(0,T;\mathcal{V}')}$.

Theorem 3.2. *For any $u_1, u_2 \in \mathcal{U}_{ad}$, we have*

$$\text{(3.4)} \quad \|Y_1(t) - Y_2(t)\|_{\mathcal{H}}^2 + \delta\int_0^t \|Y_1(s) - Y_2(s)\|_{L^2(0,S;\mathcal{V})}^2 ds$$
$$\le C\|u_1(t) - u_2(t)\|_{H^2(0,S)}^2, \quad 0 \le t \le S,$$

where $Y_1 = Y(u_1)$ and $Y_2 = Y(u_2)$ are the solutions of (3.3) with respect to u_1 and u_2, respectively.

Proof. Let Y_1 and Y_2 be the solutions of (3.3) with respect to u_1 and u_2, respectively. Then $\widetilde{Y} = Y_1 - Y_2$ satisfies the equation

$$
\begin{aligned}
(3.5) \qquad & \frac{d\widetilde{Y}}{dt} + A\widetilde{Y} = (F_{u_1}(Y_1) - F_{u_1}(Y_2)) + (F_{u_1}(Y_2) - F_{u_2}(Y_2)) \\
& \qquad + (G_{u_1}(t) - G_{u_2}(t)), \quad 0 < t \le S, \\
& \widetilde{Y}(0) = 0.
\end{aligned}
$$

From (2.2) and (3.1), we have

$$
\begin{aligned}
(3.6) \quad \|F_{u_1}(Y_2) - F_{u_2}(Y_2)\|_{\mathcal{V}'} &= \Big\| b\frac{\partial}{\partial x}\Big(y_2 \frac{\partial(\phi_1 - \phi_2)}{\partial x}\Big)\Big\|_{(H^1)'} \\
&\le \|y_2\|_{L^2} \Big\| \frac{\partial(\phi_1 - \phi_2)}{\partial x} \Big\|_{L^\infty} \le C|u_1(t) - u_2(t)| \|Y_2\|_{\mathcal{H}}
\end{aligned}
$$

and

$$
\begin{aligned}
(3.7) \quad \|G_{u_1}(t) - G_{u_2}(t)\|_{\mathcal{V}'} &= \|g_1(x,t) - g_2(x,t)\|_{L^2} \\
&\le C\Big(|u_1(t) - u_2(t)| + \Big|\frac{d(u_1(t) - u_2(t))}{dt}\Big|\Big),
\end{aligned}
$$

where, $\phi_i(x,t) = u_i(t)\frac{x^2}{2L}$ and $g_i(x,t) = d\frac{\partial^2\phi_i}{\partial x^2} - h\phi_i - \frac{\partial\phi_i}{\partial t}$ for $i = 1, 2$.

Taking the scalar product with $\widetilde{Y}$ to (3.5) and using (f.ii), (3.6), (3.7), we obtain that

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|\widetilde{Y}(t)\|_{\mathcal{H}}^2 + \frac{\delta}{2}\|\widetilde{Y}(t)\|_{\mathcal{V}}^2 \\
& \quad \le (\|Y_1(t)\|_{\mathcal{V}}^2 + \|Y_2(t)\|_{\mathcal{V}}^2 + 1)\tilde{\mu}(\|Y_1(t)\|_{\mathcal{H}}^2 + \|Y_2(t)\|_{\mathcal{H}}^2)\|\widetilde{Y}(t)\|_{\mathcal{H}}^2 \\
& \qquad + C(1 + \|Y_2(t)\|_{\mathcal{H}}^2)\Big(|u_1(t) - u_2(t)|^2 + \Big|\frac{d(u_1(t) - u_2(t))}{dt}\Big|^2\Big),
\end{aligned}
$$

where $\tilde{\mu} : [0,\infty) \to [0,\infty)$ are some increasing continuous function.

Using Gronwall's inequality, we obtain that

$$
\begin{aligned}
\|\widetilde{Y}(t)\|_{\mathcal{H}}^2 + \delta \int_0^t \|\widetilde{Y}(t)\|_{\mathcal{V}}^2 ds \\
&\le C\|u_1(t) - u_2(t)\|_{H^1(0,S)}^2 (1 + \|Y_2(t)\|_{L^\infty(0,S;\mathcal{H})}^2) \\
&\quad \times e^{\int_0^S (\|Y_1(t)\|_{\mathcal{V}}^2 + \|Y_2(t)\|_{\mathcal{V}}^2 + 1)\tilde{\mu}(\|Y_1(t)\|_{\mathcal{H}}^2 + \|Y_2(t)\|_{\mathcal{H}}^2) ds} \\
&\le C\|u_1(t) - u_2(t)\|_{H^2(0,S)}^2
\end{aligned}
$$

for all $t \in [0, S]$. Hence, (3.4) is satisfied. $\square$

4. Existence of the optimal control

Let $S > 0$ be such that for each $u \in U_{ad}$, (3.3) has a unique weak solution $Y(u) \in H^1(0,S;\mathcal{V}') \cap \mathcal{C}([0,S];\mathcal{H}) \cap L^2(0,S;\mathcal{V})$. Thus, the problem **(P)** is obviously formulated as follows:

$$(\overline{\mathbf{P}}) \qquad \text{minimize } J(u),$$

where

$$J(u) = \int_0^S \|DY(u) - Y_d\|_{\mathcal{V}}^2 dt + \gamma \|u\|_{H^2(0,S)}^2, \quad u \in U_{ad}.$$

Here, $D\binom{y}{w} = \binom{y}{0}$ and $Y_d = \binom{y_d}{0}$ is a fixed element of $L^2(0,S;\mathcal{V})$ with $y_d \in L^2(0,S;H^1(0,L))$. γ is a positive constant.

Theorem 4.1. *There exists an optimal control $\bar{u} \in \mathcal{U}_{ad}$ for $(\overline{\mathbf{P}})$ such that $J(\bar{u}) = \min_{u \in \mathcal{U}_{ad}} J(u)$.*

Proof. Let $\{u_n\} \subset U_{ad}$ be a minimizing sequence such that

$$\lim_{n \to \infty} J(u_n) = \min_{u \in U_{ad}} J(u).$$

Since $\{u_n\}$ is bounded in $H^2(0,S)$, we can assume that $u_n \to \bar{u}$ weakly in $H^2(0,S)$. From (2.3), we can choose the sequence n so that that

$$(4.1) \quad u_n \to \bar{u} \quad \text{strongly in} \quad H^1(0,S) (\text{respectively, } L^2(0,S)).$$

We call $\phi_n(x,t) = u_n(t)\frac{x^2}{2L}$ and $\bar{\phi}(x,t) = \bar{u}(t)\frac{x^2}{2L}$ the corresponding lifting functions. For simplicity, we will write $Y_n = \binom{y_n}{w_n}$ instead of the solution $Y(u_n) = \binom{y(u_n)}{w(u_n)}$ of (3.2) corresponding to u_n,

$$(4.2) \quad \begin{aligned} &\frac{\partial y_n}{\partial t} = a\frac{\partial^2 y_n}{\partial x^2} - b\frac{\partial}{\partial x}\Big(y_n \frac{\partial (w_n + \phi_n)}{\partial x}\Big) \quad \text{in } (0,L) \times (0,S],\\ &\frac{\partial w_n}{\partial t} = d\frac{\partial^2 w_n}{\partial x^2} + f y_n - h w_n + g_n(x,t) \quad \text{in } (0,L) \times (0,S],\\ &\frac{\partial y_n}{\partial x}(0,t) = \frac{\partial y_n}{\partial x}(L,t) = \frac{\partial w_n}{\partial x}(0,t) = \frac{\partial w_n}{\partial x}(L,t) = 0 \quad \text{on } (0,S],\\ &y_n(x,0) = y_0(x), \quad w_n(x,0) = w_0 \quad \text{in } (0,L). \end{aligned}$$

Here, $g_n(x,t) = d\frac{\partial^2 \phi_n}{\partial x^2} - h\phi_n - \frac{\partial \phi_n}{\partial t}$.

(4.2) is, then, formulated as an abstract equation

$$\begin{aligned} &\frac{dY_n}{dt} + AY_n = F_{u_n}(Y_n) + G_{u_n}(t), \quad 0 < t \le S,\\ &Y_n(0) = Y_0 \end{aligned}$$

in the space $\mathcal{V}'$. Here,

$$F_{u_n}(Y_n) = \begin{pmatrix} ay_n - b\frac{\partial}{\partial x}\Big(y_n \frac{\partial(w_n+\phi_n)}{\partial x}\Big) \\ fy_n \end{pmatrix} \quad \text{and} \quad G_{u_n}(t) = \begin{pmatrix} 0 \\ g_n(x,t) \end{pmatrix}.$$

As in the proof of [6, Theorm 2.1], we see that

$$\begin{aligned} &Y_n \to \overline{Y} \quad \text{weakly in} \quad L^2(0,S;\mathcal{V}), \\ &\frac{dY_n}{dt} \to \frac{d\overline{Y}}{dt} \quad \text{weakly in} \quad L^2(0,S;\mathcal{V}'), \end{aligned}$$

where $\overline{Y} = \binom{\bar{y}}{\bar{w}}$. Since $\mathcal{V}$ is compactly embedded in $\mathcal{H}$, it is shown by [4, Chap. 1, Theorem 5.1] that

$$Y_n \to \overline{Y} \text{ strongly in } L^2(0,S;\mathcal{H}). \tag{4.3}$$

Let us verify that $\overline{Y}$ is a solution to

$$\begin{aligned} &\frac{dY}{dt} + AY = F_{\bar{u}}(Y) + G_{\bar{u}}(t), \quad 0 < t \le S, \\ &Y(0) = Y_0. \end{aligned} \tag{4.4}$$

Here, $\bar{g}(x,t) = d\frac{\partial^2 \bar{\phi}}{\partial x^2} - h\bar{\phi} - \frac{\partial \bar{\phi}}{\partial t}$,

$$F_{\bar{u}}(Y) = \begin{pmatrix} ay - b\frac{\partial}{\partial x}\Big(y \frac{\partial(w+\bar{\phi})}{\partial x}\Big) \\ fy \end{pmatrix} \quad \text{and} \quad G_{\bar{u}}(t) = \begin{pmatrix} 0 \\ \bar{g}(x,t) \end{pmatrix}.$$

Let $\Psi = \binom{\psi_1}{\psi_2} \in L^2(0,S;\mathcal{V})$. Then,

$$\begin{aligned} \int_0^S \langle \frac{dY_n(t)}{dt}, \Psi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt &+ \int_0^S \langle AY_n(t), \Psi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt \\ &= \int_0^S \langle F_{u_n}(Y_n(t)), \Psi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt + \int_0^S \langle G_{u_n}(t), \Psi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt. \end{aligned}$$

First, we show that

$$\int_0^S \langle F_{u_n}(Y_n(t)), \Psi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt \to \int_0^S \langle F_{\bar{u}}(\overline{Y}(t)), \Psi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt \tag{4.5}$$

as $n \to \infty$. Indeed, for $\psi_1 \in \mathcal{C}([0,S]; H^1(0,L))$,

$$\int_0^S \Big\langle \frac{\partial}{\partial x}\Big(y_n \frac{\partial(w_n+\phi_n)}{\partial x}\Big) - \frac{\partial}{\partial x}\Big(\bar{y}\frac{\partial(\bar{w}+\bar{\phi})}{\partial x}\Big), \psi_1 \Big\rangle_{(H^1)',H^1} dt$$
$$\begin{aligned}\le C\Big[& \|y_n-\bar{y}\|_{L^2(0,S;L^2(0,L))}\|w_n\|_{L^2(0,S;H^2(0,L))}\|\psi_1\|_{\mathcal{C}([0,S];H^1(0,L))} \\ &+ \|\bar{y}\|_{L^2(0,S;H^1(0,L))}\|w_n-\bar{w}\|_{L^2(0,S;H^1(0,L))}\|\psi_1\|_{\mathcal{C}([0,S];H^1(0,L))} \\ &+ \|y_n-\bar{y}\|_{L^2(0,S;L^2(0,L))}\|u_n\|_{L^2(0,S)}\|\psi_1\|_{\mathcal{C}([0,S];H^1(0,L))} \\ &+ \|\bar{y}\|_{L^2(0,S;H^1(0,L))}\|u_n-\bar{u}\|_{L^2(0,S)}\|\psi_1\|_{\mathcal{C}([0,S];H^1(0,L))}\Big].\end{aligned}$$

From (4.1) and (4.3), we have

$$\lim_{n\to 0}\int_0^S \Big\langle \frac{\partial}{\partial x}\Big(y_n \frac{\partial(w_n+\phi_n)}{\partial x}\Big) - \frac{\partial}{\partial x}\Big(\bar{y}\frac{\partial(\bar{w}+\bar{\phi})}{\partial x}\Big), \psi_1 \Big\rangle_{(H^1)',H^1} dt = 0.$$

Therefore, (4.5) holds.

Moreover, we show that

$$\int_0^S \langle G_{u_n}(t), \Psi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt \to \int_0^S \langle G_{\bar{u}}(t), \Psi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt \tag{4.6}$$

as $n \to \infty$. Indeed, for $\psi_2(t) \in L^2(0,S;H^2(0,L))$,

$$\int_0^S \langle g_n(t) - \bar{g}(t), \psi_2\rangle_{L^2,H^2} dt$$
$$\le C\Big[\|u_n-\bar{u}\|_{L^2(0,S)} + \Big\|\frac{du_n}{dt} - \frac{d\bar{u}}{dt}\Big\|_{L^2(0,S)}\Big]\|\psi_2\|_{L^2(0,S;H^2(0,L))}.$$

From (4.1), we have

$$\lim_{n\to 0}\int_0^S \langle g_n(t) - \bar{g}(t), \psi_2\rangle_{L^2,H^2} dt = 0.$$

Therefore, (4.6) is satisfied. Therefore, we obtain that

$$\int_0^S \langle \frac{d\overline{Y}(t)}{dt}, \Psi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt + \int_0^S \langle A\overline{Y}(t), \Psi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt$$
$$= \int_0^S \langle F_{\bar{u}}(\overline{Y}(t)), \Psi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt + \int_0^S \langle G_{\bar{u}}(t), \Psi(t)\rangle_{\mathcal{V}',\mathcal{V}} dt.$$

Hence, by the uniqueness, $\overline{Y} = Y(\bar{u})$ is the unique solution to (4.4) corresponding to $\bar{u}$. Since $DY(u_n) - Y_d$ is weakly convergent to $DY(\bar{u}) - Y_d$ in $L^2(0,S;\mathcal{V})$, we have:

$$\min_{u\in U_{ad}} J(u) \le J(\bar{u}) \le \liminf_{n\to\infty} J(u_n) = \min_{u\in U_{ad}} J(u).$$

Hence, $J(\bar{u}) = \min_{u \in U_{ad}} J(u)$. $\square$

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