The Basis Number of the Lexicographic Product of Different Ladders with Paths and Cycles

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ABSTRACT. In [8] M. Y. Alzoubi and M. M. Jaradat studied the basis number of the composition of paths and cycles with Ladders, Circular ladders and Möbius ladders. Namely, they proved that the basis number of these graphs is 4 except possibly for some cases in each of them. Since the lexicographic product is noncommutative, in this paper we investigate the basis number of the lexicographic product of the different kinds of ladders with paths and cycles. In fact, we prove that the basis number of almost all of these graphs is 4.

1. Introduction

Throughout this paper, we consider only finite connected simple graphs. We use standard notations and terminology, and for undefined terms we refer the reader to [10] and [16].

Let G be a graph with n vertices and m edges. If we order the edges $e_1, e_2, \cdots, e_m$ then G is associated with a vector space as follows: If A is a subset of edges from G, then it corresponds to a $(0,1)$-vector $(a_1, a_2, \cdots, a_m)$ such that $a_i = 1$ if $e_i \in A$, and $a_i = 0$ if $e_i \notin A$. These vectors form an m-dimensional vector space over the field Z_2. The subspace generated by all the vectors that correspond to all the cycles of G is called *the cycle space of* G, which is denoted by $\mathcal{C}(G)$. The dimension is dim $\mathcal{C}(G) = m - n + 1$. Usually, we say that the cycles themselves, rather than the vectors corresponding to the cycles, generate $\mathcal{C}(G)$. A basis for the cycle space $\mathcal{C}(G)$ of a graph G is called a $k - fold$ if each edge of G occurs in at most k of the cycles in the basis. *The basis number of* G, denoted by $b(G)$, is the smallest integer k such that $\mathcal{C}(G)$ has a $k - fold$ basis.

The first important result about the basis number was given in 1937 by MacLane [17] when he proved the following theorem.

Theorem 1.1(MacLane,[17]). *A graph G is planar if and only if $b(G) \leq 2$.*

In [18] Schmeichel investigated the basis number of certain important classes of non-planar graphs, specifically, complete graphs and complete bipartite graphs.

Then J. Banks and E. Schemeichel [9] proved that for $n \geq 7$, the basis number of Q_n is 4, where Q_n is the *n-cube*. After that, many researchers were attracted to work on finding the basis number of special classes of graphs, mainly, those obtained from different kinds of graph products. We refer interested readers to [1-8], [11-15] and their references.

    The following lemmas will be used frequently in our main proofs.

Lemma 1.1(Hailat-Alzoubi[11])**.** *Let G be a graph with p vertices and q edges. If $|C|$ denotes the length of the cycle C, and $\mathcal{B} = \{C_1, C_2, \cdots, C_d \ : \ |C_i| \geq r\}$ be a k-fold basis of $\mathcal{C}(G)$ then $rd \leq \sum_{i=1}^{d} |C_i| \leq kq$, where $d = \dim \mathcal{C}(G)$.*

Lemma 1.2(Jaradat-Alzoubi-Rawashdeh, [13])**.** *Let A, B be sets of cycles of a graph G, and suppose that both A and B are linearly independent, and $E(A) \cap E(B)$ induces a forest in G (with the possibility that $E(A) \cap E(B) = \phi$). Then $A \cup B$ is linearly independent.*

Definition 1.1. The lexicographic product (composition) of two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is denoted by $G_1[G_2]$ or $G_1 \otimes G_2$. It has $V(G) = V_1 \times V_2$ as a vertex set and its edge set is

$$E = \{(u_1, v_1)(u_2, v_2) \ \text{either} \ [u_1 = u_2 \ \text{and} \ v_1 v_2 \in E_2] \ \text{or} \ [u_1 u_2 \in E_1]\}.$$

    It is worth mentioning that, in general, $G_1[G_2]$ and $G_2[G_1]$ are not isomorphic graphs since $|E(G_1[G_2])| = p_1 q_2 + p_2^2 q_1$ and $|E(G_2[G_1])| = p_2 q_1 + p_1^2 q_2$, where $|V(G_1)| = p_1$, $|V(G_2)| = p_2$, $|E(G_1)| = q_1$, $|E(G_2)| = q_2$.

    In [8] M. Y. Alzoubi and M. M. Jaradat studied the basis number of the composition of paths and cycles with Ladders, Circular ladders and Möbius ladders. Namely, they proved that the basis number of the graphs $P_n[L_m]$, $P_n[CL_m]$, $P_n[ML_m]$, $C_n[L_m]$, $C_n[CL_m]$ and $C_n[ML_m]$ is 4 except possibly for some cases in each of them.

    The purpose of this paper is to investigate the basis number of the graphs $L_n[P_m]$, $CL_n[P_m]$, $ML_n[P_m]$, $L_n[C_m]$, $CL_n[C_m]$ and $ML_n[C_m]$ taking into account that the lexicographic product is noncommutative. We use the notations P_n, C_n, L_n, CL_n and ML_n to denote a path, a cycle, a ladder, a circular ladder and a Möbius ladder, respectively.

2. The Main results

    Throughout this work, P_n and C_n denote the path $012 \cdots (n-1)$ and the cycle $012 \cdots (n-1)0$; respectively such that $E(P_n) = \{i(i+1) : 0 \leq i \leq n-2\}$ and we define the edge set of C_n by $E(C_n) = \{i(i+1) : 0 \leq i \leq n-1\}$. The circular ladder, CL_n, will be taken as a two concentric cycles $C_a = a_0 a_1 \cdots a_{n-1} a_0$ and $C_b = b_0 b_1 \cdots b_{n-1} b_0$ with the vertex-set as follows :

$$V(CL_n) = \{a_0, a_1, \cdots, a_{n-1}, b_0, b_1, \cdots, b_{n-1}\}$$

and the edge-set as follows :

$$E\left(CL_n\right) = E\left(C_a\right) \cup E\left(C_b\right) \cup \{a_i b_i : 0 \le i \le n-1\}.$$

The ladder L_n is obtained from CL_n by deleting the edges $a_0 a_{n-1}$, $b_0 b_{n-1}$. The Möbius ladder ML_n is obtained from CL_n by deleting the edges $a_0 a_{n-1}$, $b_0 b_{n-1}$ and replacing them by the two edges $a_0 b_{n-1}$ and $b_0 a_{n-1}$.

Lemma 2.1. *Let m, n be two positive integers, $(n \ge 3)$, such that $4(m^2(3n-2)+1) \le 3(2nm + m^2(3n-2))$, then $m \le 3$.*

Proof. Since $4(m^2(3n-2)+1) \le 3(2nm + m^2(3n-2))$, we have $4m^2(3n-2)+4 \le 6nm + 3m^2(3n-2)$, then $m^2(3n-2)+4 \le 6nm$, thus $m + \dfrac{4}{m(3n-2)} \le \dfrac{6n}{3n-2} \lneqq \dfrac{6n}{3(n-1)}$, therefore $m \lneqq 4$. $\qquad\square$

Lemma 2.2. *Let m, n be two positive integers; $(n \ge 3)$, and $s \le 3m(2n)$, such that $s + \left[\dfrac{6nm + 9nm^2 - 6m^2 - 3s}{4}\right] \ge 3nm^2 - 2m^2 + 1$, where $[x]$ is the greatest integer less than or equal to x. Then $m \lneqq 6$.*

Proof. Since $\left[\dfrac{6nm + 9nm^2 - 6m^2 - 3s}{4}\right] \le \dfrac{6nm + 9nm^2 - 6m^2 - 3s}{4}$ we have $3nm^2 - 2m^2 + 1 \le s + \dfrac{6nm + 9nm^2 - 6m^2 - 3s}{4}$. If we multiply this inequality by 4 and rearrange the terms we get $m^2(3n-2)+4 \le 12nm$. If we divide the inequality by $m(3n-2)$ we have $m + \dfrac{4}{m(3n-2)} \le 4\left(\dfrac{3n}{3n-2}\right)$. Since $\dfrac{3n}{3n-2}$ is a decreasing sequence for all $n \ge 3$ and $\dfrac{9}{7}$ is its maximum we have $m \lneqq 6$. $\qquad\square$

It is clear that $|V(L_n[C_m])| = 2nm$, $|E(L_n[C_m])| = 2nm + m^2(3n-2)$, and so $\dim \mathcal{C}(L_n[C_m]) = |E(L_n[C_m])| - |V(L_n[C_m])| + 1 = m^2(3n-2)+1$.

Theorem 2.1. *If $n \ge 3$ and $m \ge 5$, then $3 \le b(L_n[C_m]) \le 4$. Moreover, $b(L_n[C_m]t) = 4$ for all $n \ge 3$, $m \ge 6$.*

Proof. The graph $L_n[C_m] = C_{2n}[C_m] \bigcup (\bigcup\limits_{i=1}^{n-2} K_{(a_i,m),(b_i,m)})$, where

$$C_{2n} = a_0 a_1 \cdots a_{n-1} b_{n-1} b_{n-2} \cdots b_1 b_0 a_0.$$

Since $C_{2n}[C_m]$ is a nonplanar subgraph of $L_n[C_m]$, the graph $L_n[C_m]$ is a nonplanar graph and thus by Maclane's theorem $b(L_n[C_m]) \ge 3$.

To prove that $b(L_n[C_m]) \le 4$, we exhibit a $4 - fold$ basis for $\mathcal{C}(L_n[C_m])$. For each $1 \le r \le n-2$, define the following sets of cycles in $\mathcal{C}(L_n[C_m])$:

$$A_r \quad = \quad B(K_{(a_r,m)(b_r,m)}),$$

$$
\begin{aligned}
A_r^{'} \;\; &= \;\; \{(a_r,0)(b_r,i)(b_r,i+1)(a_r,0)\colon 0 \le i \le m-2\} \\
&\quad \cup \{(b_r,m-1)(a_r,i)(a_r,b_r)(b_r,m-1)\colon 0 \le i \le m-2\}, \\
Q \;\; &= \;\; \{Q_i = (a_i,0)(a_{i+1},0)(b_{i+1},0)(b_i,0)(a_i,0)\colon i \in Z_{n-2}\}.
\end{aligned}
$$

Let $B_r = A_r \bigcup A_r^{'}$. Consider $B(L_n[C_m]) = B(C_{2n}[C_m]) \bigcup \left(\bigcup\limits_{r=1}^{n-2} B_r \right) \bigcup Q$, where $B(C_{2n}[C_m])$ is the $4-fold$ basis of $\mathcal{C}(C_{2n}[C_m])$ which is obtained in Theorem 3.7 of [11], so $B(C_{2n}[C_m])$ is linearly independent set of cycles in $\mathcal{C}(L_n[C_m])$.

Let $B^* = B(C_{2n}[C_m]) \cup Q$, the cycles of Q enclose the finite faces of a planar ladder and form a basis to this ladder. Thus Q is linearly independent set of cycles in $\mathcal{C}(L_n[C_m])$. Moreover, $E(Q) \cap E(C_{2n}[C_m])$ induces a forest of paths, thus by Lemma 1.2 we conclude that B^* is linearly independent.

For each $1 \le r \le n-2$, A_r is Schemeichel's $4-fold$ basis of the subspace $\mathcal{C}(K_{(a_r,m),(b_r,m)})$ that obtained in Theorem 2.4 of Schemeichel [18], thus each A_r is linearly independent set of cycles in $\mathcal{C}(L_n[C_m])$. $A_r^{'}$ is linearly independent set of cycles because the cycles of $A_r^{'}$ form the set of all finite faces of their corresponding planar subgraph that obtained by pasting them together successively with increasing i. Moreover, each cycle in $A_r^{'}$ contains one edge from the set

$$
\begin{aligned}
H \;\; &= \;\; \{(a_i,0)(a_i,1),(a_i,1)(a_i,2),\cdots,(a_i,m-1)(a_i,m)\} \\
&\quad \cup \{(b_i,0)(b_i,1),(b_i,1)(b_i,2),\cdots,(b_i,m-1)(b_i,m)\},
\end{aligned}
$$

and this edge doesnot occur in any cycle of A_r, then the cycles of $A_r^{'}$ are linearly independent with the cycles of A_r, thus $B_r = A_r \cup A_r^{'}$ is linearly independent set of cycles in $\mathcal{C}(L_n[C_m])$. $\bigcup\limits_{r=1}^{n-2} B_r$ is linearly independent set of cycles in $\mathcal{C}(L_n[C_m])$ because $E(B_r) \cap E(B_k) = \phi$, for all $1 \le r,k \le n-2$ and $r \ne k$. Moreover, $E(B^*) \cap E\left(\bigcup\limits_{r=1}^{n-2} B_r \right)$ is a forest, thus by Lemma 1.2, $B(L_n[C_m]) = B^* \cup \left(\bigcup\limits_{r=1}^{n-2} B_r \right)$ is a linearly independent set of cycles in $\mathcal{C}(L_n[C_m])$. Since

$$
\begin{aligned}
|B(L_n[C_m])| \;\; &= \;\; |B(C_{2n}[C_m])| + \left| \bigcup\limits_{r=1}^{n-2} B_r \right| + |Q| \\
&= \;\; 2nm^2 + 1 + (m^2-1)(n-2) + n - 2 \\
&= \;\; 3nm^2 - 2m^2 + 1 = \dim \mathcal{C}(L_n[C_m]).
\end{aligned}
$$

it follows that $B(L_n[C_m])$ is a basis for $\mathcal{C}(L_n[C_m])$. It is easy to verify that $B(L_n[C_m])$ is a $4-fold$ basis for $\mathcal{C}(L_n[C_m])$. Hence, $3 \le \mathrm{b}(L_n[C_m]) \le 4$ for all $n \ge 3$ and $m \ge 5$.

On the other hand, suppose that $\mathcal{C}(L_n[C_m])$ has a $3-fold$ basis $\mathcal{B}$. Then, to prove that $\mathcal{B}$ can not exist for all $n \ge 3$ and $m \ge 6$, we have three cases:

Case 1. Suppose that $\mathcal{B}$ contains only 3-cycles. Then $|\mathcal{B}| \le 3m(2n) = 6nm$, since

every 3-cycles in $\mathcal{B}$ must contain an edge from the set

$$S = \{(a_i, r)(a_i, r+1)\colon r \in \ Z_m, \, i \ \in Z_n\} \cup \{(b_i, r)(b_i, r+1)\colon r \in \ Z_m, \, i \ \in Z_n\},$$

where $|S| = 2nm$, and the fold of every edge of S is at most 3. But $|\mathcal{B}| = m^2(3n - 2) + 1 = \dim(\mathcal{C}(L_n\,[C_m]))$, so that $m^2(3n-2) + 1 \leq 6nm$, and $m^2(3n-2) + 1 \lneqq m^2(3n-2) + 1 \leq 6nm$, since $m \geq 5$ then $5^2(3n-2) \lneqq 30n$, which implies that $75n - 30n - 50 \lneqq 0 \Leftrightarrow 45n - 50 \lneqq 0$, which doesnot hold for all $n \geq 2$. Therefore, $m^2(3n-2) + 1 \leq 6nm$ doesnot hold for all $n \geq 3$ and $m \geq 5$. Hence, $\mathcal{B}$ cannot be a basis of $\mathcal{C}(L_n\,[C_m])$, a contradiction.

Case 2. Suppose that $\mathcal{B}$ consists only of cycles of length greater than or equal to 4, then by Lemma 1.1 we have $4(m^2\,(3n-2\,)\,+1) \leq 3(2nm + m^2\,(3n\,-2))$ because $\dim \mathcal{C}(L_n\,[C_m]) = m^2(\,3n-2) + 1$, $|E(L_n\,[C_m])| = m^2(3n\,-2) + 2nm$, and $|C_i| \geq 4$ for every $C_i \in \mathcal{B}$. But, by Lemma 2.1, we have a contradiction for any $n \geq 3$ and $m \geq 6$.

Case 3. Suppose that $\mathcal{B}$ consists of s 3-cycles and t cycles of length greater than or equal to 4. Then $s \leq 3m\,(2n)$ because we have at most $3m(2n)$ $3-cycles$ in $\mathcal{B}$ as we explained in Case 1. Since $|E(L_n\,[C_m])| = m^2\,(3n\,-2)\,+2nm$, and the fold of every edge of $L_n\,[C_m]$ is at most 3 in $\mathcal{B}$ and $3s$ edges are joined to make the s 3-cycles, we have $t \leq \left\lceil \dfrac{6nm + 9nm^2 - 6m^2 - 3s}{4} \right\rceil$. Then, $m^2(3n-2) + 1 = dim(\mathcal{C}(L_n\,[C_m])) = |\mathcal{B}| = s + t \leq s + \left\lceil \dfrac{6nm + 9nm^2 - 6m^2 - 3s}{4} \right\rceil$, so that $m^2(3n-2) + 1 \leq s + \left\lceil \dfrac{6nm + 9nm^2 - 6m^2 - 3s}{4} \right\rceil$. But, this inequality implies a contradiction for all $n \geq 3$ and $m \geq 6$ as proved in Lemma 2.2.

From the above three cases we deduce that $\mathcal{C}(L_n\,[C_m])$ has no $3-fold$ basis for all $n \geq 3$ and $m \geq 6$. Hence, $\mathrm{b}(L_n\,[C_m]) = 4$ for all $n \geq 3$ and $m \geq 6$. $\square$

Lemma 2.3 *Let m, n be two positive integers, $n \geq 3$, such that $4(3nm^2 - 2m^2 - 2n + 1) \leq 3(2nm - 2n + 3nm^2 - 2m^2)$, then this inequality holds for $m \lneqq 4$.*

Proof. Let $4(3nm^2 - 2m^2 - 2n + 1) \leq 3(2nm - 2n + 3nm^2 - 2m^2)$. We can simplify it to get the inequality $m^2(3n-2) \leq 2n(3m+1) - 4$. Dividing by $m(3n-2)$ implies $m \leq \dfrac{2n(3m+1) - 4}{m(3n-2)} \leq 2\left(\dfrac{3n}{3n-2}\right) + \dfrac{2(n-2)}{m(3n-2)}$. Since $\dfrac{3n}{3n-2}$ is a decreasing sequence for all $n \geq 3$ and $\dfrac{9}{7}$ is its maximum, we conclude that $m \lneqq 4$. $\square$

Lemma 2.4 *Let m, n be two positive integers, $(n \geq 3)$, and $s \leq 3(2n)(m-1)$, such that $m^2(3n-2) - 2n + 1 \leq s + \left\lceil \dfrac{3m^2(3n-2) + 6n(m-1) - 3s}{4} \right\rceil$, then $m \lneqq 6$.*

Proof. Since $\left\lceil \dfrac{3m^2(3n-2) + 6n(m-1) - 3s}{4} \right\rceil \leq \dfrac{3m^2(3n-2) + 6n(m-1) - 3s}{4}$,

we have $m^2(3n-2) - 2n + 1 \leq s + \dfrac{3m^2(3n-2) + 6n(m-1) - 3s}{4}$. If we multiply by 4, rearrange the terms and use the fact that $s \leq 6n(m-1)$, we reach the inequality $m^2(3n-2) \leq 12nm - 4(n-1)$. Dividing by $m(3n-2)$ implies $m \leq 4\dfrac{3n}{3n-2} - 4\dfrac{n-1}{m(3n-2)} \lneqq 6$ being $\dfrac{3n}{3n-2}$ is a decreasing sequence for each $n \geq 3$ and $\dfrac{9}{7}$ is its maximum. $\hfill\square$

Corollary 2.1. *For every $n \geq 3$ and $m \geq 5$, we have $3 \leq b(L_n[P_m]) \leq 4$. Moreover, if $n \geq 3$ and $m \geq 6$, then $b(L_n[P_m]) = 4$.*

Proof. The graph $L_n[P_m]$ is a subgraph of $L_n[C_m]$ consists of $3n-2$ copies of $K_{m,m}$.

Let $B(L_n[P_m]) = B(L_n[C_m]) - M$, where

$$\begin{aligned}
M &= \{(a_i,0)(a_i,1)(a_i,2)\cdots(a_i,m-1)(a_i,0)\colon i \in Z_n\} \\
&\quad \cup \{(b_i,0)(b_i,1)(b_i,2)\cdots(b_i,m-1)(b_i,0)\colon i \in Z_n\}.
\end{aligned}$$

and $B(L_n[C_m])$ is the $4-fold$ basis of $\mathcal{C}(L_n[C_m])$ that obtained in previous theorem. Because $B(L_n[C_m])$ is linearly independent set and $B(L_n[P_m]) \subseteq B(L_n[C_m])$, $B(L_n[P_m])$ is linearly independent set of cycles in $\mathcal{C}(L_n[P_m])$. Since $|B(L_n[P_m])| = 3nm^2 - 2m^2 + 1 - 2n = m^2(3n-2) - 2n + 1 = \dim \mathcal{C}(L_n[P_m])$ we have $B(L_n[P_m])$ is a basis for $\mathcal{C}(L_n[P_m])$. The fold of any edge of $L_n[P_m]$ in $B(L_n[P_m])$ doesn't excess in $B(L_n[C_m])$, thus $B(L_n[P_m])$ is a $4-fold$ basis of $\mathcal{C}(L_n[P_m])$. Hence, $3 \leq b(L_n[P_m]) \leq 4$.

On the other hand, suppose that $\mathcal{C}(L_n[P_m])$ has a $3-fold$ basis $\mathcal{B}$, then to prove that such $\mathcal{B}$ cannot exist for all $n \geq 3$ and $m \geq 4$ we have three cases:

Case 1. Suppose that $\mathcal{B}$ consists only of 3-cycles, then $|\mathcal{B}| \leq 3(2n)(m-1) = 6nm - 6n$, because every 3-cycle in $\mathcal{B}$ must contain an edge from the set $S = \{(a_i,r)(a_i,r+1)\colon r \in Z_{m-1},\, i \in Z_n\} \cup \{(b_i,r)(b_i,r+1)\colon r \in Z_{m-1},\, i \in Z_n\}$, and the fold of every edge is at most 3. Since $|\mathcal{B}| = m^2(3n-2) - 2n + 1$, we have $m^2(3n-2) - 2n + 1 \leq 6nm - 6n$. This inequality reduces to $m^2(3n-2) + 1 \leq 6nm - 4n$. Dividing by $m(3n-2)$ and simplifying this inequality gives $m + \dfrac{1}{m(3n-2)} \leq \dfrac{2}{3}\left(\dfrac{3n}{3n-2}\right)\left(\dfrac{3m-2}{m}\right) < 3$ because $\dfrac{3n}{3n-2}$ is a decreasing sequence for each $n \geq 3$ and $\dfrac{9}{7}$ is its maximum. Hence, $\mathcal{B}$ cannot be a basis of $\mathcal{C}(L_n[P_m])$ for all $n \geq 3$ and $m \geq 4$, a contradiction.

Case 2. Suppose that $\mathcal{B}$ consists only of cycles of length greater than or equal to 4, then by Lemma 1.1 we have $4(3nm^2 - 2m^2 - 2n + 1) \leq 3(2nm - 2n + 3nm^2 - 2m^2)$ because $\dim \mathcal{C}(L_n[P_m]) = 3nm^2 - 2m^2 - 2n + 1$, $|E(L_n[P_m])| = 2nm - 2n + 3nm^2 - 2m^2$, and $|C_i| \geq 4$, for every $C_i \in \mathcal{B}$. But by Lemma 2.3, we have a contradiction for any $n \geq 3$ and $m \geq 6$.

Case 3. Suppose that $\mathcal{B}$ consists of s 3-cycles and t cycles of length greater than or equal to 4, then $s \leq 3(m-1)(2n)$, because we have at most $3(2n)(m-1)$ 3-cycles in $\mathcal{B}$ as we explained in Case 1. Since $|E(L_n[P_m])| = m^2(3n-2) + 2n(m-1)$, and the fold of every edge of $L_n[P_m]$ is at most 3 in $\mathcal{B}$ and $3s$ edges are joined to make the s 3-cycles we have $t \leq \left[\dfrac{3(m^2(3n-2) + 2n(m-1)) - 3s}{4} \right]$, then $m^2(3n-2) -$

$2n + 1 = \dim \mathcal{C}(L_n[P_m]) = |\mathcal{B}| = s + t \leq s + \left[\dfrac{3(m^2(3n-2) + 2n(m-1)) - 3s}{4} \right]$ so

that $m^2(3n-2) - 2n + 1 \leq s + \left[\dfrac{3(m^2(3n-2) + 2n(m-1)) - 3s}{4} \right]$. But this

inequality implies a contradiction for all $n \geq 3$ and $m \geq 6$ as proved in Lemma 2.4.

From the above three cases we deduce that $\mathcal{C}(L_n[P_m])$ has no $3-fold$ basis for all $n \geq 3$ and $m \geq 6$. Hence, $\mathrm{b}(L_n[P_m]) = 4$ for all $n \geq 3$ and $m \geq 6$. $\qquad \square$

Lemma 2.5. *Let m, n be two positive integers, $(n \geq 4)$, such that $4(3nm^2 + 1) \leq 3(3nm^2 + 2nm)$. Then $m < 2$.*

Proof. Since $4(3nm^2 + 1) \leq 3(3nm^2 + 2nm)$, we have $12nm^2 + 4 \leq 6nm + 9nm^2$. So that $3nm^2 + 4 \leq 6nm$, and $3m + \dfrac{4}{nm} \leq 6$, this implies that $3m < 6$. Thus $m < 2$. $\square$

Lemma 2.6 *Let m, n be two positive integers, $(n \geq 4)$, $s \leq 6nm$, such that $3nm^2 + 1 \leq s + \left[\dfrac{6nm + 9nm^2 - 3s}{4} \right]$, then $m < 4$.*

Proof. Since $\left[\dfrac{6nm + 9nm^2 - 3s}{4} \right] \leq \dfrac{6nm + 9nm^2 - 3s}{4}$, we have $3nm^2 + 1 \leq$

$s + \dfrac{6nm + 9nm^2 - 3s}{4}$. If we multiply by 4 rearrange the terms and use the fact

that $s \leq 6nm$, we get $m + \dfrac{4}{3nm} \leq 4$, so that $m < 4$. $\qquad \square$

Theorem 2.2. *If $n \geq 4$ and $m \geq 5$, then $b(CL_n[C_m]) = 4$.*

Proof. The graph $CL_n[C_m] = L_n[C_m] \cup K_{(a_{n-1},m),(a_0,m)} \cup K_{(b_{n-1},m),(b_0,m)}$ consists of $3n$ copies of the nonplanar graph $K_{m,m}$. It is clear that $CL_n[C_m]$ is a nonplanar graph because $L_n[C_m]$ is a nonplanar subgraph of it, thus by MacLane's Theorem $\mathrm{b}(CL_n[C_m]) \geq 3$.

To prove $b(CL_n[C_m]) \leq 4$, we exhibit a 4-fold basis for $\mathcal{C}(CL_n[C_m])$. Define the following sets of cycles in $\mathcal{C}(CL_n[C_m])$:

$$
\begin{aligned}
A_1 &= \{(a_{n-1},0)(a_0,i)(a_0,i+1)(a_{n-1},0): 0 \leq i \leq m-2\} \\
&\quad \cup \{(a_0,m-1)(a_{n-1},i)(a_{n-1},i+1)(a_0,m-1): 0 \leq i \leq m-2\}, \\
A_2 &= \{(b_{n-1},0)(b_0,i)(b_0,i+1)(b_{n-1},0): 0 \leq i \leq m-2\} \\
&\quad \cup \{(b_0,m-1)(b_{n-1},i)(b_{n-1},i+1)(b_0,m-1): 0 \leq i \leq m-2\},
\end{aligned}
$$

$$
\begin{aligned}
A_3 &= B(K_{(a_{n-1,m}),(a_{0,m})}), \\
A_4 &= B(K_{(b_{n-1,m}),(b_{0,m})}), \\
Q_1 &= (a_0, m-1)(a_1, m-1)(a_2, m-1), \cdots, (a_{n-1}, m-1)(a_0, m-1), \\
Q_2 &= (b_0, m-1)(b_1, m-1)(b_2, m-1), \cdots, (b_{n-1}, m-1)(b_0, m-1).
\end{aligned}
$$

Let $B_1 = A_1 \cup A_3$ and $B_2 = A_2 \cup A_4$ and define $B(CL_n[C_m]) = B(L_n[C_m])$ $\cup B_1 \cup B_2 \cup Q_1 \cup Q_2$, where $B(L_n[C_m])$ is the 4-fold basis of $\mathcal{C}(L_n[C_m])$ that was exhibited in Theorem 2.1. So $B(L_n[C_m])$ is linearly independent set of cycles in $\mathcal{C}(CL_n[C_m])$. Since Q_1 contains the edge $(a_{n-1}, m-1)(a_0, m-1)$ and Q_2 contains the edge $(b_{n-1}, m-1)(b_0, m-1)$ and each of these edges doesn't occur in any cycle of $B(L_n[C_m])$ then $B^* = B(L_n[C_m]) \cup Q_1 \cup Q_2$ is linearly independent.

Note that A_3 and A_4 are Schemeichel's $4-fold$ bases of the subspaces $\mathcal{C}(K_{(a_{n-1,m})},$ $_{(a_{0,m})})$ and $\mathcal{C}(K_{(b_{n-1,m}),(b_{0,m})})$, respectively, and they were obtained in Theorem 2.4 of [18]. Thus, A_3 and A_4 are linearly independent set of cycles in $\mathcal{C}(CL_nC_m)$, $A_3 \cap A_4 = \phi$, so $A_3 \cup A_4$ is linearly independent set of cycles. A_1 and A_2 are linearly independent sets of cycles because each of them represents the set of all finite faces of the corresponding planar graph that formed by pasting these cycles successively with increasing i, and $A_1 \cap A_2 = \phi$, so $A_1 \cup A_2$ is linearly independent set of cycles. Since each linear combination of the cycles of A_1 contains at least one edge from the set

$$
\begin{aligned}
H_1 &= \{(a_0, 0)(a_0, 1), (a_0, 1)(a_0, 2), \cdots, (a_0, m-1)(a_0, m)\} \\
&\cup \{(a_{n-1}, 0)(a_{n-1}, 1), (a_{n-1}, 1)(a_{n-1}, 2), \cdots, (a_{n-1}, m-1)(a_{n-1}, m)\}
\end{aligned}
$$

which doesnot appear in any cycle of A_3, then the cycles of A_1 are linearly independent with the cycles of A_3. Thus $B_1 = A_1 \cup A_3$ is a linearly independent set of cycles in $\mathcal{C}(CL_n[C_m])$. Similarly, each linear combination of the cycles of A_2 contains at least one edge from the set

$$
\begin{aligned}
H_2 &= \{(b_0, 0)(b_0, 1), (b_0, 1)(b_0, 2), \cdots, (b_0, m-1)(b_0, m)\} \\
&\cup \{(b_{n-1}, 0)(b_{n-1}, 1), (b_{n-1}, 1)(b_{n-1}, 2), \cdots, (b_{n-1}, m-1)(b_{n-1}, m)\}
\end{aligned}
$$

and this edge doesnot appear in any cycle of A_4, then the cycles of A_2 are linearly independent of the cycles of A_4. Thus $B_2 = A_2 \cup A_4$ is a linearly independent set of cycles in $\mathcal{C}(CL_n[C_m])$. Since $E(B_1) \cap E(B_2) = \phi$, then $B_1 \cup B_2$ is linearly independent. Moreover, $E(B_1 \cup B_2) \cap E(B^*)$ is a forest, thus by Lemma 1.2, we conclude that $B(CL_n[C_m]) = B^* \cup B_1 \cup B_2$ is linearly independent. Since

$$
\begin{aligned}
|B(CL_n[C_m])| &= |B(L_n[C_m])| + |B_1| + |B_2| + |Q_1| + |Q_2| \\
&= 3nm^2 - 2m^2 + 1 + m^2 - 1 + m^2 - 1 + 1 + 1 \\
&= 3nm^2 + 1 = \dim \mathcal{C}(CL_n[C_m]),
\end{aligned}
$$

it follows that $B(CL_n[C_m])$ is a basis for $\mathcal{C}(CL_n[C_m])$. It is easy to verify that $B(CL_n[C_m])$ is a 4-fold basis for $\mathcal{C}(CL_n[C_m])$. Hence, b $(CL_n[C_m]) = 4$ for all $n \geq 4$ and $m \geq 5$.

On the other hand, suppose that $\mathcal{C}(CL_n[C_m])$ has a $3-fold$ basis $\mathcal{B}$. Then we have three cases to prove that $\mathcal{B}$ can't exist for all $n \geq 4$ and $m \geq 5$:

Case 1. Suppose that $\mathcal{B}$ contains only 3-cycles then $|\mathcal{B}| \leq 3m(2n) = 6nm$, because every $3-cycles$ in $\mathcal{B}$ must contain an edge from the set

$$
\begin{aligned}
S \;=\; & \{(a_i,0)(a_i,1),\,(a_i,1)(a_i,2),\cdots,(a_i,m-1)(a_i,0)\colon i \in Z_n\} \\
& \cup \{(b_i,0)(b_i,1),\,(b_i,1)(b_i,2),\cdots,(b_i,m-1)(b_i,0)\colon i \in Z_n\},
\end{aligned}
$$

and the fold of every edge of S is at most 3. But $|\mathcal{B}| = 3nm^2+1$, so that $3nm^2+1 \leq 3m(2n) = 6nm$, but this inequality doesnot hold for all $n \geq 2$ and $m \geq 2$. Hence, $\mathcal{B}$ cannot be a basis of $\mathcal{C}(CL_n[C_m])$ for all $n \geq 4$ and $m \geq 5$, a contradiction.

Case 2. Suppose that $\mathcal{B}$ consists only of cycles of length greater than or equal to 4 then by Lemma 1.1 we have $4(3nm^2+1) \leq 3(3nm^2+2nm)$, because dim $\mathcal{C}(CL_n[C_m]) = 3nm^2+1$, $|E(L_n[C_m])| = 3nm^2+2nm$, and $|C_i| \geq 4$ for every $C_i \in \mathcal{B}$. But by Lemma 2.5, we have a contradiction for any $n \geq 4$ and $m \geq 5$.

Case 3. Suppose that $\mathcal{B}$ consists of $s\,3-cycles$ and t cycles of length greater than or equal to 4, then $s \leq 3m(2n)$, because we have at most $3m(2n)$ $3-cycles$ in $\mathcal{B}$ as we explained in Case 1. Since $|E(L_n[C_m])| = 3nm^2+2nm$, and the fold of every edge of $CL_n[C_m]$ is at most 3 in $\mathcal{B}$ and $3s$ edges are joined to make the $s\,3-cycles$, we have $t \leq \left\lceil \dfrac{6nm+9nm^2-3s}{4} \right\rceil$. Then $3nm^2+1 = \dim\mathcal{C}(CL_n[C_m]) = |\mathcal{B}| = s+t \leq s+\left\lceil \dfrac{6nm+9nm^2-3s}{4} \right\rceil$. Thus $3nm^2+1 \leq s+\left\lceil \dfrac{6nm+9nm^2-3s}{4} \right\rceil$. But this inequality implies a contradiction for all $n \geq 4$ and $m \geq 5$, as it was proved in Lemma 2.6. $\square$

Lemma 2.7. *Let m, n be two positive integers, $(n \geq 4)$, such that $4(3nm^2-2n+1) \leq 3(3nm^2+2nm-2n)$. Then $m \lneqq 3$.*

Proof. Since $4(3nm^2-2n+1) \leq 3(3nm^2+2nm-2n)$ we have $12nm^2-8n+4 \leq 9nm^2+6nm-6n$, then $3nm^2+4-2n \leq 6nm$, thus $m \leq 2+\left(\dfrac{2}{3m}-\dfrac{4}{3nm}\right)$. Therefore, $m \lneqq 3$. $\square$

Lemma 2.8. *Let m, n be two positive integers, $(n \geq 4)$, $s \leq 6nm-6n$ such that $3nm^2-2n+1 \leq s+\left\lceil \dfrac{6nm+9nm^2-6n-3s}{4} \right\rceil$. Then $m \lneqq 5$.*

Proof. Since $\left\lceil \dfrac{6nm+9nm^2-6n-3s}{4} \right\rceil \leqq \dfrac{6nm+9nm^2-6n-3s}{4}$ we get $3nm^2-2n+1 \leq s+\dfrac{6nm+9nm^2-6n-3s}{4}$. If we multiply by 4 and rearrange the terms and use the fact that $s \leq 6nm-6n$ we get the inequality $3nm^2 \leq 2n+12nm-4$. Dividing by $3nm$ implies $m \leq 4+\left(\dfrac{2}{3m}-\dfrac{4}{3nm}\right) = 4+\dfrac{2n-4}{3nm}$. since $n \geq 4$ this

inequality holds only for $m \lneqq 5$.   □

Corollary 2.2 *For every $n \geq 4$ and $m \geq 5$, $b(CL_n [P_m]) = 4$.*

Proof. The graph $CL_n [P_m]$ is a subgraph of $CL_n [C_m]$ that consists of $3n$ copies of $K_{m,m}$.

Let $B(CL_n [P_m]) = B(L_n [C_m]) - M$, where

$$
\begin{aligned}
M \quad = \quad & \{(a_i, 0)(a_i, 1)(a_i, 2) \cdots (a_i, m-1)(a_i, 0): \ i \in Z_n\} \\
& \cup \{(b_i, 0)(b_i, 1)(b_i, 2) \cdots (b_i, m-1)(b_i, 0): \ i \in Z_n\}
\end{aligned}
$$

and $B(CL_n [C_m])$ is the $4 - fold$ basis of $\mathcal{C}(CL_n [C_m])$ that was exhibited in Theorem 2.1. Since $B(CL_n [C_m])$ is linearly independent set and $B(CL_n [P_m]) \subseteq B(CL_n [C_m])$, then $B(CL_n [P_m])$ is linearly independent set of cycles in $\mathcal{C}(CL_n [P_m])$. Since $|B(CL_n [P_m])| = 3nm^2 + 1 - 2n = \dim \mathcal{C}(CL_n [P_m])$, we conclude that $B(CL_n [P_m])$ is a basis for $\mathcal{C}(CL_n [P_m])$. The fold of any edge of $CL_n [P_m]$ in $B(CL_n [P_m])$ is at most as it is in $B(CL_n [C_m])$. Thus $B(CL_n [P_m])$ is 4-$fold$ basis of $\mathcal{C}(CL_n [P_m])$. Hence, $b(CL_n [P_m]) = 4$ for each $n \geq 4$ and $m \geq 5$.

On the other hand, suppose that $\mathcal{C}(CL_n [P_m])$ has a $3 - fold$ basis $\mathcal{B}$, then we have the following three cases to prove that such $\mathcal{B}$ doesnot exist for all $n \geq 4$ and $m \geq 5$:

Case1. Suppose that $\mathcal{B}$ contains only 3 -cycles, then $|\mathcal{B}| \leq 3(m-1)(2n) = 6nm - 6n$, because every 3-cycles in $\mathcal{B}$ must contain an edge from the set

$$
\begin{aligned}
S \quad = \quad & \{(a_i, 0)(a_i, 1), (a_i, 1)(a_i, 2), \cdots, (a_i, m-2)(a_i, m-1): \ i \in Z_n\} \\
& \cup \{(b_i, 0)(b_i, 1), (b_i, 1)(b_i, 2), \cdots ., (b_i, m-2)(b_i, m-1): \ i \in Z_n\},
\end{aligned}
$$

and the fold of every edge of S is at most 3. But $|\mathcal{B}| = \dim \mathcal{C}(CL_n [P_m]) = 3nm^2 - 2n + 1$, so $3nm^2 - 2n + 1 \leq 6nm - 6n$. Then $3nm^2 + 1 \leq 6nm - 4n$. If we divide by $3nm$ we get $m + \dfrac{1}{3nm} + \dfrac{4}{3m} \leq 2$, which implies $m \lneqq 1$. Hence $\mathcal{B}$ cannot be a basis of $\mathcal{C}(CL_n [P_m])$ for all $n \geq 4$ and $m \geq 5$, a contradiction.

Case 2. Suppose that $\mathcal{B}$ consists only of cycles of length greater than or equal to 4, then by Lemma 1.1 we have $4(3nm^2 - 2n + 1) \leq 3(3nm^2 + 2nm - 2n)$, because $\dim \mathcal{C}(CL_n [P_m]) = 3nm^2 - 2n + 1$, $|E(L_n [P_m])| = 3nm^2 + 2nm - 2n$, and $|C_i| \geq 4$ for every $C_i \in \mathcal{B}$. But by Lemma 2.7 we have a contradiction for any $n \geq 4$ and $m \geq 5$.

Case 3. Suppose that $\mathcal{B}$ consists of $s3 - cycles$ and t $cycles$ of length greater than or equal to 4, then $s \leq 3(m-1)(2n)$, because we have at most $3(2n)(m -1)$ $3 - cycles$ in $\mathcal{B}$ as we explained in Case 1. Since $|E(L_n [P_m])| = 3nm^2 + 2nm - 2n$, and the fold of every edge of $CL_n [P_m]$ is at most 3 in $\mathcal{B}$ and $3s$ edges are joined to make the $s3 - cycles$, we have $t \leq \left\lceil \dfrac{6nm + 9nm^2 - 6n - 3s}{4} \right\rceil$. Then

$$
3nm^2 - 2n + 1 = \dim \mathcal{C}(CL_n [P_m]) = |\mathcal{B}| = s + t \leq s + \left\lceil \dfrac{6nm + 9nm^2 - 6n - 3s}{4} \right\rceil.
$$

So that $3nm^2 - 2n + 1 \leq s + \left\lceil \dfrac{6nm + 9nm^2 - 6n - 3}{4} \right\rceil$. But, this inequality implies a contradiction for all $n \geq 4$ and $m \geq 5$ as we proved in Lemma 2.8. $\square$

Theorem 2.3. *For every $n \geq 4$ and $m \geq 5$, $b(ML_n\,[C_m]) = 4$.*

Proof. The graph $ML_n\,[C_m] = L_n\,[C_m] \cup K_{(a_{n-1},m),(b_{0,m})} \cup K_{(b_{n-1},m),(a_{0,m})}$ consists of $3n$ copies of the nonplanar graph $K_{m,m}$. It is clear that $ML_n\,[C_m]$ is a nonplanar graph because $L_n\,[C_m]$ is a nonplanar subgraph of it, thus by Maclane's theorem $b(ML_n\,[C_m]) \geq 3$.

To prove b$(ML_n\,[C_m]) \leq 4$, we exhibit a $4-fold$ basis for $\mathcal{C}(ML_n\,[C_m])$. Define the following sets of cycles in $\mathcal{C}(ML_n\,[C_m])$:

$$
\begin{aligned}
A_1 &= \{(a_{n-1},0)(b_0,i)(b_0,i+1)(a_{n-1},0) \colon 0 \leq i \leq m-2\} \\
&\quad \cup \{(b_0,m-1)(a_{n-1},i)(a_{n-1},i+1)(b_0,m-1) \colon 0 \leq i \leq m-2\}, \\
A_2 &= \{(b_{n-1},0)(a_0,i)(a_0,i+1)(b_{n-1},0) \colon 0 \leq i \leq m-2\} \\
&\quad \cup \{(a_0,m-1)(b_{n-1},i)(b_{n-1},i+1)(a_0,m-1) \colon 0 \leq i \leq m-2\}, \\
A_3 &= B(K_{(a_{n-1},m),(b_{0,m})}), \\
A_4 &= B(K_{(b_{n-1,m}),(a_{0,m})}), \\
Q_1 &= (a_0,m-1)(a_1,m-1), \cdots, (a_{n-1},m-1)\ (b_0,m-1\ )(a_0,m-1), \\
Q_2 &= (b_0,m-1)(b_1,m-1)(b_2,m-1), \cdots, (b_{n-1},m-1)(a_0,m-1)(b_0,m-1).
\end{aligned}
$$

Let $B_1 = A_1 \cup A_3$, $B_2 = A_2 \cup A_4$. Define $B(ML_n\,[C_m]) = B(L_n\,[C_m]) \cup B_1 \cup B_2 \cup Q_1 \cup Q_2$, where $B(L_n\,[C_m])$ is the 4 -fold basis of $\mathcal{C}(L_n\,[C_m])$ that was exhibited in Theorem 2.1. So $B(L_n\,[C_m])$ is linearly independent set of cycles in $\mathcal{C}(ML_n\,[C_m])$. Since Q_1 contains the edge $(a_{n-1},m-1)(b_0,m-1)$ and Q_2 contains the edge $(b_{n-1},m-1)(a_0,m-1)$ and each of these edges doesn't occur in any cycle of $B(L_n\,[C_m])$ then $B^* = B(L_n\,[C_m]) \cup Q_1 \cup Q_2$ is linearly independent.
Note that A_3 and A_4 are Schemeichel's $4-fold$ bases of the subspaces $\mathcal{C}(K_{(a_{n-1},m)}, \\ _{(b_{0,m})})$ and $\mathcal{C}\ (K_{(b_{n-1},m),(a_{0,m})})$, respectively, and they were obtained in Theorem 2.4 of [18]. Thus A_3 and A_4 are linearly independent set of cycles in $\mathcal{C}\ (ML_n\,[C_m])$, and $A_3 \cap A_4 = \phi$, so $A_3 \cup A_4$ is linearly independent sets of cycles. A_1 and A_2 are linearly independent sets of cycles because each of them represents the set of all finite faces of the corresponding planar graph that formed by pasting these cycles successively with increasing i, and $A_1 \cap A_2 = \phi$, so $A_1 \cup A_2$ is linearly independent set of cycles. Since each linear combination of cycles of A_1 contains at least one edge from the set

$$
\begin{aligned}
H_1 &= \{(b_0,0)(b_0,1), (b_0,1)(b_0,2), \cdots, (b_0,m-1)(b_0,m)\} \\
&\quad \cup \{(a_{n-1},0)(a_{n-1},1), (a_{n-1},1)(a_{n-1},2), \cdots, (a_{n-1},m-1)(a_{n-1},m)\}
\end{aligned}
$$

and this edge doesnot appear in any cycle of A_3 then the cycles of A_1 are linearly independent of the cycles of A_3. Thus $B_1 = A_1 \cup A_3$ is linearly independent set of cycles in $\mathcal{C}(CL_n\,[C_m])$. Similarly, each linear combination of cycles of A_2 contains

at least one edge from the set

$$
\begin{aligned}
H_2 &= \{(a_0,0)(a_0,1),(a_0,1)(a_0,2),\cdots,(a_0,m-1)(a_0,m)\} \\
&\cup \{(b_{n-1},0)(b_{n-1},1),(b_{n-1},1)(b_{n-1},2),\cdots,(b_{n-1},m-1)(b_{n-1},m)\},
\end{aligned}
$$

and this edge doesnot appear in any cycle of A_4 then the cycles of A_2 are linearly independent of the cycles of A_4, thus $B_2 = A_2 \cup A_4$ is linearly independent set of cycles in $\mathcal{C}(ML_n[C_m])$. Since $E(B_1) \cap E(B_2) = \phi$ the set $B_1 \cup B_2$ is linearly independent. Moreover, $E(B_1 \cup B_2) \cap E(B^*)$ is a forest, then by Lemma 1.2 we conclude that $B(ML_n[C_m]) = B^* \cup B_1 \cup B_2$ is linearly independent set of cycles. Since

$$
\begin{aligned}
|B(ML_n[C_m])| &= |B(L_n[C_m])| + |B_1| + |B_2| + |Q_1| + |Q_2| \\
&= 3nm^2 - 2m^2 + 1 + (m^2 - 1) + (m^2 - 1) + 1 + 1 \\
&= 3nm^2 + 1 = \dim \mathcal{C}(ML_n[C_m]),
\end{aligned}
$$

it follows that $B(ML_n[C_m])$ is a basis for $\mathcal{C}(ML_n[C_m])$. It is easy to see that $B(ML_n[C_m])$ is a $4 - fold$ basis for $\mathcal{C}(ML_n[C_m])$. Hence, $b(ML_n[C_m]) = 4$ for all $n \geq 4$ and $m \geq 5$.

On the other hand, suppose that $\mathcal{C}(ML_n[C_m])$ has a $3 - fold$ basis $\mathcal{B}$. Then we have three cases to prove that $\mathcal{B}$ can't exist for all $n \geq 4$ and $m \geq 5$. They are identical to the three cases of Theorem 2.2, which we omit here. $\qquad \square$

Corollary 2.3. *For every $n \geq 4$ and $m \geq 5$, $(bML_n[P_m]) = 4$.*

Proof. The graph $ML_n[P_m]$ is a subgraph of $ML_n[C_m]$ that consists of $3n$ copies of $K_{m,m}$.

Let $B(ML_n[P_m]) = B(ML_n[C_m]) - M$, where

$$
\begin{aligned}
M &= \{(a_i,0)(a_i,1),(a_i,1)(a_i,2),\cdots,(a_i,m-1)(a_i,0) : i \in Z_n\} \\
&\cup \{(b_i,0)(b_i,1),(b_i,1)(b_i,2),\cdots,(b_i,m-1)(b_i,0) : i \in Z_n\},
\end{aligned}
$$

and $B(ML_n[C_m])$ is the $4 - fold$ basis of $\mathcal{C}(ML_n[C_m])$ that was exhibited in Theorem 2.3. Since $B(ML_n[C_m])$ is linearly independent set and $B(ML_n[P_m]) \subseteq B(ML_n[C_m])$, then $B(ML_n[P_m])$ is linearly independent set of cycles in $\mathcal{C}(ML_n[P_m])$. Since $|B(ML_n[P_m])| = 3nm^2 - 2n + 1 = \dim \mathcal{C}(ML_n[P_m])$, we conclude that $B(ML_n[P_m])$ is a basis for $\mathcal{C}(ML_n[P_m])$. The fold of any edge of $ML_n[P_m]$ in $B(ML_n[P_m])$ is the same as it is in $B(ML_n[C_m])$, thus $B(ML_n[P_m])$ is a 4-$fold$ basis of $\mathcal{C}(ML_n[P_m])$. Hence, $b(ML_n[P_m]) = 4$ for each $n \geq 4$ and $m \geq 5$.

On the other hand, suppose that $\mathcal{C}(ML_n[P_m])$ has a 3-$fold$ basis $\mathcal{B}$. Then we have three cases to prove that $\mathcal{B}$ can't exist for all $n \geq 4$ and $m \geq 5$. They are identical to the three cases of Theorem 2.2, which we omit here. $\qquad \square$

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