KYUNGPOOK Math. J. 48(2008), 317-321

Normal Families and Shared Values of Meromorphic Functions

Chao Meng
Department of Mathematics, Shandong University, Jinan 250100, Shandong, China
e-mail : mengchao@mail.sdu.edu.cn

Abstract. Some criteria for determining the normality of the family F of meromorphic functions in the unit disc, which share values depending on $f \in F$ with their derivatives is obtained. The new results in this paper improve some earlier related results given by Pang and Zalcman [3], Fang and Zalcman [2], A. P. Singh and A. Singh [5].

1. Introduction, definitions and main results

Let f and g be meromorphic functions on a domain D in C, and let a and b be complex numbers. If $g(z) = b$ whenever $f(z) = a$, we write

$$f(z) = a \Rightarrow g(z) = b.$$

In a different notation, we have $\overline{E}_f(a) \subset \overline{E}_g(b)$, where $\overline{E}_h(c) = h^{-1}(c) \cap D = \{z \in D : h(z) = c\}$. If $f(z) = a \Rightarrow g(z) = b$ and $g(z) = b \Rightarrow f(z) = a$, we write

$$f(z) = a \Leftrightarrow g(z) = b.$$

If $f(z) = a \Leftrightarrow g(z) = a$, we say that f and g share a on D.

Schwick is probably the first to find a connection between the normality criterion and shared values of meromorphic functions. He proved the following theorem.

Theorem A([4]). *Let F be a family of meromorphic functions in the unit disc Δ, and let a_1, a_2, a_3 be distinct complex numbers. If f and f' share a_1, a_2 and a_3 for every $f \in F$, then F is normal in Δ.*

Pang and Zalcman extended the above result as follows.

Theorem B([3]). *Let F be a family of meromorphic functions in the unit disc Δ and let a and b be distinct complex numbers and c be a nonzero complex number. If for every $f \in F$, $f(z) = 0 \Leftrightarrow f'(z) = a$ and $f(z) = c \Leftrightarrow f'(z) = b$, then F is normal in Δ.*

Received December 18, 2007.
2000 Mathematics Subject Classification: 30D45, 30D35.
Key words and phrases: meromorphic function, shared value, normal family.

In 2001, Fang and Zalcman proved the following result.

Theorem C([2]). *Let F be a family of meromorphic functions in the unit disc Δ and let b, c and d be nonzero complex numbers such that $d \neq b$. Suppose that for each $f \in F$, $f(z) = 0 \Rightarrow f'(z) = b$ and $f'(z) = d \Rightarrow f(z) = c$. Then F is normal in Δ so long as $b \neq (m+1)d$, $m = 1, 2, 3, \cdots$.*

In Theorem B and Theorem C the constants are the same for each $f \in F$. In 2004, A. P. Singh and A. Singh proved that the condition for the constants to be the same can be relaxed to some extent. More precisely, they proved the following theorem.

Theorem D([5]). *Let F be a family of meromorphic functions in the unit disc Δ. For each $f \in F$ let a_f, b_f, c_f be distinct nonzero complex numbers such that $(a_f b_f / c_f^2) = M$ for some constant M. Let the spherical distance σ between the points a_f, b_f, c_f satisfy*

$$\min\{\sigma(a_f, b_f), \sigma(b_f, c_f), \sigma(c_f, a_f)\} \geq m$$

for some $m > 0$. Let $f(z) = 0 \Leftrightarrow f'(z) = a_f$ and $f(z) = c_f \Leftrightarrow f'(z) = b_f$. Let $M = (ab/c^2)$, where a, b, c are distinct. If the elements of $\overline{E}_f(c_f)$ and $\overline{E}_f(0)$ are the only solutions of

$$f'(z) = \frac{a_f b}{a}(1 - (\frac{1}{c_f} - \frac{a}{c a_f}) f(z))^2$$

and

$$f'(z) = a_f (1 - (\frac{1}{c_f} - \frac{a}{c a_f}) f(z))^2$$

respectively, then F is normal in Δ.

Now the following problem is considered: Is it possible to relax the nature of sharing values in Theorem D ? In this paper, we prove the following theorem which answers the above question.

Theorem 1. *Let F be a family of meromorphic functions in the unit disc Δ. For each $f \in F$ let a_f, b_f, c_f be distinct nonzero complex numbers such that $(a_f b_f / c_f^2) = M$ for some constant M. Let the spherical distance σ between the points a_f, b_f, c_f satisfy*

$$\min\{\sigma(a_f, b_f), \sigma(b_f, c_f), \sigma(c_f, a_f)\} \geq m$$

for some $m > 0$. Let $f(z) = 0 \Rightarrow f'(z) = a_f$ and $f'(z) = b_f \Rightarrow f(z) = c_f$. Let $M = (ab/c^2)$, where a, b, c are distinct and $a \neq (m+1)b$, $m = 1, 2, 3, \cdots$. If the elements of $\overline{E}_f(c_f)$ are the only solutions of

$$f'(z) = \frac{a_f b}{a}(1 - (\frac{1}{c_f} - \frac{a}{c a_f}) f(z))^2,$$

then F is normal in Δ.

Remark. Theorem 1 removes the restriction on $\overline{E}_f(0)$ of Theorem D and the nature of sharing values is relaxed.

If the meromorphic functions in F and their derivatives share a_f bounded by some constant M, and αa_f respectively, then we shall prove the following theorem.

Theorem 2. *Let F be a family of meromorphic functions on the unit disc Δ. For each $f \in F$ let there exist a_f ($0 < |a_f| \leq M$ for some constant M) and αa_f ($\alpha \neq 1, 2, 3, \cdots$ is a constant) and $f(z) = a_f \Rightarrow f'(z) = a_f$, $f'(z) = \alpha a_f \Rightarrow f(z) = \alpha a_f$. Further let the elements of $\overline{E}_f(\alpha a_f)$ be the only solutions of*

$$f'(z) = \frac{a_f b}{a}(1 - A_f(f(z) - a_f))^2,$$

where $A_f = ((1/(\alpha - 1)a_f) - (a/ca_f))$, a is any nonzero constant, $b = \alpha a$ and $c = (\alpha - 1)a$. Then F is normal in Δ.

2. Some lemmas

We need the following lemmas in the proof of Theorem 1 and Theorem 2.

Lemma 1([1]). *The Mobius map $g(z) = \dfrac{az+b}{cz+d}$, $ad - bc = 1$ satisfies the Lipschitz condition*

$$\sigma(g(z), g(w)) \leq \frac{\pi}{2}\|g\|^2 \sigma(z, w),$$

where $\|g\|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2$.

The proof is immediate since from [5]

$$\sigma_0(g(z), g(w)) \leq \|g\|^2 \sigma_0(z, w),$$

where σ_0 is the spherical metric, and the spherical metric and chordal metric σ are related by

$$\frac{2}{\pi}\sigma_0(z, w) \leq \sigma(z, w) \leq \sigma_0(z, w).$$

Lemma 2([1]). *Let m be any positive number. Then the Mobius transformation g which satisfies $\sigma(g(a), g(b)) \geq m$, $\sigma(g(b), g(c)) \geq m$, $\sigma(g(c), g(a)) \geq m$ for some constant a, b and c, also satisfies the uniform Lipschitz condition*

$$\sigma(g(z), g(w)) \leq k_m \sigma(z, w),$$

where k_m is a constant depending on m.

3. Proof of Theorem 1

For each $f \in F$, define a Mobius map g_f by

$$g_f(z) = \frac{z}{Az+B},$$

where $A = \frac{1}{c_f} - \frac{a}{ca_f}$ and $B = \frac{a}{a_f}$. Then clearly we have

$$g_f^{-1}(z) = \frac{Bz}{1-Az}$$

and

$$(g_f^{-1})'(z) = \frac{B}{(1-Az)^2}$$

so that $g_f^{-1}(0) = 0$, $g_f^{-1}(c_f) = c$, $(g_f^{-1})'(0) = \frac{a}{a_f}$, $(g_f^{-1})'(c_f) = \frac{b}{b_f}$.

Now if z_0 is such that $f(z_0) = 0$ then since $f(z) = 0 \Rightarrow f'(z) = a_f$ we have $f'(z_0) = a_f$ and so $(g_f^{-1} \circ f)(z_0) = g_f^{-1}(0) = 0$ and $(g_f^{-1} \circ f)'(z_0) = (g_f^{-1})'(f(z_0))f'(z_0) = a$.

Now we show that $g_f^{-1} \circ f(z) = 0 \Rightarrow (g_f^{-1} \circ f(z))' = a$. Let z_1 is such that $(g_f^{-1} \circ f)(z_1) = 0$, then $(g_f^{-1} \circ f)(z_1) = 0 = g_f^{-1}(0)$. Also g_f^{-1} being a Mobius map, is one-to-one so that $f(z_1) = 0$ and so $f'(z_1) = a_f$. Thus $(g_f^{-1} \circ f(z_1))' = a$.

Next we show that $(g_f^{-1} \circ f(z))' = b \Rightarrow g_f^{-1} \circ f(z) = c$. Let z_2 is such that $(g_f^{-1} \circ f(z_2))' = b$. Then $(g_f^{-1})'(f(z_2))f'(z_2) = b$ and so

$$f'(z_2) = \frac{a_f b}{a}(1 - (\frac{1}{c_f} - \frac{a}{ca_f})f(z_2))^2 \,. \tag{1}$$

Since only the elements of $\overline{E}_f(c_f)$ satisfy (1), it follows that $f(z_2) = c_f$ and so $(g_f^{-1} \circ f)(z_2) = c$. Thus $(g_f^{-1} \circ f(z))' = b \Rightarrow g_f^{-1} \circ f(z) = c$.

Thus by Theorem C, the family $G = \{(g_f^{-1} \circ f) : f \in F\}$ is normal and hence equicontinuous in Δ. Therefore given $(\epsilon/k_m) > 0$, where k_m is the constant of Lemma 2, there exist $\delta > 0$ such that for the spherical distance $\sigma(x,y) < \delta$,

$$\sigma((g_f^{-1} \circ f)(x), (g_f^{-1} \circ f)(y)) < \frac{\epsilon}{k_m}$$

for each $f \in F$. Hence by Lemma 2

$$\begin{aligned} \sigma(f(x), f(y)) &= \sigma((g_f \circ g_f^{-1} \circ f)(x), (g_f \circ g_f^{-1} \circ f)(y)) \\ &\leq k_m \sigma((g_f^{-1} \circ f)(x), (g_f^{-1} \circ f)(y)) < \epsilon \,. \end{aligned} \tag{2}$$

Thus the family F is equicontinuous in Δ. This completes the proof of Theorem 1.

4. Proof of Theorem 2

Let $b_f = \alpha a_f$ and $c_f = (\alpha - 1)a_f$ so that $a_f b_f / c_f^2$ is the constant $\alpha/(\alpha-1)^2$. For each $f \in F$, define $g_f(z) = f(z) - a_f$. Let $G = \{g_f : f \in F\}$. Then clearly

$$g_f(z) = 0 \Rightarrow f(z) = a_f \Rightarrow f'(z) = a_f \Rightarrow g_f'(z) = a_f \tag{3}$$

and

$$g_f'(z) = b_f \Rightarrow f'(z) = b_f \Rightarrow f(z) = b_f \Rightarrow g_f(z) = c_f\,. \tag{4}$$

From the assumption of Theorem 2, we get that the elements of $\overline{E}_{g_f}(c_f)$ are the only solutions of

$$g_f'(z) = \frac{a_f b}{a}(1 - A_f g_f(z))^2.$$

Hence by Theorem 1, G is normal in Δ, and hence given $\epsilon > 0$, there exists $\delta > 0$ for all x, y such that $\sigma(x, y) < \delta$, then we have $\sigma(g_f(x), g_f(y)) < \epsilon$ for every $f \in F$. Now define $R_f(z) = z + a_f$, then each $R_f(z)$ is a Mobius map and $R_f(g_f(z)) = f(z)$. Hence by Lemma 1

$$\begin{aligned} \sigma(f(x), f(y)) &= \sigma(R_f(g_f(x)), R_f(g_f(y))) \\ &\le \|R_f\|^2 \frac{\pi}{2} \sigma(g_f(x), g_f(y)) \\ &\le (2 + M^2)\frac{\pi}{2}\epsilon\,. \end{aligned} \tag{5}$$

Hence F is normal in Δ. This completes the proof of Theorem 2.

References

[1] A. F. Beardon, Iteration of Rational Functions, Springer-Verlag, New York, 1991.

[2] M. L. Fang and L. Zalcman, *Normal families and shared values of meromorphic functions II*, Comput. Methods Funct. Theory, **1**(2001), 289-299.

[3] X. C. Pang and L. Zalcman, *Normality and shared values*, Arkiv Math., **38**(2000), 171-182.

[4] W. Schwick, *Sharing values and normality*, Arch Math.(Basel), **59**(1992), 50-54.

[5] A. P. Singh and A. Singh, *Sharing values and normality of meromorphic fucntions*, Complex Variables Theory Appl., **49**(2004), 417-425.