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Subsethood Measures Defined by Choquet Integrals

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Abstract

In this paper, we consider concepts of subsethood measure introduced by Fan et al. [2]. Based on this, we give various subsethood measure defined by Choquet integral with respect to a fuzzy measure on fuzzy sets which is often used in information fusion and data mining as a nonlinear aggregation tool and discuss some properties of them. Furthermore, we introduce simple examples.

Key Words : subsethood measure, fuzzy set, Choquet integral.

1. Introduction

Fuzzy sets were suggested for the first time by Zadeh[13]. In fuzzy set theory, fuzzy entropy, distance measure, and similarity measure are three basic concepts. Xuechang [11] had systematically studied these three concepts. Fuzzy implication operator is another basic concept in fuzzy set theory. Various fuzzy implication operators have been proposed [8]. A concept that closely relates to the above concepts is subsethood measure. Fan[3] commented on the subsethood measure and weak subsethood measure defined by Young[12] and gave some new definition of subsethood measure. They also presented some subsethood measure formulas from the point of set theoretic approach, and from fuzzy implication operator.

In this paper, we give various subsethood measure defined by Choquet integral with respect to a fuzzy measure and discuss some properties of them. The Choquet integral with respect to a fuzzy measure is often used in information fusion and data mining as a nonlinear aggregation tool(see [1,6,10]). The nonadditivity of fuzzy measures can effectively describe the interaction among the contribution from each attribute toward some target(see Example 3.14 (1) and (2)).

In section 2, the elementary definitions and some notations are introduced. In section 3, we introduce Choquet integral and their properties. Based on this, we give various subsethood measures defined by Choquet integral with respect to a fuzzy measure on fuzzy sets, and discuss some properties of them. Furthermore, we introduce simple examples which compare Choquet subsethood measures with Fan's subsethood measures.

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2. Definitions and Notations

The way to study the definition of subsethood measure is roughly divided into two as follows:

(i) *Subsethood measure must be two-valued for crisp sets*: Subsethood measure must be consistent with the contained relation between crisp sets. That is $c(A,B) \in \{0,1\}$ whenever A and B are crisp sets, c is a subsethood measure.

(ii) *Subsethood measure is the degree of membership in power set* : In ordinary crisp set theory, a crisp set A is called a subset of another crisp set B, if A belongs to the power set of B. Thus, the fuzzification is immediate: the degree to which A is a subset of B is the degree of membership of A in the power set $\wp(B)$ of B. We can formalize all this as follows: $c(A,B) = m_{\wp(B)}(A)$.

Throughout this paper, we write X to denote the universal set,

$$F(X) = \{A|A : X \to [0,1] \text{ is a function}\}$$

stands for the set of fuzzy sets in X, $\wp(X)$ stands for the set of crisp subsets in X. m_A expresses the membership function of a fuzzy set A, A^c is the complement of A, that is, $m_{A^c}(x) = 1 - m_A(x)$ for all $x \in X$. We write $A \subset B$ to mean that $m_A(x) \leq m_B(x)$ for all $x \in X$. For fuzzy set A, define $[A] = \{x \in X | m_A(x) > 0\}$, $n(A)$ is the cardinal number of crisp set $[A]$ (we note that n is called a cardinality measure), and $M(A)$is the fuzzy cardinal number of a fuzzy set A in a finite universal set X, that is, $M(A) = \sum_{x \in X} m_A(x)$ (we note that M is called a fuzzy cardinality measure).

We note that subsethood measure has been applied to image processing[9], neural network architecture[5], feature selection[4], and defuzzification [7]. Now, we introduce various subsethood measures in Fan [3].

Definition 2.1 A real function $c : F(X) \times F(X) \longrightarrow [0, 1]$ is called a strong subsethood measure if c has the following properties :

(C_1^s) if $A \subset B$, then $c(A, B) = 1$;

(C_2^s) if $A \neq \emptyset$ and $A \cap B = \emptyset$, then $c(A, B) = 0$;

(C_3^s) if $A \subset B \subset C$, then $c(C, A) \leq c(B, A)$ and $c(C, A) \leq c(C, B)$.

Definition 2.2 A real function $c : F(X) \times F(X) \longrightarrow [0, 1]$ is called a subsethood measure if c has the following properties :

(C_1) if $A \subset B$, then $c(A, B) = 1$;

(C_2) $c(X, \emptyset) = 0$;

(C_3) if $A \subset B \subset C$, then $c(C, A) \leq c(B, A)$ and $c(C, A) \leq c(C, B)$.

Definition 2.3 A real function $c : F(X) \times F(X) \longrightarrow [0, 1]$ is called a weak subsethood measure if c has the following properties :

(C_1^w) $c(\emptyset, \emptyset) = 1$, $c(\emptyset, X) = 1$ and $c(X, X) = 1$;

(C_2^w) if $A \neq \emptyset$ and $A \cap B = \emptyset$, then $c(A, B) = 0$;

(C_3^w) if $A \subset B \subset C$, then $c(C, A) \leq c(B, A)$ and $c(C, A) \leq c(C, B)$.

3. Choquet Subsethood Measures

Choquet integral with respect to a classical measure was first introduced in capacity theory by Choquet[1]. Fuzzy measure and Choquet integral with respect to a fuzzy measure was then proposed by Sugeno and et. al([6]). We introduce fuzzy measure and Choquet integrals(see [6,10]).

Definition 3.1 Let (X, Ω) be a measurable space. A fuzzy measure on X is a real-valued function $\mu : \Omega \longrightarrow [0, 1]$ satisfying

(i) $\mu(\emptyset) = 0$, $\mu(X) = 1$

(ii) $\mu(A) \leq \mu(B)$, whenever $A, B \in \Omega$ and $A \subset B$.

Definition 3.2 (1) The Choquet integral of a measurable function f with respect to a fuzzy measure μ is defined by

$$C_\mu(f) = \int_0^\infty \mu_f(r)dr$$

where $\mu_f(r) = \mu(\{x \in X | f(x) > r\})$ and the integral on the right-hand side is an ordinary one.

(2) If X is a finite set, that is, $X = \{x_1, \cdots, x_n\}$, then the Choquet integral of f on X is defined by

$$C_\mu(f) = \sum_{i=1}^{n} x_{(i)}[\mu(A_{(i)}) - \mu(A_{(i+1)})]$$

where $(\cdot)$ indicates a permutation on $\{1, 2, \cdots, n\}$ such that $x_{(1)} \leq \cdots \leq x_{(n)}$. Also $A_{(i)} = \{(i), \cdots, (n)\}$ and $A_{(n+1)} = \emptyset$.

(3) A measurable functions f is said to be c-integrable if $C_\mu(f)$ exists.

We introduce the concept of comonotonic between two functions and some characterizations of Choquet integral which are used to define Choquet subsethood measure. We recall that Choquet integral has comonotonically additivity(see Theorem 3.5(iii)).

Definition 3.3 Let $f, g : X \to I$ be measurable functions. We say that f and g are comonotonic, the symbol $f \sim g$ if and only if

$$f(x) < f(x') \Longrightarrow g(x) \leq g(x') \quad \text{for all } x, x' \in X.$$

Theorem 3.4 Let $f, g, h : X \longrightarrow I$ be measurable functions. Then we have the followings:

(i) $f \sim g$,

(ii) $f \sim g \Longrightarrow g \sim f$,

(iii) $f \sim a$ for all $a \in I$,

(iv) $f \sim g$ and $f \sim h \Longrightarrow f \sim (g+h)$.

Theorem 3.5 Let $f, g : X \longrightarrow I$ be measurable functions.

(i) If $f \leq g$, then $C_\mu(f) \leq C_\mu(g)$.

(ii) If $f \sim g$ and $a, b \in I$, then

$$C_\mu(af + bg) = aC_\mu(f) + bC_\mu(g).$$

(iii) If we define $(f \vee g)(x) = f(x) \vee g(x)$ for all $x \in X$, then

$$C_\mu(f \vee g) \geq \mu(f) \vee C_\mu(g).$$

(iv) If we define $(f \wedge g)(x) = f(x) \wedge g(x)$ for all $x \in X$, then

$$C_\mu(f \wedge g) \leq C_\mu(f) \wedge C_\mu(g).$$

We give the following formulas to calculate various subsethood measures on fuzzy sets.

Remark 3.6 If we let

$$F^*(X) = \{A \in F(X) | A \text{ has } c\text{-integrable membership function } m_A\}$$

and if X is a finite set, then m_A is measurable and hence $F(X) = F^*(X)$.

Theorem 3.7 Let X be an universal set. If we define a function as the followings: for $A, B \in F^*(X)$,

$$c_1(A, B) = \begin{cases} 1, & A = B = \emptyset \\ \frac{C_\mu(m_B)}{C_\mu(m_A \vee m_B)}, & \text{elsewhere} \end{cases}$$

then c_1 is a subsethood measure on $F^*(X)$, we note that it is called a Choquet subsethood measure.

. **Proof**. (C_1) If $A = B = \emptyset$, then $c_1(A, B) = 1$. And if $A \subset B$, then $m_A \leq m_B$. Thus,

$$c_1(A, B) = \frac{C_\mu(m_B)}{C_\mu(m_A \vee m_B)} = \frac{C_\mu(m_B)}{C_\mu(m_B)} = 1.$$

(C_2)

$$c_1(X, \emptyset) = \frac{C_\mu(m_\emptyset)}{C_\mu(m_X \vee m_\emptyset)} = \frac{C_\mu(0)}{C_\mu(1)} = \frac{1}{\mu(X)} = 1.$$

(C_3) If $A \subset B \subset C$, then $m_A \leq m_B \leq m_C$. Hence, by Theorem 3.5 (iii), we have

$$c_1(C, A) = \frac{C_\mu(m_A)}{C_\mu(m_C \vee m_A)} \leq \frac{C_\mu(m_A)}{C_\mu(m_B \vee m_A)} = c_1(B, A)$$

and

$$c_1(C, A) = \frac{C_\mu(m_A)}{C_\mu(m_C \vee m_A)} = \frac{C_\mu(m_A)}{C_\mu(m_C \vee m_B)} = c_1(C, B).$$

Therefore c_1 is a subsethood measure.

Theorem 3.8 Let X be an universal set. If we define a function as followings: for $A, B \in F^*(X)$,

$$c_2(A, B) = \begin{cases} 1, & A = \emptyset \\ \frac{C_\mu(m_A \wedge m_B)}{C_\mu(m_A)}, & A \neq \emptyset \end{cases}$$

then c_2 is a strong subsethood measure on $F^*(X)$, we note that it is called a Choquet strong subsethood measure.

Proof. (C_1^s) Let $A \subset B$. If $A = \emptyset$,then $c_2(\emptyset, B) = 1$. If $A \neq \emptyset$, then $m_A \leq m_B$ and hence $C_\mu(m_A \wedge m_B) = C_\mu(m_A)$. Thus

$$c_2(A, B) = \frac{C_\mu(m_A)}{C_\mu(m_A)} = 1.$$

(C_2^s) If $A \neq \emptyset$ and $A \cap B = \emptyset$, then

$$0 = m_{A \cap B} = m_A \wedge m_B$$

and hence $C_\mu(m_A \wedge m_B) = 0$. Thus

$$c_2(A, B) = \frac{C_\mu(m_A \wedge m_B)}{C_\mu(m_A)} = 0.$$

(C_3^s) If $A \subset B \subset C$, then $m_A \leq m_B \leq m_C$. Thus, we obtain

$$\begin{aligned} c_2(C, A) &= \begin{cases} 1, & C = \emptyset \\ \frac{C_\mu(m_C \wedge m_A)}{C_\mu(m_A)}, & C \neq \emptyset \end{cases} \\ &\leq \begin{cases} 1, & B = \emptyset \\ \frac{C_\mu(m_B \wedge m_A)}{C_\mu(m_A)}, & B \neq \emptyset \end{cases} \\ &= c_2(B, A). \end{aligned}$$

$$c_2(C, A) = \begin{cases} 1, & C = \emptyset \\ \frac{C_\mu(m_C \wedge m_A)}{C_\mu(m_A)}, & C \neq \emptyset \end{cases}$$

$$\leq \begin{cases} 1, & A = \emptyset \\ \frac{C_\mu(m_B \wedge m_A)}{C_\mu(m_C)}, & A \neq \emptyset \end{cases} = c_2(B, A).$$

Therefore c_2 is a strong subsethood measure.

Theorem 3.9 Let X be an universal set. If we define a function as followings: for $A, B \in F^*(X)$,

$$c_3(A, B) = \frac{C_\mu(m_{A^c}) \vee C_\mu(m_B)}{C_\mu(m_A \vee m_{A^c} \vee m_B \vee m_{B^c})},$$

then the real function c_3 is a weak subsethood measure, we note that it is called a Choquet weak subsethood measure.

Proof. (C_1^w)

$$\begin{aligned} c_3(\emptyset,\emptyset) &= \frac{C_\mu(m_{\emptyset^c}) \vee C_\mu(m_{\emptyset^c})}{C_\mu(m_\emptyset \vee m_{\emptyset^c} \vee m_\emptyset \vee m_{\emptyset^c})} \\ &= \frac{C_\mu(m_X) \vee C_\mu(m_X)}{C_\mu(m_X)} \\ &= 1. \end{aligned}$$

Similarly, we obtain $c_3(\emptyset, X) = 1$ and $c_3(X, X) = 1$.

(C_2^w)

$$c_3(X,\emptyset) = \frac{C_\mu(m_{X^c}) \vee C_\mu(m_\emptyset)}{C_\mu(m_X \vee m_{X^c} \vee m_\emptyset \vee m_{\emptyset^c})} = \frac{0}{1} = 0.$$

(C_3^w) If $A \subset B \subset C$, then $m_A \leq m_B \leq m_C$. We note that $m_{C^c} \leq m_{B^c}$ and hence

$$C_\mu(m_C \vee m_{C^c} \vee m_A \vee m_{A^c}) \geq C_\mu(m_B \vee m_{B^c} \vee m_A \vee m_{A^c}).$$

Thus, we obtain

$$\begin{aligned} c_3(C,A) &= \frac{C_\mu(m_{C^c}) \vee C_\mu(m_A)}{C_\mu(m_C \vee m_{C^c} \vee m_A \vee m_{A^c})} \\ &\leq \frac{C_\mu(m_{B^c}) \vee C_\mu(m_A)}{C_\mu(m_B \vee m_{B^c} \vee m_A \vee m_{A^c})} \\ &= c_3(B,A). \end{aligned}$$

Similarly, $c_3(C,A) \leq c_3(C,B)$. Therefore c_3 is a weak subsethood measure.

Theorem 3.10 If μ is a cardinality measure n, then C_μ is a fuzzy cardinality measure on $F^*(X)$, that is, $C_\mu(m_A) = M(A)$ for all $A \in F^*(X)$.

Proof. Let $A \in F^*(X)$ and m_A be the c-integrable membership function of A. Then by Definition 3.2(2), we have

$$\begin{aligned} C_\mu(m_A) &= \textstyle\sum_{i=1}^n m_A(x_{(i)})\{\mu(A_{(i)}) - \mu(A_{(i+1)})\} \\ &= \textstyle\sum_{i=1}^n m_A(x_{(i)}) \\ &= \textstyle\sum_{i=1}^n m_A(x_i) = M(A). \end{aligned}$$

Therefore C_μ is a fuzzy cardinality measure on $F^*(X)$.

Theorem 3.11 If μ is a cardinality measure on a finite set X, then we have the followings: for all $A, B \in F(X)$,

$$c_1(A,B) = c_L(A,B), c_2(A,B) = c_K(A,B), \text{ and } c_3(A,B) = c_F(A,B).$$

Proof. Theorem 3.10 implies

$$C_\mu(m_A) = M(A), C_\mu(m_{A^c}) = M(A^c) \text{ and } C_\mu(m_B) = M(B),$$

$$C_\mu(m_A \vee m_B) = M(A \cup B),$$

$$C_\mu(m_A \wedge m_B) = M(A \cap B),$$

$$C_\mu(m_A \vee m_{A^c} \vee m_B \vee m_{B^c}) = M(A \cup A^c \cup B \cup B^c).$$

Thus, clearly, we obtain

$$\begin{aligned} c_1(A,B) &= \frac{C_\mu(m_B)}{C_\mu(m_A \vee m_B)} \\ &= \frac{M(B)}{M(A \cup B)} = c_L(A,B); \end{aligned}$$

$$\begin{aligned} c_2(A,B) &= \begin{cases} 1, & A = \emptyset \\ \frac{C_\mu(m_A \wedge m_B)}{C_\mu(m_A)}, & A \neq \emptyset \end{cases} \\ &= \begin{cases} 1, & A = \emptyset \\ \frac{M(A \cap B)}{M(A)}, & A \neq \emptyset \end{cases} = c_K(A,B); \end{aligned}$$

$$\begin{aligned} c_3(A,B) &= \frac{C_\mu(m_{A^c}) \vee C_\mu(m_B)}{C_\mu(m_A \vee m_{A^c} \vee m_B \vee m_{B^c})} \\ &= \frac{M(A^c) \vee M(B)}{M(A \cup A^c \cup B \cup B^c)} = c_F(A,B). \end{aligned}$$

For a finite set X, we give a simple example of C_μ which is not a fuzzy cardinality measure on $F(X)$.

Example 3.12 Let n be a cardinality measure, X be a finite set and $\mu = n^2$. It is easily to see that μ is a fuzzy measure, but C_μ is not a fuzzy cardinality measure on $F(X)$.

It is easily to know that Choquet strong subsethood measure $\Rightarrow$ Choquet subsethood measure $\Rightarrow$ Choquet weak subsethood measure. But in order to prove that the converses do not hold, we give the following examples.

Example 3.13 (1) Let $X = \{x_1, x_2\}$ and $\mu = n$. If $A = \{(x_1, 0.2), (x_2, 0.6)\}$ and $B = \{(x_1, 0.3), (x_2, 0.8)\}$, then $c_1(A,B) = \frac{1.2 \vee 1.1}{1.6} = 0.75 \neq 1$. Thus c_3 is not Choquet subsethood measure.

(2) Let $X = \{x_1, x_2, x_3\}$ and $\mu = n$. If $A = \{(x_1, 0), (x_2, 0.5), (x_3, 0.6)\}$ and $B = \{(x_1, 0.5), (x_2, 0), (x_3, 0)\}$, then clearly we have $A \neq \emptyset$ and $A \cap B = \emptyset$. But we can obtain

$$c_1(A,B) = \frac{0 \times (3-2) + 0 \times (2-1) + 0.5 \times 1}{0.5 \times (3-2) + 0.5 \times (2-1) + 0.6 \times 1} = \frac{5}{16} \neq 0.$$

Thus, c_1 is not Choquet strong subsethood measure.

In order to compare various Choquet subsethood measures and Fan's subsethood measures, we give the following examples.

Example 3.14 (1) Let $X = \{x_1, x_2\}$ and $\mu = n^2$. If $A = \{(x_1, 0.2), (x_2, 0.6)\}$ and $B = \{(x_1, 0.1), (x_2, 0.9)\}$, then by Theorem 3.2(2), we can have the followings:

$$\begin{aligned} c_1(A,B) &= \tfrac{0.1\times(2^2-1)+0.9\times 1}{0.2\times(2^2-1)+0.9\times 1} = \tfrac{4}{5} \\ &< \tfrac{10}{11} = \tfrac{0.1+0.9}{0.2+0.9} = c_L(A,B); \end{aligned}$$

$$\begin{aligned} c_2(A,B) &= \tfrac{0.1\times(2^2-1)+0.6\times 1}{0.2\times(2^2-1)+0.6\times 1} = \tfrac{3}{4} \\ &< \tfrac{7}{8} = \tfrac{0.1+0.6}{0.2+0.6} = c_K(A,B); \end{aligned}$$

$$\begin{aligned} c_3(A,B) &= \tfrac{[0.4\times(2^2-1)+0.8\times 1]\vee[0.1\times(2^2-1)+0.9\times 1]}{0.9\times(2^2-1)+0.9\times 1)} = \tfrac{1}{18} \\ &< \tfrac{2}{3} = \tfrac{[0.8+0.4]\vee[0.1+0.9]}{0.9+0.9} = c_F(A,B). \end{aligned}$$

(2) Let $X = \{x_1, x_2\}$ and $\mu = n^2$. If $A = \{(x_1, 0.2), (x_2, 0.6)\}$ and $B = \{(x_1, 0.1), (x_2, 0.9)\}$ and $\sqrt{2} \approx 1.4$, then by Theorem 3.2(2), we can have the followings:

$$\begin{aligned} c_1(A,B) &= \tfrac{0.1\times(\sqrt{2}-1)+0.9\times 1}{0.2\times(\sqrt{2}-1)+0.9\times 1} \approx \tfrac{48}{49} \\ &> \tfrac{10}{11} = c_L(A,B); \end{aligned}$$

$$\begin{aligned} c_2(A,B) &= \tfrac{0.1\times(\sqrt{2}-1)+0.6\times 1}{0.2\times(\sqrt{2}-1)+0.6\times 1} \approx \tfrac{16}{17} \\ &> \tfrac{7}{8} = c_K(A,B); \end{aligned}$$

$$\begin{aligned} c_3(A,B) &= \tfrac{[0.4\times(\sqrt{2}-1)+0.8\times 1]\vee[0.1\times(\sqrt{2}-1)+0.9\times 1]}{0.9\times(\sqrt{2}-1)+0.9\times 1} \approx \tfrac{16}{21} \\ &> \tfrac{2}{3} = \tfrac{[0.8+0.4]\vee[0.1+0.9]}{0.9+0.9} = c_F(A,B). \end{aligned}$$

Remark 3.15 Example 3.14 (1) means that the degree of Choquet subsethood measure is less than the degree of Fan's subsethood measure, but Example 3.14 (2) means that the degree of Choquet subsethood measure is larger than the degree of Fan's subsethood measure, because of fuzzy measure's effect. These are very interesting facts for Choquet subsethood measures on fuzzy sets.

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