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Pascal Triangle and Properties of Bipartite Steinhaus Graphs

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Abstract. In this paper, we investigate the number of ones in rows of Pascal's Rectangle. Using these results, we determine the existence of regular bipartite Steinhaus graphs. Also, we give an upper bound for the minimum degree of bipartite Steinhaus graphs.

1. Introduction

Let $T = a_{11}a_{12}\cdots a_{1n}$ be an n-long string of zeros and ones with $a_{11} = 0$. The *Steinhaus graph* G, generated by T has as its adjacency matrix, the *Steinhaus matrix*, $A(G) = [a_{ij}]$ which is obtained from the following, called the *Steinhaus property*: $a_{i,j} \equiv a_{i-1,j-1} + a_{i-1,j} \pmod 2$ if $1 < i < j \leq n$. In this case, T is call the *generating string* of G. A *Steinhaus triangle* is the upper-triangular part of a Steinhaus matrix (excluding the diagonal) and hence, is generated by the first row (which is the generating string) in the triangle. It is obvious that there are exactly 2^{n-1} Steinhaus graphs of order n. The vertices of a Steinhaus graph are usually

$$\begin{array}{c c}
 & \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}
\end{array}$$

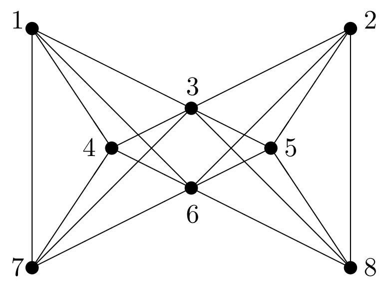

Figure 1 Steinhaus graph with the generating string 00110110

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labelled by their corresponding row numbers. In Figure 1, the Steinhaus graph generated by 00110110 is pictured.

Steinhaus in [9] asked if there were Steinhaus triangles containing the same number of zeros and ones and Harborth [7] answered this affirmatively by showing that for each n, $n \equiv 0, 1 \pmod 4$, there are at least four strings of length $n-1$ that generate such triangles. In particular, bipartite Steinhaus graphs were studied in [3], [4] and [6]. Also, conditions and conjectures on the existence for regular Steinhaus graphs were given in [2].

We now present some facts concerning Pascal's rectangle modulo two (see Figure 2). The rows of the rectangle are labelled $R_1^*, R_2^*, \cdots$, and so the kth element of R_n^* is 0 if $k > n$ and is $\binom{n-1}{k-1} \pmod 2$ if $1 \le k \le n$. We denote by $R_{n,k}$ the string formed by the first k elements of R_n^* and we set $R_n = R_{n,n}$. If T is a string of zeros and ones, then T^k is the string T concatenated with itself $k-1$ times. For example, if $T = 01$, then $T^4 = 01010101$.

$R_{1,8} \rightarrow$	1	0	0	0	0	0	0	0
$R_{2,8} \rightarrow$	1	1	0	0	0	0	0	0
$R_{3,8} \rightarrow$	1	0	1	0	0	0	0	0
$R_{4,8} \rightarrow$	1	1	1	1	0	0	0	0
$R_{5,8} \rightarrow$	1	0	0	0	1	0	0	0
$R_{6,8} \rightarrow$	1	1	0	0	1	1	0	0
$R_{7,8} \rightarrow$	1	0	1	0	1	0	1	0
$R_{8,8} \rightarrow$	1	1	1	1	1	1	1	1

Figure 2 Pascal's rectangle of length 8

In this paper, $\lfloor x \rfloor$ is the floor of x and $\lceil x \rceil$ is the ceiling of x. We denote $log_2(x)$ by $lg(x)$. Also, if k is a positive integer, then let $K = 2^{\lceil lg(k) \rceil}$ and $T = R_{K-k+1,K}$. In [4], the generating strings for bipartite Steinhaus graphs were described as follows.

Theorem 1.1([4]). *A Steinhaus graph is bipartite if and only if its generating string is a prefix of either* $0^k T^{i2^m} 0^{K2^m}$ *or* $0^k T^{2^j} 0^m$ *for each positive integer* k, *odd positive integer* i *larger than* 1, *non-negative integers* j, m.

In [6], the tight bound for number of bipartite Steinhaus graphs was described as follow.

Theorem 1.2([6]). *Let* $b(n)$ *be the number of bipartite Steinhaus graph with* n

vertices. For $n \geq 4$

$$\lceil \frac{1}{8}(17n-22) \rceil \leq b(n) \leq \lfloor \frac{1}{2}(5n-7) \rfloor .$$

2. Pascal's triangle and some results of bipartite Steinhaus graphs

First, among rows of the same length in Pascal's rectangle, we want to determine the numbers of ones in two consecutive rows. For a positive integer k and $0 \leq r \leq 3$, let the string of R_{4k+r} be denoted by $T_{k,1,r} \cdots T_{k,i,r} \cdots T_{k,k,r} T_r$, where $T_{k,i,r}$ is a string of length 4 consisting of $(4i-3)^{th}, \cdots, (4i)^{th}$ digits and T_r the last r digits in R_{4k+r} respectively. For example, $R_4 = T_{1,1,0}$ is 1^4 i.e. 1111. Also, $R_9 = T_{2,1,1}T_{2,2,1}T_1$ is $1^4 1^4$. Hereafter, we denote $T_{k,i,0}$ to $T_{k,i}$. By induction on k, we get to the following two facts.

If both $T_{k-1,i-1}$ and $T_{k-1,i}$ are either 1^4 or 0^4, then $T_{k,i}$ is 0^4.

If either $T_{k-1,i-1}$ or $T_{k-1,i}$ is 1^4 and the other is 0^4, then $T_{k,i}$ is 1^4.

This gives a recurrence relation for $T_{k,i}$ similar to the binomial coefficient recurrence, $\binom{k-1}{i-1} + \binom{k-1}{i} = \binom{k}{i}$. Note that $T_{k,i}$ is 0^4 if $\binom{k-1}{i-1}$ is even, is 1^4 if $\binom{k-1}{i-1}$ is odd. If we regard $T_{k,i}$ $-$ 1^4 or 0^4 as either 1 or 0 respectively, then $T_{k,i}{}'s$ satisfy the binomial coefficient recurrence. It is straightforward to show that $T_{k,i,1}$ is either 0000 or 1000, $T_{k,i,2}$ is either 0000 or 1100, and that $T_{k,i,3}$ is either 0000 or 1010. Next, we compute the numbers of ones in some consecutive rows in Pascal's Rectangle. First, we start with Lucas's Theorem.

Theorem 2.1([8]). *Let p be prime and let $n = \sum a_i p^i$ and $m = \sum b_i p^i$ be the p-ary expansions of positive integers n and m. Then*

$$\binom{n}{m} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1} \cdots \pmod{p}.$$

Lemma 2.2. *For $0, k < 2^m$, $R_{k+2^m} = R_k 0^{2^m-k} R_k$.*

Proof. When $p = 2$, we get to the following by Theorem 2.1. $\binom{n}{m}$ is odd, i.e. $\binom{n}{m} \equiv 1 \pmod 2$ if and only if whenever m has a 1 as its $i-$th binary digit, then so does n. Since for $k < j \leq 2^m$, j does not have 2^m in binary expansion, the fact above gives $\binom{k+2^m}{j-1} \equiv 0 \pmod 2$. For $1 \leq j \leq k$ and $k+1 \leq j \leq k+2^m$, use above fact again. □

Let n be a positive integer such that the number of ones in R_n is equal to the number of ones in R_{n+1}. Note that neither n nor $n+1$ can be a power of 2. Let $n = k + 2^m$ where $k < 2^m$. By Lemma 2.2, the number of ones in R_n is two times of the number of ones in R_k. So, the number of ones in R_k is equal to the number of ones in R_{k+1}. By continuing this process, we get $n = 4k+2$, for some non-negative integer k. The next Theorem will show that the converse is also true.

Theorem 2.3. *Let n be a positive integer. Then the number of ones in R_n is equal to the number of ones in R_{n+1} if and only if $n = 4k+2$, for some non-negative integer k.*

Proof. Let $n = 4k + r$ for some k and r, where $0 \le r \le 3$.

Case 1 $r = 1$.
Note that $T_{k,i,1}$ is either 1000 or 0000 for $1 \le i \le k$ and $T_1 = 1$. If $T_{k,i,1}$ is 1000, then $T_{k,i,2}$ is 1100 for $1 \le i \le k$. If $T_{k,i,1}$ is 0000, then $T_{k,i,2}$ is 0000 for $2 \le i \le k$. Moreover, $T_2 = 11$. So, the number of ones in R_{4k+1} is exactly half of the number of ones in R_{4k+2}.

Case 2 $r = 2$.
Note that $T_{k,i,2}$ is either 1100 or 0000 for $1 \le i \le k$. If $T_{k,i,2}$ is 1100, then $T_{k,i,3}$ is 1010 for $1 \le i \le k$. If $T_{k,i,2}$ is 0000, then $T_{k,i,3}$ is 0000 for $2 \le i \le k$. Moreover, $T_3 = 101$. So, the number of ones in R_{4k+2} is equal to the number of ones in R_{4k+3}.

Case 3 $r = 3$.
Note that $T_{k,i,3}$ is either 1010 or 0000 for $2 \le i \le k$. If $T_{k,i,3}$ is 1010 then $T_{k,i,4}$ is 1111 for $1 \le i \le k$. If $T_{k,i,3}$ is 0000 then $T_{k,i,4}$ is 0000 for $2 \le i \le k$. Moreover, $T_4 = 1111$. So, the number of ones in R_{4k+3} is not equal to (in fact, half of) the number of ones in R_{4k+4}.

Case 4 $r = 0$.
This case is clear.

Hence by combining all cases, the proof is completed. □

We divide all generating strings of bipartite Steinhaus graphs into two types, $0^kT^j0^m$ for $m > 0$ and 0^kT^j. First, let $a_{11}a_{12}\cdots a_{1n}$ be a generating string of a bipartite steinhaus graph. Let $a_{11}a_{12}\cdots a_{1n}$ be $0^kT^j0^m$ for $m > 0$. Since $T = R_{K-k+1}$, the degrees of vertices $1, 2$, and 3 are not equal by Theorem 2.3. If $a_{11}a_{12}\cdots a_{1n}$ is a prefix of 0^kT^j, then the degree of vertex $k+1$ is k and the degree of vertex $k+2$ is at most $\lceil \frac{k}{2} \rceil$. So the degrees of three vertices are not equal. Hence, we get to the following result:

Theorem 2.4. *There are no nontrivial regular bipartite Steinhaus graphs with at least three vertices.*

Finally, we find an upper bound for minimum degrees of bipartite Steinhaus graphs. Let G be a nonempty bipartite Steinhaus graph with $n \ge 8$ vertices. Observe that the first k-long string $a_{1,k+1}a_{2,k+1}\cdots a_{k,k+1}$ in $(k+1)^{th}$ column of $A(G)$ are all ones. So, the first k-long string $a_{1,k+2}a_{2,k+2}\cdots a_{k,k+2}$ in $(k+2)^{th}$ column of $A(G)$ are alternatively zero and one.

Lemma 2.5. *If* $a_{11}a_{12}\cdots a_{1n}$ *is* $0^kT^j0^m$ *for some* $m>0$*, then the minimum degree* $\delta(G)$ *is at most* $\frac{n}{4}$*.*

Proof. Note that the (n-k)-long string $a_{k,k+1}a_{k,k+2}\cdots a_{k,n}$ in the k^{th} row of $A(G)$ is $1^{K^j}0^m$. So degree of vertex $k+2$ is given by

$$\sum_{i=1,\cdots,k+1} a_{i,k+2} + \sum_{i=k+3,\dots,n} a_{k+2,i},$$

which is at most $\frac{k+4}{2}$. But when $j=1$, the vertex $k-(\frac{K}{2}+1)$ is of degree 2. If $j\geq 2$, it is not difficult to deduce that $n\geq 2k+8$ from $n\geq k+K+m$ for $k\geq 5$. When $1\leq k\leq 4$, it is straightforward. This gives that the minimum degree $\delta(G)$ of G is at most $\frac{n}{4}$. □

Theorem 2.6. *For any bipartite Steinhaus graph G with at least* 3 *vertices,* $\delta(G)$ *is less than* $\lfloor\frac{n}{4}\rfloor$*.*

Proof. Assume that $a_{11}a_{12}\cdots a_{1n}$ is a prefix of 0^kT^j. If $a_{11}a_{12}\cdots a_{1n}$ contains 0^kT, the degree of vertex $k+K$ is one because all entries below k^{th} row in upper triangle of $A(G)$ are zeros and the string $T=R_{K-k+1,K}$ generates the k-long string $a_{1,k+K}a_{2,k+K}\cdots a_{k,k+K}$ which is $0^{k-1}1$. If $a_{11}a_{12}\cdots a_{1n}$ is a proper prefix of 0^kT, the vertex $k-(\frac{K}{2}+1)$ is of degree 2. By combining Lemma 2.5, the proof is completed. □

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