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ON WEAKLY γ-CONTINUOUS FUNCTIONS

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Abstract. We introduce the concepts of weakly γ-continuity, strongly γ-closed graph and γ-T_2 spaces. And we study some characterizations and properties of such concepts.

1. Introduction

Let X, Y and Z be topological spaces on which no separation axioms are assumed unless explicit stated. Let S be a subset of X. The closure (resp. interior) of S will be denoted by clS (resp. $intS$). A subset S of X is called *semi-open set* [2] (resp. α-set [4]) if $S \subset cl(int(S))$ (resp. $S \subset int(cl(int(S))))$. The complement of a semi-open set (resp. α-set) is called *semi-closed set* (resp. α-closed set).

A subset $M(x)$ of a space X is called a *semi-neighborhood* of a point $x \in X$ if there exists a semi-open set S such that $x \in S \subset M(x)$. In [1], Latif introduced the notion of semi-convergence of filters. And he investigated some characterizations related to semi-open continuous functions. Now we recall the concept of semi-convergence of filters. Let $S(x) = \{A \in SO(X) : x \in A\}$ and let $S_x = \{A \subset X :$ there exists $\mu \subset S(x)$ such that μ is finite and $\cap\mu \subset A\}$. Then S_x is called the *semi-neighborhood filter* at x. For any filter F on X, we say that F semi-converges to x if and only if F is finer than the semi-neighborhood filter at x. A subset U of X is called a *γ-open set* [4] in X if whenever a filter F semi-converges to x and $x \in U$, then $U \in F$. The class of all γ-open sets in X will be denoted by $\gamma(X)$.

The *γ-interior* [4] of a set A in X, denoted by $I_\gamma(A)$, is the union of all γ-open sets contained in A.
The *γ-closure* [4] of a set A in X, denoted by $Cl_\gamma(A)$, $Cl_\gamma(A) = \{x \in X : A \cap U \neq \emptyset$ for all $U \in S_x\}$.

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Theorem 1.1 ([4]). *Let (X, τ) be a topological space and $A \subset X$.*
(a) $I_\gamma(A) = \{x \in A : A \in S_x\}$.
(b) A is γ-open set if and only if $A = I_\gamma(A)$.
(c) A set G is γ-closed if and only if whenever F semi-converges to x and $A \in F$, then $x \in A$.

Theorem 1.2 ([4]). *Let (X, τ) be a topological space and A be a subset of X.*
(1) $A \subset Cl_\gamma(A)$.
(2) A is γ-closed if and only if $A = Cl_\gamma A$.
(3) $I_\gamma(A) = X - Cl_\gamma(X - A)$.
(4) $Cl_\gamma(A) = X - I_\gamma(X - A)$.

2. Weakly γ-continuous functions

Definition 2.1. Let (X, τ) and (Y, μ) be two topological spaces. Then $f : X \to Y$ is said to be *weakly γ-continuous* if for $x \in X$ and each open subset V containing $f(x)$, there is a γ-open subset U containing x such that $f(U) \subset cl(V)$.

We get the following implications but the converses are not true:

continuous $\Rightarrow$ semi-continuous $\Rightarrow$ γ-continuous $\Rightarrow$ weakly γ-continuous

Example 2.2. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{a, b\}, X\}$ be a topology on X. Then $\gamma(X) = \{\emptyset, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$. Consider a function $f : (X, \tau) \to (Y, \mu)$ defined as follows: $f(a) = c$, $f(b) = d$, $f(c) = a$ and $f(d) = b$. Then f is weakly γ-continuous. But f is not γ-continuous because for a γ-open set $\{a, b\}$, $f^{-1}(\{a, b\}) = \{c, d\}$ is not γ-open.

Theorem 2.3. *Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces (X, τ) and (Y, μ). Then the following statements are equivalent:*
(1) f is weakly γ-continuous.
(2) $f^{-1}(V) \subseteq I_\gamma(f^{-1}(cl(V)))$ for every open subset V of Y.
(3) $Cl_\gamma(f^{-1}(int(A))) \subseteq f^{-1}(A)$ for every closed set A of Y.
(4) $Cl_\gamma(f^{-1}(int(cl(B)))) \subseteq f^{-1}(cl(B))$ for every set B of Y.
(5) $f^{-1}(int(B)) \subseteq I_\gamma(f^{-1}(cl(int(B))))$ for every set B of Y.
(6) $Cl_\gamma(f^{-1}(V)) \subseteq f^{-1}(cl(V))$ for every open subset V of Y.

Proof. (1) $\Rightarrow$ (2) Let V be an open subset in Y and $x \in f^{-1}(V)$. There exists a γ-open set U of X containing x such that $f(U) \subseteq cl(V)$.

Since $x \in U \subseteq f^{-1}(cl(V))$, by definition of γ-interior, $x \in I_\gamma(f^{-1}(cl(V)))$. Hence $f^{-1}(V) \subseteq I_\gamma(f^{-1}(cl(V)))$.

(2) $\Rightarrow$ (3) Let A be a closed subset in Y. Then $Y - A$ in open in Y and, by (2)

$$\begin{aligned} f^{-1}(Y-A) &\subseteq I_\gamma(f^{-1}(cl(Y-A))) \\ &= I_\gamma(f^{-1}(Y - int(A))) \\ &\subseteq X - Cl_\gamma(f^{-1}(int(A))). \end{aligned}$$

Thus $Cl_\gamma(f^{-1}(int(A))) \subseteq f^{-1}(A)$.

(3) $\Rightarrow$ (4) Let B be a subset of Y. Since $cl(B)$ is closed in Y, from (3), it follows $Cl_\gamma(f^{-1}(int(cl(B)))) \subseteq f^{-1}(cl(B))$.

(4) $\Rightarrow$ (5) Let B be a subset of Y. Then

$$\begin{aligned} f^{-1}(int(B)) &= X - f^{-1}(cl(Y-B)) \\ &\subseteq X - Cl_\gamma(f^{-1}int(cl(Y-B))) \\ &= I_\gamma(f^{-1}cl(int(B))). \end{aligned}$$

Thus we get the result.

(5) $\Rightarrow$ (6) Let V be an open subset of Y. Suppose $x \notin f^{-1}(cl(V))$. Then $f(x) \notin cl(V)$ and so there exists an open set U containing $f(x)$ such that $U \cap V = \emptyset$ and so $cl(U) \cap V = \emptyset$. By (5), $x \in f^{-1}(U) \subseteq I_\gamma(f^{-1}(cl(U)))$. Then by definition of γ-interior, there exists an open set G containing x such that $x \in G \subseteq f^{-1}(cl(U))$. Since $cl(U) \cap V = \emptyset$ and $f(G) \subseteq cl(U)$, we have $G \cap f^{-1}(V) = \emptyset$ and so $x \notin Cl_\gamma(f^{-1}(V))$. Hence $Cl_\gamma(f^{-1}(V)) \subseteq f^{-1}(cl(V))$.

(6) $\Rightarrow$ (1) Let $x \in X$ and V an open set in Y containing $f(x)$. Since $V = int(V) \subseteq int(cl(V))$, by (6),

$$\begin{aligned} x \in f^{-1}(V) &\subseteq f^{-1}(int(cl(V))) \\ &= X - f^{-1}(cl(Y - cl(V))) \\ &\subseteq X - Cl_\gamma(f^{-1}(Y - cl(V))) \\ &= I_\gamma(f^{-1}(cl(V))). \end{aligned}$$

So there exists a γ-open subset U in X such that $U \subseteq f^{-1}(cl(V))$. Hence f is weakly γ-continuous. □

We recall that a point x of a topological space X is said to be θ-adherent of A if $A \cap cl(V) \neq \emptyset$ for every open set V containing x. The set of all θ-adherent points of A is called θ-closure of A [6] and is denoted by

$cl_\theta(A)$. If $A = cl_\theta(A)$, then A is called θ-closed. The complement of a θ-closed set is said to be θ-open. It is shown in [6] that $cl(A) = cl_\theta(A)$ for every open set A and $cl_\theta(B)$ is closed for every subset B of X.

Theorem 2.4. *Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces (X, τ) and (Y, μ). Then the following statements are equivalent:*
(1) f is weakly γ-continuous.
(2) $Cl_\gamma(f^{-1}(int(cl_\theta(B)))) \subseteq f^{-1}(cl_\theta(B))$ for every set B of Y.
(3) $Cl_\gamma(f^{-1}(int(cl(B)))) \subseteq f^{-1}(cl_\theta(B))$ for every set B of Y.
(4) $Cl_\gamma(f^{-1}(int(cl(G)))) \subseteq f^{-1}(cl(G))$ for every open subset G of Y.
(5) $f(Cl_\gamma(A)) \subseteq cl_\theta(f(A))$ for every set A of X.
(6) $Cl_\gamma(f^{-1}(B)) \subseteq f^{-1}(cl_\theta(B))$ for every set B of Y.

Proof. (1) $\Rightarrow$ (2) Let B be any subset in Y; then $cl_\theta(B)$ is closed, by Theorem 2.3 (3), we get the result.

(2) $\Rightarrow$ (3) It is obvious since $cl(B) \subseteq cl_\theta(B)$ for every subset B of Y.

(3) $\Rightarrow$ (4) It is obvious since $cl(G) = cl_\theta(G)$ for every open subset G of Y.

(4) $\Rightarrow$ (1) Since $G \subseteq int(cl(G))$ for every open set G of Y, from Theorem 2.3 (6), it follows f is weakly γ-continuous.

(1) $\Rightarrow$ (5) Let A be a subset of X. Let $x \in Cl_\gamma(A)$ and G be an open subset of Y containing $f(x)$. Since f is weakly γ-continuous, there exists a γ-open set U containing x in X such that $f(U) \subseteq cl(G)$. Since $x \in Cl_\gamma(A)$, we have $U \cap A \neq \emptyset$ and so $\emptyset \neq f(U) \cap f(A) \subseteq cl(G) \cap f(A)$. Hence $f(x) \in cl_\theta(f(A))$.

(5) $\Rightarrow$ (6) Let B be a subset of Y; then by (5), we have $f(Cl_\gamma(f^{-1}(B))) \subseteq cl_\theta(f(f^{-1}(B))) \subseteq cl_\theta(B)$ and so we get the result.

(6) $\Rightarrow$ (1) Let B be a subset of Y; then by (6),

$$\begin{aligned} Cl_\gamma(f^{-1}(int(cl(B)))) &\subseteq f^{-1}(cl_\theta(int(cl(B)))) \\ &= f^{-1}(cl(int(cl(B)))) \\ &\subseteq f^{-1}(cl(B)). \end{aligned}$$

Hence f is weakly γ-continuous by Theorem 2.3 (4). □

Theorem 2.5. *Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces (X, τ) and (Y, μ). Then the following statements are equivalent:*
(1) f is weakly γ-continuous.
(2) $Cl_\gamma(f^{-1}(int(K))) \subseteq f^{-1}(K)$ for every regular closed set K of Y.
(3) $Cl_\gamma(f^{-1}(int(cl(G)))) \subseteq f^{-1}(cl(G))$ for every β-open set G of Y.

(4) $Cl_\gamma(f^{-1}(int(cl(G)))) \subseteq f^{-1}(cl(G))$ for every semiopen set G of Y.

Proof. (1) $\Rightarrow$ (2) Let K be any regular closed set of Y. Then by Theorem 2.3(6), we have $Cl_\gamma(f^{-1}(int(K))) \subseteq f^{-1}(cl(int(K)))$. Since K is regular closed, we have $Cl_\gamma(f^{-1}(int(K))) \subseteq f^{-1}(K)$.

(2) $\Rightarrow$ (3) Let G be any β-open set. From $cl(G) \subseteq cl(int(cl(G))) \subseteq cl(G)$, it follows $cl(G)$ is regular closed. By (2), we have $Cl_\gamma(f^{-1}(int(cl(G)))) \subseteq f^{-1}(cl(G))$.

(3) $\Rightarrow$ (4) It is obvious since every semiopen set is β-open.

(4) $\Rightarrow$ (1) Let V be any open set of Y. Then by (4),

$$Cl_\gamma(f^{-1}(V)) \subseteq Cl_\gamma(f^{-1}(int(cl(V)))) \subseteq f^{-1}(cl(V)).$$

Hence from Theorem 2.3(6), f is weakly γ-continuous. □

Theorem 2.6. *Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces (X, τ) and (Y, μ). Then the following statements are equivalent:*
(1) f is weakly γ-continuous.
(2) $Cl_\gamma(f^{-1}(int(cl(G)))) \subseteq f^{-1}(cl(G))$ for every preopen set G of Y.
(3) $Cl_\gamma(f^{-1}(G)) \subseteq f^{-1}(cl(G))$ for every preopen set G of Y.
(4) $f^{-1}(G) \subseteq Int_\gamma(f^{-1}(cl(G)))$ for every preopen set G of Y.

Proof. (1) $\Rightarrow$ (2) Let G be any preopen set in Y. Then $cl(G) = cl(int(cl(G)))$. Let $A = int(cl(G))$. Then from Theorem 2.3 (6), it follows that $Cl_\gamma(f^{-1}(A)) \subseteq f^{-1}(cl(A))$. Since $cl(A) = cl(G)$, we have $Cl_\gamma(f^{-1}(int(cl(G)))) \subseteq f^{-1}(cl(G))$.

(2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (4) Let G be any preopen set in Y. Then from definition of preopen sets and (3), it follows that

$$\begin{aligned} f^{-1}(G) &\subseteq f^{-1}(int(cl(G))) \\ &= X - f^{-1}(cl(Y - cl(G))) \\ &\subseteq X - (Cl_\gamma(f^{-1}(Y - cl(G)))) \\ &= Int_\gamma(f^{-1}(cl(G))). \end{aligned}$$

Hence we have (4).

(4) $\Rightarrow$ (1) Since every open set is preopen, from (4) and Theorem 2.3(6), f is weakly γ-continuous. □

Definition 2.7. Let X be a topological space. Then X is said to be γ-T_2 if for every two distinct points x and y in X, there exist two disjoint γ-open sets U and V such that $x \in U$ and $y \in V$.

Let X be a topological space. Then X is said to be *Urysohn* if for every two distinct points x and y in X, there exist two open sets U and V such that $cl(U) \cap cl(V) = \emptyset$.

Theorem 2.8. *Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces (X, τ) and (Y, μ). If f is a weakly γ-continuous injection and Y is Urysohn, then X is γ-T_2.*

Proof. Let x_1 and x_2 be two distinct elements in X, then $f(x_1) \neq f(x_2)$. There exist two open sets U and V in Y containing $f(x_1)$, $f(x_2)$, respectively, such that $cl(U) \cap cl(V) = \emptyset$. Since f is weakly γ-continuous, there exist γ-open sets U_1, V_2 containing x_1, x_2, respectively, such that $f(U_1) \subseteq cl(U)$, $f(V_2) \subseteq cl(V)$. It follows $U_1 \cap V_2 = \emptyset$. Hence X is γ-T_2. □

Definition 2.9. Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces (X, τ) and (Y, μ). We call f has a *strongly γ-closed graph* if for each $(x, y) \notin G(f)$, there exist a γ-open set U and an open set V containing x and y, respectively, such that $(U \times cl(V)) \cap G(f) = \emptyset$.

Lemma 2.10. Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces (X, τ) and (Y, μ). Then f has a strongly γ-closed graph if for each $(x, y) \notin G(f)$, there exist a γ-open set U containing and an open set V containing x and y, respectively, such that $f(U) \cap cl(V) = \emptyset$.

Proof. Obvious. □

Theorem 2.11. *Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces (X, τ) and (Y, μ). If f is weakly γ-continuous and Y is Urysohn, then f has a strongly γ-closed graph.*

Proof. Let $(x, z) \notin G(f)$. Then $z \neq f(x)$ and since Y is Urysohn, there exist two open sets U and V containing z and $f(x)$, respectively, such that $cl(U) \cap cl(V) = \emptyset$. Since f is weakly γ-continuous, there exists a γ-open set H containing x such that $f(H) \subseteq cl(V)$. It implies $f(H) \cap cl(U) = \emptyset$. Hence f has a strongly γ-closed graph. □

Theorem 2.12. *Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces (X, τ) and (Y, μ). If f is a weakly γ-continuous injection with a strongly γ-closed graph, then X is γ-T_2.*

Proof. Let x_1 and x_2 be two distinct elements in X, then $f(x_1) \neq f(x_2)$. This implies that $(x_1, f(x_2)) \in (X \times Y) - G(f)$. Since f has a strongly γ-closed graph, there exist a γ-open set U and an open set V containing x_1 and $f(x_2)$, respectively, such that $f(U) \cap cl(V) = \emptyset$. Since f is weakly γ-continuous, there exists a γ-open set W containing x_2 such that $f(W) \subseteq cl(V)$. It implies $f(W) \cap f(U) = \emptyset$. Therefore $W \cap U = \emptyset$ and so X is a γ-T_2 space. □

Definition 2.13. A subset A of a topological space (X, τ) is called *γ-compact* relative to A if every collection $\{U_i : i \in J\}$ of γ-open subsets of X such that $A \subset \cup\{U_i : i \in J\}$, there exists a finite subset J_0 of J such that $A \subset \cup\{U_i : i \in J_0\}$.

A subset A of a topological space X is said to be *quasi H-closed* relative to A [6] if every collection $\{U_i : i \in J\}$ of open subsets of X such that $A \subseteq \cup\{U_i : i \in J\}$, there exists a finite subset J_0 of J such that $A \subseteq \cup\{cl(U_i) : i \in J_0\}$.

Theorem 2.14. *Let $f : (X, \tau) \to (Y, \mu)$ be a function on topological spaces (X, τ) and (Y, μ). If f is weakly γ-continuous and A is a γ-compact subset of X, then $f(A)$ is quasi H-closed relative to (Y, μ).*

Proof. Let $\{V_i : i \in J\}$ be a cover of $f(A)$ by open subsets of Y. For each $x \in A$, there exists $i(x) \in J$ such that $f(x) \in V_{i(x)}$. Since f is weakly γ-continuous, there exists a γ-open set $U(x)$ containing x such that $f(U(x)) \subseteq cl(V_{i(x)})$. The family $\{U(x) : x \in A\}$ is a cover of A by γ-open sets in X. Since A is γ-compact, there is a finite subcover $\{U(x_1), U(x_2), \cdots, U(x_n) : x_j \in A, j = 1, 2, \cdots, n\}$ such that $A \subseteq \cup U(x_j)$. Then

$$f(A) \subseteq f(\cup U(x_j)) \subseteq \cup f(U(x_j)) \subseteq \cup cl(V_{i(x_j)}),$$

$1 \leq j \leq n$.

Thus $f(A)$ is quasi H-closed relative to (Y, μ). □

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