The Properties of Implications and Conjunctions

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Abstract

We investigate the properties of (forcing)-implications, conjunctions and adjointness in a sense Morsi et.al [1,5].

Key words : (forcing)-implications, conjunctions, adjointness

1. Introduction and Preliminaries

Recently, Morsi et.al [1,5] introduced the theory of implications and conjunctions (generalized by t-norm) related by adjointness in many valued logics.

In this paper, we introduce characterizations of (forcing)-implications, conjunctions and adjointness. We investigate the relations of them. In particular, we study the (forcing)-implications, conjunctions and adjointness induced by functions. Let L be a completely distributive lattice with a top 1 and a bottom 0.

Definition 1.1. ([1,5]) A binary operation $A : L \times L \to L$ is called an implication if it satisfies:

(A1) if $x \leq y$, then $A(x,z) \geq A(y,z)$.
(A2) if $y \leq z$, then $A(x,y) \leq A(x,z)$.
(A3) $A(1,z) = z$.

A binary operation $A : L \times L \to L$ is called a forcing-implication if it satisfies (A1), (A2) and
(H) $y \leq z$ iff $H(y,z) = 1$.

Definition 1.2. ([1,5]) A binary operation $K : L \times L \to L$ is called a conjunction if it satisfies:

(K1) if $x \leq y$, then $K(x,z) \leq K(y,z)$.
(K2) if $y \leq z$, then $K(x,y) \leq K(x,z)$.
(K3) $K(1,z) = z$.

Definition 1.3. ([1,5]) (1) A binary operation K is called a left adjoint of A, denoted by $K \dashv A$, if it satisfies: for all $x, y, z \in L$,

$$\text{(adjointness)} \;\; K(x,y) \leq z \;\text{ iff }\; y \leq A(x,z).$$

(2) A binary operation H is called a left adjoint of A , denoted by $H \dashv^{op} A$, if it satisfies: for all $x, y, z \in L$,

$$\text{(adjointness)} \;\; H(y,z) \leq^{op} x \;\text{ iff }\; y \leq A(x,z)$$

where $\leq^{op} = \geq$.

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Definition 1.4. ([1,5]) A function $N : L \to L$ is called a negation if it satisfies:

(N1) $N(0) = 1$ and $N(1) = 0$.
(N2) if $x \leq y$, then $N(x) \geq N(y)$.
(N3) $N(N(x)) = x$.

2. Implications and Conjunctions

Theorem 2.1. Let $f : L \to L$ be an order-isomorphic function (f is bijective and $x \leq y$ iff $f(x) \leq f(y)$) with $f(1) = 1$. Define a binary operation $A : L \to L$ by

$$A(x,y) = f^{-1}(N(f(x)) \vee f(y)).$$

Then A is an implication. Moreover, if L is a Boolean algebra, then A is an implication and a forcing-implication.

Proof. It is easily proved

$$A(1,z) = f^{-1}(N(f(1)) \vee f(z)) = f^{-1}(f(z)) = z.$$

If L is a Boolean algebra, then $1 = N(a) \vee b$ iff $a \leq b$. Thus

$$\begin{aligned} & 1 = A(x,y) = f^{-1}(N(f(x)) \vee f(y)) \\ & \text{iff } \; 1 = N(f(x)) \vee f(y) \\ & \text{iff } \; f(x) \leq f(y) \; \text{ iff } \; x \leq y. \end{aligned}$$

Hence A is a forcing-implication.

□

Example 2.2. Let $(P(U), \subset, \emptyset, U)$ be a completely distributive lattice.

(1) We define an operator $A : P(U) \to P(U)$ as follows:

$$A(X,Y) = Y.$$

Then A is an implication operator.

(2) We define an operator $H : P(U) \to P(U)$ as follows:

$$H(X,Y) = \begin{cases} U & \text{if } X \subset Y, \\ \emptyset & \text{if } X \not\subset Y. \end{cases}$$

Then H is a forcing-implication.

(3) We define an operator $A : P(U) \to P(U)$ as follows

$$A(X,Y) = X^c \cup Y.$$

Then A is an implication and forcing implication operator.

Theorem 2.3. Let $f : [0,1] \to [f(0),1]$ be a bijective strictly-increasing function and $p > 0$. Define binary operations $A_1, A_2 : [0,1] \times [0,1] \to [0,1]$ by

$$A_1(x,y) = f^{-1}\Big(\frac{f(y)}{f(x)^p} \wedge 1\Big), \quad f(0) \neq 0$$

$$A_2(x,y) = f^{-1}\Big((1 - f(x)^p + f(y)) \wedge 1\Big), \quad f(0) = 0$$

Then we have the following properties:

(1) A_1 and A_2 are implications.

(2) If $p = 1$, then A_1 and A_2 are implications and forcing-implications.

Proof. (1) Since $f(1) = 1$, we have:

$$A_1(1,y) = f^{-1}\Big(\frac{f(y)}{f(1)^p} \wedge 1\Big) = y,$$

$$A_2(1,y) = f^{-1}\Big((1 - f(1)^p + f(y)) \wedge 1\Big) = y.$$

(2) If $p = 1$, then

$$A_1(x,y) = f^{-1}\Big(\tfrac{f(y)}{f(x)} \wedge 1\Big) = 1$$
$$\Leftrightarrow \tfrac{f(y)}{f(x)} \geq 1 \Leftrightarrow x \leq y$$

$$A_2(x,y) = f^{-1}\Big((1 - f(x) + f(y)) \wedge 1\Big) = 1$$
$$\Leftrightarrow 1 - f(x) + f(y) \geq 1 \Leftrightarrow x \leq y.$$

□

Example 2.4. (1) Let $f : [0,1] \to [f(0),1]$ be a bijective strictly-increasing function as $f(x) = \frac{1}{2}x^2 + \frac{1}{2}$. From Theorem 2.3(1), we define an operator

$$\begin{aligned} A_1(x,y) &= f^{-1}\Big(\tfrac{f(y)}{f(x)^p} \wedge 1\Big) \\ &= \sqrt{(2^p \tfrac{y^2+1}{(x^2+1)^p} - 1) \wedge 1}. \end{aligned}$$

If $p = 1$, then A_1 is an implication and forcing-implication.

(2) Let $f : [0,1] \to [0,1]$ be a bijective strictly-increasing function as $f(x) = x^2$. From Theorem 2.3(1), we define an operator

$$\begin{aligned} A_2(x,y) &= f^{-1}\Big((1 - f(x)^p + f(y)) \wedge 1\Big) \\ &= \sqrt{(1 - x^{2p} + y^2) \wedge 1}. \end{aligned}$$

If $p = 1$, then A_2 is implications and forcing-implications.

Theorem 2.5. Let $f : L \to L$ be an order-isomorphic function with $f(1) = 1$. Define a binary operation $K : L \to L$ by

$$K(x,y) = f^{-1}(f(x) \wedge f(y)).$$

Then K is a conjunction.

Proof. It is easily proved from

$$K(1,y) = f^{-1}(f(1) \wedge f(y)) = y.$$

□

Example 2.6. Let $(P(U), \subset, \emptyset, U)$ be a completely distributive lattice. We define an operator $K : P(U) \to P(U)$ as follows:

$$K(X,Y) = X \cap Y.$$

Then K is a conjunction.

Theorem 2.7. Let $f : [0,1] \to [f(0),1]$ be a bijective strictly-increasing function and $p > 0$. Define binary operations $K_1, K_2 : [0,1] \times [0,1] \to [0,1]$ by

$$K_1(x,y) = f^{-1}\Big(f(x)^p f(y) \vee f(0)\Big), \quad f(0) \neq 0$$

$$K_2(x,y) = f^{-1}\Big((f(x)^p + f(y) - 1) \vee 0\Big), \quad f(0) = 0.$$

Then K_1 and K_2 are conjunctions.

Proof. Since $f(1) = 1$, we have:

$$K_1(1,y) = f^{-1}\Big(f(1)^p f(y) \vee f(0)\Big) = y,$$

$$K_2(1,y) = f^{-1}\Big((f(1)^p + f(y) - 1) \vee 0\Big) = y.$$

□

Example 2.8. (1) Let $f : [0,1] \to [f(0),1]$ be a bijective strictly-increasing function as $f(x) = \frac{1}{2}x^2 + \frac{1}{2}$. From Theorem 2.7, we define an operator

$$\begin{aligned} K_1(x,y) &= f^{-1}\Big(f(x)^p f(y) \vee f(0)\Big) \\ &= \sqrt{(2^{-p}(y^2+1)(x^2+1)^p - 1) \vee 0}. \end{aligned}$$

(2) Let $f : [0,1] \to [0,1]$ be a bijective strictly-increasing function as $f(x) = x^2$. From Theorem 2.7, we define an operator

$$\begin{aligned} K_2(x,y) &= f^{-1}\Big((f(x)^p + f(y) - 1) \vee 0\Big) \\ &= \sqrt{(x^{2p} + y^2 - 1) \vee 0}. \end{aligned}$$

3. The Adjointness for Fuzzy Logics

Theorem 3.1. (1) A binary operation K is a left adjoint of A iff for all $x, y, z \in L$,

$$y \leq A(x, K(x, y)), \quad K(x, A(x, z)) \leq z.$$

(2) A binary operation H is a left adjoint of A, $H \dashv^{op} A$, iff for all $x, y, z \in L$,

$$y \leq A(H(y, z), z), \quad H(A(x, z), z) \geq x.$$

Proof. (1) Since $K(x, y) \leq K(x, y)$ and $A(x, z) \leq A(x, z)$, by adjointness, we have

$$y \leq A(x, K(x, y)), \quad K(x, A(x, z)) \leq z.$$

Conversely, let $K(x, y) \leq z$. By (A2), we have

$$A(x, z) \geq A(x, K(x, y)) \geq y.$$

Let $A(x, z) \geq y$. By (K2), we have

$$K(x, y) \leq K(x, A(x, z)) \leq z.$$

(2) Since $H(x, y) \leq^{op} H(x, y)$ and $A(x, z) \leq A(x, z)$, by adjointness, we have

$$y \leq A(H(x, y), y), \quad H(A(x, z), z) \leq^{op} x.$$

Conversely, let $H(y, z) \leq^{op} x$. By (A1), we have

$$A(x, z) \geq A(H(y, z), z) \geq y.$$

Let $A(x, z) \geq y$. By (A1), we have

$$H(y, z) \geq H(A(x, z), z) \geq x.$$

Hence $H(y, z) \leq^{op} x$.

□

Theorem 3.2. Let $(L, \leq)$ be a distributive complete lattice.
(1) An implication A satisfies $A(x, \bigwedge z_i) = \bigwedge A(x, z_i)$ iff there exists a conjunction K with $K \dashv A$ defined by

$$K(x, y) = \bigwedge\{z \in L \mid y \leq A(x, z)\}.$$

(2) A conjunction K satisfies $K(x, \bigvee z_i) = \bigvee K(x, z_i)$ iff there exists an implication A with $K \dashv A$ defined by

$$A(x, y) = \bigvee\{z \in L \mid K(x, z) \leq y\}.$$

(3) An implication A satisfies $A(\bigvee x_i, z) = \bigwedge A(x_i, z)$ iff there exists a forcing-implication H with $H \dashv^{op} A$ defined by

$$H(x, y) = \bigvee\{z \in L \mid x \leq A(z, y)\}.$$

Proof. (1) ($\Rightarrow$) (K1) If $x_1 \leq x_2$, then $A(x_1, z) \geq A(x_2, z) \geq y$ implies $K(x_1, y) \leq K(x_2, y)$.

(K2) If $y_1 \leq y_2$, then $y_1 \leq y_2 \leq A(x, z)$ implies $K(x, y_1) \leq K(x, y_2)$.

(K3) $K(1, y) = \bigwedge\{z \in L \mid y \leq A(1, z) = z\} = y$. Hence K is a conjunction. Let $y \leq A(x, z)$. Then $K(x, y) \leq z$. Let $K(x, y) \leq z$. Then $A(x, K(x, y)) \leq A(x, z)$ and

$$\begin{aligned} A(x, K(x, y)) &= A(x, \bigwedge\{z \in L \mid y \leq A(x, z)\}) \\ &= \bigwedge\{A(x, z) \mid y \leq A(x, z)\} \\ &\geq y. \end{aligned}$$

So, $A(x, z) \geq y$. Hence $K \dashv A$.

($\Leftarrow$) Enough to $\bigwedge A(x, z_i) \leq A(x, \bigwedge z_i)$. It follows from:

$$\begin{aligned} & K(x, \bigwedge A(x, z_i)) \leq K(x, A(x, z_i)) \leq z_i \\ & \Rightarrow K(x, \bigwedge A(x, z_i)) \leq \bigwedge z_i \\ & \Rightarrow \bigwedge A(x, z_i) \leq A(x, \bigwedge z_i). \end{aligned}$$

(2) ($\Rightarrow$) (A1) If $x_1 \leq x_2$, then $K(x_1, z) \leq K(x_2, z)$. So, $A(x_1, y) \geq A(x_2, y)$.

(A2) If $y_1 \leq y_2$, then $K(x, z) \leq y_1 \leq y_2$ implies $A(x, y_1) \leq A(x, y_2)$.

(A3) $A(1, y) = \bigvee\{z \in L \mid K(1, z) = z \leq y\} = y$. Hence A is an implication. Let $K(x, y) \leq z$. By the definition of A, $y \leq A(x, z)$. Let $z \leq A(x, y)$. Then $K(x, A(x, y)) \geq K(x, z)$ and

$$\begin{aligned} K(x, A(x, y)) &= K(x, \bigvee\{z \in L \mid K(x, z) \leq y\}) \\ &= \bigvee\{K(x, z) \mid K(x, z) \leq y\} \\ &\leq y. \end{aligned}$$

So, $K(x, z) \leq y$. Hence $K \dashv A$.

($\Leftarrow$) Enough to $\bigvee K(x, z_i) \geq K(x, \bigvee z_i)$. It follows from:

$$\begin{aligned} & A(x, \bigvee K(x, z_i) \geq A(x, K(x, z_i)) \geq z_i \\ & \Rightarrow A(x, \bigvee K(x, z_i) \geq \bigvee z_i \\ & \Rightarrow \bigvee K(x, z_i) \geq K(x, \bigvee z_i). \end{aligned}$$

(3) ($\Rightarrow$) Let $H(y, z) = 1$ be given. Then $y \leq z$ from:

$$\begin{aligned} & z = A(1, z) = A(H(y, z), z) \\ & = A(\bigvee\{x_i \mid y \leq A(x_i, z)\}, z) \\ & \geq \bigwedge\{A(x_i, z) \mid y \leq A(x_i, z)\} \\ & \geq y. \end{aligned}$$

Let $y \leq z$. Since $y \leq z = A(1, z)$, we have

$$H(y, z) = \bigvee\{x_i \mid y \leq A(x_i, z)\} = 1.$$

Hence H is a forcing-implication. Let $y \leq A(x, z)$. By the definition of H, $x \leq H(y, z)$. Let $x \leq H(y, z)$. Then $A(H(y, z), z) \leq A(x, z)$ and

$$\begin{aligned} A(H(y, z), z) &= A(\bigvee\{x_i \in L \mid y \leq A(x_i, z)\}, z) \\ &= \bigwedge\{A(x_i, z) \mid y \leq A(x_i, z)\} \\ &\geq y. \end{aligned}$$

So, $A(x,z) \geq y$. Hence $H \dashv^{op} A$.

($\Leftarrow$) Enough to $\bigwedge A(x_i,z) \leq A(\bigvee x_i, z)$. It follows from:

$$\begin{aligned} &H(\bigwedge A(x_i,z),z) \geq H(A(x_i,z)z) \geq x_i \\ &\Rightarrow H(\bigwedge A(x_i,z),z) \geq \bigvee x_i \\ &\Rightarrow \bigwedge A(x_i,z) \leq A(\bigvee x_i, z). \end{aligned}$$

□

Theorem 3.3. Let $f : [0,1] \to [f(0),1]$ be a bijective strictly-increasing continuous function. Define an implication $A : [0,1] \times [0,1] \to [0,1]$ by

$$A(x,y) = f^{-1}\Big(\frac{f(y)}{f(x)} \wedge 1\Big), \quad f(0) \neq 0.$$

Then there exists a forcing-implication H such that $A = H$ and conjunction K such that

$$K(x,y) = f^{-1}(f(x)f(y) \vee f(0)).$$

Proof. Since A satisfies $A(\bigvee x_i, z) = \bigwedge A(x_i,z)$, by Theorem 3.2(3), there exists a forcing-implication H defined by

$$H(x,y) = \bigvee\{z \in L \mid x \leq A(z,y)\}.$$

Since $x \leq A(z,y) = f^{-1}(\frac{f(y)}{f(z)} \wedge 1)$, we have $z \leq f^{-1}(\frac{f(y)}{f(x)} \wedge 1)$. Since A is continuous from pasting lemma, we have

$$H(x,y) = f^{-1}\Big(\frac{f(y)}{f(x)} \wedge 1\Big).$$

Hence $A = H$. Since A is continuous, we have $A(x, \bigwedge z_i) = \bigwedge A(x,z_i)$. By Theorem 3.2(1), there exists a conjunction K defined by $K(x,y) = \bigwedge\{z \in L \mid y \leq A(x,z)\}$. Since $y \leq f^{-1}(\frac{f(z)}{f(x)} \wedge 1)$, we have

$$z \geq f^{-1}(f(x)f(y) \vee f(0).$$

Hence $K(x,y) = f^{-1}(f(x)f(y) \vee f(0))$. □

Example 3.4. Let $f : [0,1] \to [f(0),1]$ be a bijective strictly-increasing function as $f(x) = \frac{1}{2}x + \frac{1}{2}$. From Theorem 3.3, we define an operator

$$A(x,y) = f^{-1}\Big(\frac{f(y)}{f(x)} \wedge 1\Big) = (\frac{2y-x+1}{x+1}) \wedge 1.$$

Equivalently,

$$A(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{2y-x+1}{x+1} & \text{if } x > y. \end{cases}$$

(1) A is an implication satisfying $A(x, \bigwedge z_i) = \bigwedge A(x,z_i)$. Hence we can obtain a conjunction K as follows

$$\begin{aligned} K(x,y) &= \bigwedge\{z \in L \mid y \leq A(x,z) = (\tfrac{2z-x+1}{x+1}) \wedge 1\} \\ &= (\tfrac{xy+x+y-1}{2}) \vee 0. \end{aligned}$$

Furthermore, $A(x,K(x,y)) = y \vee \frac{1-x}{1+x}$ and $K(x,A(x,z)) \leq z$ from:

Since $x \geq K(x,y)$,

$$A(x,K(x,y)) = \frac{2K(x,y)-x+1}{x+1} = y \vee \frac{1-x}{1+x}.$$

If $x > z$,

$$\begin{aligned} K(x,A(x,z)) &= \tfrac{xA(x,z)+x+A(x,z)-1}{2} \vee 0 \\ &= \tfrac{x\frac{2z-x+1}{x+1}+x+\frac{2z-x+1}{x+1}-1}{2} \vee 0 \\ &= z. \end{aligned}$$

If $x \leq z$, then $K(x,A(x,z)) = x \leq z$.

(2) A is an implication satisfying $A(\bigvee x_i, z) = \bigwedge A(x_i,z)$. Hence we can obtain a forcing implication H as follows

$$\begin{aligned} H(x,y) &= \bigvee\{z \in L \mid x \leq A(z,y) = (\tfrac{2y-z+1}{z+1}) \wedge 1\} \\ &= (\tfrac{2y-x+1}{x+1}) \wedge 1. \end{aligned}$$

Furthermore, $A(H(y,z),z) = y \vee \frac{z(y+1)}{z+1}$ and $H(A(x,z),z) \geq x$.

Theorem 3.5. Let $f : [0,1] \to [0,1]$ be a bijective strictly-increasing continuous function. Define an implication $A : [0,1] \times [0,1] \to [0,1]$ by

$$A(x,y) = f^{-1}\Big((1 - f(x) + f(y)) \wedge 1\Big).$$

Then there exists a forcing-implication H such that $A = H$ and conjunction K such that

$$K(x,y) = f^{-1}\Big((f(x) + f(y) - 1) \vee f(0)\Big).$$

Proof. Since A satisfies $A(\bigvee x_i, z) = \bigwedge A(x_i,z)$, by Theorem 3.2(3), there exists a forcing-implication H defined by

$$H(x,y) = \bigvee\{z \in L \mid x \leq A(z,y)\}.$$

Since $x \leq A(z,y) = f^{-1}\Big((1-f(z)+f(y)) \wedge 1\Big)$, we have $z \leq f^{-1}\Big((1 - f(x) + f(y)) \wedge 1\Big)$. Since A is continuous from pasting lemma, we have

$$H(x,y) = f^{-1}\Big((1 - f(x) + f(y)) \wedge 1\Big).$$

Hence $A = H$. Since A is continuous, we have $A(x, \bigwedge z_i) = \bigwedge A(x,z_i)$. By Theorem 3.2(2), there exists a conjunction K defined by $K(x,y) = \bigwedge\{z \in L \mid y \leq A(x,z)\}$. Since $y \leq f^{-1}\Big((1-f(x)+f(z)) \wedge 1\Big)$, we have

$$z \geq f^{-1}\Big((f(x) + f(y) - 1) \vee f(0)\Big).$$

Hence $K(x,y) = f^{-1}\Big((f(x) + f(y) - 1) \vee f(0)\Big)$. □

Example 3.6. Let $f : [0,1] \to [0,1]$ be a bijective strictly-increasing function as $f(x) = x^p (p > 0)$. From Theorem 3.5, we define an implication

$$A(x,y) = \Big((1 - x^p + y^p) \wedge 1\Big)^{\frac{1}{p}}.$$

Since A is an implication satisfying $A(x, \bigwedge z_i) = \bigwedge A(x, z_i)$ and $A(\bigvee x_i, z) = \bigwedge A(x_i, z)$. Hence we can obtain a forcing implication $H = A$ and a conjunction K as follows

$$K(x,y) = \Big((x^p + y^p - 1) \vee 0\Big)^{\frac{1}{p}}.$$

Theorem 3.7. Let $f : L \to L$ be an order-isomorphic function with $f(1) = 1$. Define a conjunction $K : L \times L \to L$ with $K(x, \bigvee z_i) = \bigvee K(x, z_i)$ and

$$K(x,y) = f^{-1}(f(x) \wedge f(y)).$$

Then there exists a forcing-implication H such that $A = H$ with $K \dashv A$ defined as

$$H(x,y) = \begin{cases} 1 & \text{if } x \le y, \\ y & \text{if } x \not\le y. \end{cases}$$

Proof. It is easily proved from Theorem 3.2. □

Example 3.8. Let $(P(U), \subset, \emptyset, U)$ be a completely distributive lattice. We define an operator $K : P(U) \to P(U)$ as follows:

$$K(X,Y) = X \cap Y.$$

Then K is a conjunction with $K(X, \cup Y_i) = \cup K(X, Y_i)$. We obtain an implication operator $A = H$ as follows:

$$\begin{aligned} A(X,Y) &= \bigcup\{Z \in P(U) \mid X \cap Z \subset Y\} \\ &= \bigcup\{Z \in P(U) \mid Z \subset X^c \cup Y\} \\ &= X^c \cup Y. \end{aligned}$$

Furthermore, $X \cap Z \subset Y$ iff $Z \subset X^c \cup Y$.

Example 3.9. We define an operator $A : [0,1] \times [0,1] \to [0,1]$ as follows:

$$A(x,y) = \begin{cases} 1 & \text{if } y > 2x - 1, \\ (1 - x) \vee y & \text{if } y \le 2x - 1. \end{cases}$$

Then A is an implication operator which does not satisfy $A(x, \bigwedge z_i) \ne \bigwedge A(x, z_i)$ and $A(\bigvee x_i, z) \ne \bigwedge A(x_i, z)$ because

$$\begin{aligned} 1 &= \textstyle\bigwedge_{n \in N} A(\frac{3}{4} - \frac{1}{n+1}, \frac{1}{2}) \\ &\ne A(\textstyle\bigvee_{n \in N} \frac{3}{4} - \frac{1}{n+1}, \frac{1}{2}) = A(\frac{3}{4}, \frac{1}{2}) = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} 1 &= \textstyle\bigwedge_{n \in N} A(\frac{3}{4}, \frac{1}{2} + \frac{1}{n+1}) \\ &\ne A(\frac{3}{4}, \textstyle\bigwedge_{n \in N} \frac{1}{2} + \frac{1}{n+1}) = A(\frac{3}{4}, \frac{1}{2}) = \frac{1}{2}. \end{aligned}$$

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