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Weakly Complementary Cycles in 3-Connected Multipartite Tournaments

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Abstract. The vertex set of a digraph D is denoted by $V(D)$. A c-partite tournament is an orientation of a complete c-partite graph. A digraph D is called cycle complementary if there exist two vertex disjoint cycles C_1 and C_2 such that $V(D) = V(C_1) \cup V(C_2)$, and a multipartite tournament D is called weakly cycle complementary if there exist two vertex disjoint cycles C_1 and C_2 such that $V(C_1) \cup V(C_2)$ contains vertices of all partite sets of D. The problem of complementary cycles in 2-connected tournaments was completely solved by Reid [4] in 1985 and Z. Song [5] in 1993. They proved that every 2-connected tournament T on at least 8 vertices has complementary cycles of length t and $|V(T)| - t$ for all $3 \leq t \leq |V(T)|/2$. Recently, Volkmann [8] proved that each regular multipartite tournament D of order $|V(D)| \geq 8$ is cycle complementary. In this article, we analyze multipartite tournaments that are weakly cycle complementary. Especially, we will characterize all 3-connected c-partite tournaments with $c \geq 3$ that are weakly cycle complementary.

1. Terminology

In this paper all digraphs are finite without loops and multiple arcs. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively. If xy is an arc of a digraph D, then we write $x \to y$ and say x *dominates* y, and if X and Y are two disjoint vertex sets or subdigraphs of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. Furthermore, $X \rightsquigarrow Y$ denotes the fact that there is no arc leading from Y to X.

If D is a digraph, then the *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x. Therefore, if the arc $xy \in E(D)$ exists, then y is an *outer neighbor* of x and x is an *inner neighbor* of y. The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are called the *outdegree* and the *indegree* of x, respectively. Furthermore, the numbers $\delta_D^+ = \delta^+ = \min\{d^+(x)|x \in V(D)\}$ and $\delta_D^- = \delta^- = \min\{d^-(x)|x \in V(D)\}$ are the *minimum outdegree* and the *minimum*

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indegree, respectively.

For a vertex set X of D, we define $D[X]$ as the subdigraph induced by X. If we replace in a digraph D every arc xy by yx, then we call the resulting digraph the *converse* of D, denoted by D^{-1}.

If we speak of a *cycle*, then we mean a directed cycle, and a cycle of length n is called an *n-cycle*. The length of a cycle C is denoted by $L(C)$. A digraph D is called *pancyclic* if it contains cycles of length n for all $n \in \{3, 4, \dots, |V(D)|\}$. If $x \in V(C)$ ($x \in V(P)$, respectively) for a cycle C (a path P), then we denote the successor of x in the given cycle (path) by x^+ and the predecessor by x^-. A digraph D is *cycle complementary* if there exist two vertex-disjoint cycles C and C' such that $V(D) = V(C) \cup V(C')$.

A digraph D is *strongly connected* or *strong* if for each pair of vertices u and v, there is a path from u to v in D. A digraph D with at least $k+1$ vertices is *k-connected* if for any set A of at most $k-1$ vertices, the subdigraph $D - A$ is strong. The *connectivity*, denoted by $\kappa(D)$, is then defined to be the largest value of k such that D is k-connected. If $\kappa(D) = 1$ and x is a vertex of D such that $D - x$ is not strong, then we say that x is a *cut-vertex* of D.

A digraph D is called *c-partite*, if its underlying graph G is c-partite. Especially, a *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. A *tournament* is a c-partite tournament with exactly c vertices. If $V_1, V_2, \cdots, V_c$ are the partite sets of a c-partite tournament D and the vertex x of D belongs to the partite set V_i, then we define $V(x) = V_i$. If D is a c-partite tournament with the partite sets $V_1, V_2, \dots, V_c$ such that $|V_1| \leq |V_2| \leq \cdots \leq |V_c|$, then $|V_c| = \alpha(D)$ is the *independence number* of D.

2. Introduction and preliminary results

There is an extensive literature about the existence of complementary cycles in digraphs. In 1985, Reid investigated 2-connected tournaments. In this class of digraphs he found an example of a 3-connected regular tournament with seven vertices, which is not cycle complementary.

Example 2.1(Reid [4])**.** Let T_7 be the 3-regular and 3-connected tournament presented in Figure 1. Then it is well-known that T_7 doesn't contain a 3-cycle C_3 and a 4-cycle C_4 such that $V(T_7) = V(C_3) \cup V(C_4)$.

The tournament T_7 is the sole exception of a 2-connected tournament with at least 6 vertices that is not cycle complementary.

Theorem 2.2(Reid [4])**.** *Let T be a 2-connected tournament with at least $n \geq 6$ vertices. Then either T contains a 3-cycle and an $(n-3)$-cycle which are vertex disjoint or T is the 7-tournament T_7.*

In 1993, Song [5] extended this result.

Theorem 2.3(Song [5])**.** *If T is a 2-connected tournament with at least eight* *vertices, then T contains two complementary cycles of length t and $|V(T)| - t$ for all $3 \leq t \leq |V(T)|/2$.*

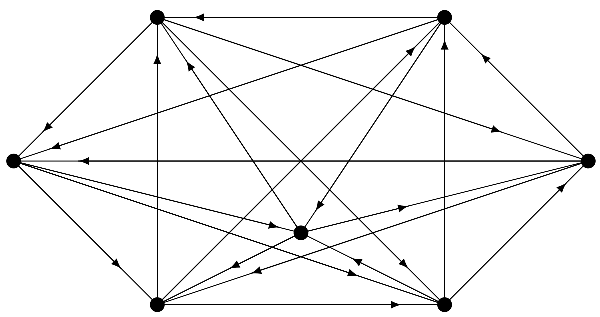


Figure 1: The 3-regular, 3-connected tournament T_7

The problem of complementary cycles in multipartite tournaments is much more difficult to analyze than in tournaments. This is why up to now only regular multipartite tournaments were considered. Even not all digraphs of this class are cycle complementary as the following example demonstrates.

Example 2.4(Volkmann [8])**.** Let $V_1 = \{x_1, x_2\}$, $V_2 = \{y_1, y_2\}$ and $V_3 = \{u_1, u_2\}$ be the partite sets of the 2-connected 3-partite tournament $D_{3,2}$ presented in Figure 2. Then it is a simple matter to verify that $D_{3,2}$ doesn't contain two vertex disjoint cycles.

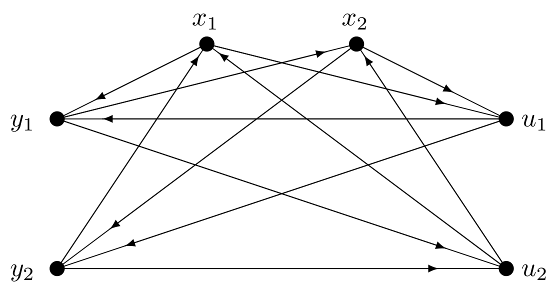


Figure 2: The 2-connected 3-partite tournament $D_{3,2}$

In 2004, Volkmann [8] proved the following result for regular multipartite tournaments.

Theorem 2.5(Volkmann [8])**.** *Let D be a regular c-partite tournament. If $c = 2$*

and $|V(D)| \geq 8$ *or* $c \geq 3$ *and* $|V(D)| \geq 6$*, then* D *is cycle complementary, unless* D *is isomorphic to* T_7 *in Figure 1 or to* $D_{3,2}$ *in Figure 2.*

This theorem could make believe that the following conjecture of Yeo [11] is valid.

Conjucture 2.6(Yeo [11])**.** A regular c-partite tournament D with $c \geq 4$ has a pair of vertex disjoint cycles of length t and $|V(D)| - t$ for all $t \in \{3, 4, \cdots, |V(D)| - 3\}$.

In [10], Volkmann showed that this conjecture is valid for $t = 3$ with exception of three special digraphs. Moreover, in this article and in [9] he treated the case that $t = 4$ in Conjecture 2.6. And in a recent article Korneffel, Meierling, Volkmann and Winzen [3] have shown that Conjecture 2.6 is true for $t = 5$.

There is still another unsolved conjecture by Volkmann [7] concerning complementary cycles.

Conjucture 2.7(Volkmann [7])**.** A multipartite tournament D with $\kappa(D) \geq \alpha(D) + 1$ is cycle complementary, unless D is a member of a finite family of multipartite tournaments.

The aim of this article is to weaken the condition that D is cycle complementary in the following way.

Definition 2.8. Let D be a c-partite digraph with the partite sets $V_1, V_2, \cdots, V_c$. Two vertex disjoint cycles C and C' are called *weakly complementary*, if they contain vertices of all partite sets of D, which means that $(V(C) \cup V(C')) \cap V_i \neq \emptyset$ for all $1 \leq i \leq c$ and $V(C) \cap V(C') = \emptyset$. A c-partite digraph D with such two cycles is *weakly cycle complementary.*

Note that a tournament is weakly cycle complementary, if and only if it is cycle complementary. This definition leads to a new problem.

Problem 2.9. Find necessary and/or sufficient conditions for a c-partite digraph to be weakly cycle complementary.

Using the weaker Definition 2.8 of cycle complementarity it is possible to characterize k-connected multipartite tournaments which are weakly cycle complementary. Especially, in this article we will prove that a 3-strong c-partite tournament D with $c \geq 3$ and at least six vertices is weakly cycle complementary unless D is isomorphic to T_7 in Figure 1. The following results play an important role to prove this characterization.

Theorem 2.10(Bondy [1])**.** *Each strong c-partite tournament contains an m-cycle for each* $m \in \{3, 4, \cdots, c\}$.

Theorem 2.11(Goddard, Oellermann [2])**.** *Let* D *be a strongly connected multipartite tournament with the partite sets* $V_1, V_2, \cdots, V_c$*. Then every vertex of* D *belongs to a cycle that contains vertices from exactly* m *partite sets for each* $m \in \{3, 4, \cdots, c\}$.

Corollary 2.12. *Every vertex of a strongly connected c-partite tournament D with $c \geq 2$ belongs to a cycle that contains vertices from all c partite sets.*

In the following we call a cycle containing the vertex x_0 and vertices from all partite sets of a multipartite tournament a *GodOel-cycle* $C(x_0)$.

Theorem 2.13(Tewes, Volkmann [6])**.** *If D is a non-strong c-partite tournament with the partite sets $V_1, V_2, \cdots, V_c$, then there exists a unique decomposition of $V(D)$ into pairwise disjoint subsets $D_1, D_2, \cdots, D_p$, where D_i is the vertex set of a strong component of D or $D_i \subseteq V_l$ for some $l \in \{1, 2, \cdots, c\}$ such that $D_i \rightsquigarrow D_j$ for $1 \leq i < j \leq p$ and there are $x_i \in D_i$ and $x_{i+1} \in D_{i+1}$ such that $x_i \to x_{i+1}$ for $1 \leq i < p$.*

3. Main result

Theorem 3.1. *Let D be a c-partite tournament with $c \geq 3$, $|V(D)| \geq 6$ and $\kappa(D) \geq 3$. Then D is weakly cycle complementary unless D is isomorphic to T_7 in Figure 1.*

Proof. Let D be a c-partite tournament with $c \geq 3$, $|V(D)| \geq 6$ and $\kappa(D) \geq 3$. According to Theorem 2.10, D contains a 3-cycle $C = v_1v_2v_3v_1$. Let $D - V(C)$ consist of the partite sets $V_1, V_2, \cdots, V_{c'}$.

First, let $\kappa(D) \geq 4$. In this case $D - V(C)$ is strong and thus it contains a GodOel-cycle C'. Since C and C' are two vertex disjoint cycles in D with vertices from all partite sets, we conclude that D is weakly cycle complementary.

Second, we may assume that $\kappa(D) = 3$. If $D - V(C)$ is strong, then as above we see that D is weakly cycle complementary. Hence let $D - \{v_1, v_2, v_3\}$ be non-strong. Theorem 2.13 implies that there is a unique decomposition of $V(D) - V(C)$ in subsets $D_1, D_2, \cdots, D_p$, where D_i is the vertex set of a strong component of $D - V(C)$ or $D_i \subseteq V_l$ for some $l \in \{1, 2, \cdots, c'\}$ such that $D_i \rightsquigarrow D_j$ for $1 \leq i < j \leq p$ and there are $x_i \in D_i$ and $x_{i+1} \in D_{i+1}$ such that $x_i \to x_{i+1}$ for $1 \leq i < p$. Suppose that D is not weakly cycle complementary.

If D_1 is an independent set of vertices and there are vertices $v_1' \in D_1$ and $v_i \in V(C)$ such that $v_1' \rightsquigarrow v_i$ (without loss of generality, let $i = 1$), then the fact that $d^-_{D-\{v_2,v_3\}}(v_1') = 0$ yields that $D - \{v_2, v_3\}$ is not strong, a contradiction to $\kappa(D) = 3$. Hence it remains to treat the case that $V(C) \to D_1$, if $D_1 \subseteq V_j$ for some $1 \leq j \leq c'$. Analogously, we see that $D_p \to V(C)$, if D_p is an independent set of vertices. If D_1 is the vertex set of a strong component with $|D_1| \geq 3$, then the fact that $\kappa(D) \geq 3$ implies that there are three pairwise non-incident arcs leading from $V(C)$ to D_1. Analogously, if D_p is the vertex set of a strong component with $|D_p| \geq 3$, then there are three pairwise non-incident arcs leading from D_p to $V(C)$.

To prove this theorem we distinguish different cases.

Case 1. Assume that there are at least two vertex sets in $D_1, D_2, \cdots, D_p$ that induce a non-trivial strong component.

Subcase 1.1. Assume that at least one of these vertex sets is D_i with $1 < i < p$. Let $x_1 \in N^+(v_1) \cap D_1$ and $y_1 \in N^-(v_2) \cap D_p$. Let us define $C_1 = C(x_1)$ if $D[D_1]$ is a non-trivial strong component and $C_1 = x_1$ otherwise. Analogously, let $C_p = C(y_1)$ if $D[D_p]$ is a non-trivial strong component and $C_p = y_1$ otherwise. Similarly, we define C_j $(2 \leq j \leq p-1)$ as an arbitrary GodOel-cycle of $D[D_j]$, if D_j induces a non-trivial strong component and $C_j = v'_j$ with $v'_j \in D_j$ otherwise. Now it is obvious that C_i and $(v_1C_1C_2 \cdots C_{i-1}C_{i+1} \cdots C_pv_2v_3v_1$ or $v_1C_1C_2 \cdots C_{i-2}C_{i+1} \cdots C_pv_2v_3v_1$ or $v_1C_1C_2 \cdots C_{i-1}C_{i+2} \cdots C_pv_2v_3v_1)$ are two weakly complementary cycles of D, if we interpret the second cycle in the following way:

If D_1 is the vertex set of a non-trivial strong component, then we walk from v_1 to x_1 and along the cycle C_1 until we reach the vertex x_1^- and then we walk to a vertex of C_2. If however D_2 is an independent set of vertices such that $v'_2 \in V(x_1^-)$ for all $v'_2 \in D_2$, then we walk along the cycle C_1 until x_1^{--}. In the case that D_1 is an independent set of vertices we walk from v_1 to x_1 and then to a vertex of D_2.

If we arrive at the vertex v'_j of a GodOel-cycle C_j, then we walk along the cycle until we reach the vertex v'^-_j and then we pass over to a vertex of the next component. In the case that the next component is an independent set of vertices that belong to the same partite set as v'^-_j then we stop at the vertex v'^{--}_j and pass over to the next component.

Finally, if D_p induces a non-trivial strong component, then we pay attention that we reach the cycle C_p in the vertex y_1^+, then we walk the cycle along until y_1 and pass over to v_2, v_3 and we finish the cycle with v_1.

These two cycles lead to a contradiction to our assumption that D is not weakly cycle complementary.

Subcase 1.2. Assume that only the two vertex sets D_1 and D_p induce a non-trivial strong component. Let $v_1 \to x_s$, $v_2 \to x_t$ and $v_3 \to x_m$ be the three pairwise non-incident arcs leading from $V(C)$ to D_1. Analogously, let $y_s \to v_1$, $y_t \to v_2$ and $y_m \to v_3$ be three pairwise non-incident arcs from D_p to $V(C)$. Let $v'_i \in D_i$ $(2 \leq i \leq p-1)$. If one of the vertices x_s, x_t or x_m does not belong to any GodOel-cycle C_1 of D_1, say $x_s \notin C_1$, then C_1 and $(v_1x_sv'_2v'_3 \cdots v'_{p-1}y_t^+y_t^{++} \cdots y_tv_2v_3v_1$ or $v_1x_sv'_2v'_3 \cdots v'_{p-1}y_t^{++} \cdots y_tv_2v_3v_1)$ are two weakly complementary cycles, a contradiction. Hence we have $\{x_s, x_t, x_m\} \subseteq V(C_1)$. Analogously, we observe that $\{y_s, y_t, y_m\} \subseteq V(C_p)$ for a GodOel-cycle C_p of $D[D_p]$.

Subcase 1.2.1. Assume that $p = 2$.

Subcase 1.2.1.1. Assume that x_t is on the oriented path along C_1 from x_m to x_s and that y_m is on the path along C_2 from y_s to y_t. Then we have one of the cycles $v_1x_sx_s^+ \cdots x_t^-y_t^+y_t^{++} \cdots y_mv_3v_1$ and $v_1x_sx_s^+ \cdots x_t^{--}y_t^+y_t^{++} \cdots y_mv_3v_1$. Let this cycle be called C'_1. If we are not in the case that $x_t^+ = x_s$, $y_m^+ = y_t$ and $V(x_t) = V(y_t)$, then there are the cycles $v_2x_tx_t^+ \cdots x_s^-y_m^+y_m^{++} \cdots y_tv_2$ or $v_2x_tx_t^+ \cdots x_s^{--}y_m^+y_m^{++} \cdots y_tv_2$ or $v_2x_tx_t^+ \cdots x_s^-y_m^{++} \cdots y_tv_2$ and C'_1, which contain vertices from all partite sets of D, a contradiction.

Hence, let $x_t^+ = x_s$, $y_m^+ = y_t$ and $V(x_t) = V(y_t)$. Now we have one of the cycles $v_1v_2x_tx_t^+\cdots x_m^-y_ty_t^+\cdots y_sv_1$ and $v_1v_2x_tx_t^+\cdots x_m^{--}y_ty_t^+\cdots y_sv_1$. Let this cycle be called C_1''. If we are not in the case that $x_m^+ = x_t$, $y_m^- = y_s$ and $V(x_m) = V(y_m)$, then D contains one of the cycles $v_3x_mx_m^+\cdots x_t^-y_s^+y_s^{++}\cdots y_mv_3$, $v_3x_mx_m^+\cdots x_t^{--}y_s^+y_s^{++}\cdots y_mv_3$ and $v_3x_mx_m^+\cdots x_t^-y_s^{++}\cdots y_mv_3$. This cycle and C_1'' are weakly complementary cycles, a contradiction.

Hence, let $x_m^+ = x_t$, $y_m^- = y_s$ and $V(x_m) = V(y_m)$. Now we have the cycle $C_1''' = v_2v_3x_mx_tv_2$. If we are not in the case that $x_s^+ = x_m$, $y_t^+ = y_s$ and $V(x_s) = V(y_s)$, then D contains one of the cycles $v_1x_sx_s^+\cdots x_m^-y_t^+y_t^{++}\cdots y_sv_1$, $v_1x_sx_s^+\cdots x_m^{--}y_t^+y_t^{++}\cdots y_sv_1$ and $v_1x_sx_s^+\cdots x_m^-y_t^{++}\cdots y_sv_1$. This cycle and C_1''' are weakly complementary cycles a contradiction.

Hence, let $x_s^+ = x_m$, $y_t^+ = y_s$ and $V(x_s) = V(y_s)$. But this implies that $|V(C_1)| = |V(C_2)| = 3$ and C_1 and C_2 consist of vertices from the same partite sets. Consequently we arrive at the weakly complementary cycles C_1 and C, also a contradiction.

Subcase 1.2.1.2. Assume that x_t is on the path along C_1 from x_s to x_m and that y_m is on the path along C_2 from y_s to y_t. Then there is one of the cycles $v_2x_tx_t^+\cdots x_s^-y_m^+y_m^{++}\cdots y_tv_2$ and $v_2x_tx_t^+\cdots x_s^{--}y_m^+y_m^{++}\cdots y_tv_2$, say this cycle is C_1'. Furthermore, there is one of the cycles $v_3v_1x_sx_s^+\cdots x_t^-y_t^+y_t^{++}\cdots y_mv_3$ and $v_3v_1x_sx_s^+\cdots x_t^-y_t^{++}\cdots y_mv_3$. This cycle and C_1' are weakly complementary cycles, a contradiction.

Subcase 1.2.1.3. Assume that x_t is on the path along C_1 from x_s to x_m and that y_m is on the path along C_2 from y_t to y_s. Then there is one of the cycles $v_2x_tx_t^+\cdots x_s^-y_m^+y_m^{++}\cdots y_tv_2$ and $v_2x_tx_t^+\cdots x_s^-y_m^{++}\cdots y_tv_2$. Say this cycle is C_1'. If we are not in the case that $x_s^+ = x_t$, $y_t^+ = y_m$ and $V(x_s) = V(y_m)$, then D has one of the cycles $v_3v_1x_sx_s^+\cdots x_t^-y_t^+y_t^{++}\cdots y_mv_3$, $v_3v_1x_sx_s^+\cdots x_t^{--}y_t^+y_t^{++}\cdots y_mv_3$ and $v_3v_1x_sx_s^+\cdots x_t^-y_t^{++}\cdots y_mv_3$. This cycle and C_1' are weakly complementary cycles of D, a contradiction.

Hence, let $x_s^+ = x_t$, $y_t^+ = y_m$ and $V(x_s) = V(y_m)$. Now we have one of the cycles $v_1x_sx_s^+\cdots x_m^-y_my_m^+\cdots y_sv_1$ and $v_1x_sx_s^+\cdots x_m^{--}y_my_m^+\cdots y_sv_1$. Let this cycle be called C_1''. If we are not in the case that $x_m^+ = x_s$, $y_s^+ = y_t$ and $V(x_m) = V(y_t)$, then there is one of the cycles $v_2v_3x_mx_m^+\cdots x_s^-y_s^+y_s^{++}\cdots y_tv_2$, $v_2v_3x_mx_m^+\cdots x_s^{--}y_s^+y_s^{++}\cdots y_tv_2$ and $v_2v_3x_mx_m^+\cdots x_s^-y_s^{++}\cdots y_tv_2$. This cycle and C_1'' are weakly complementary cycles, a contradiction.

Hence, let $x_m^+ = x_s$, $y_s^+ = y_t$ and $V(x_m) = V(y_t)$. Then D contains the cycle $C_1''' = v_3x_my_mv_3$. If we are not in the case that $x_t^+ = x_m$, $y_m^+ = y_s$ and $V(x_t) = V(y_s)$, then there is one of the cycles $v_1v_2x_tx_t^+\cdots x_m^-y_m^+y_m^{++}\cdots y_sv_1$, $v_1v_2x_tx_t^+\cdots x_m^{--}y_m^+y_m^{++}\cdots y_sv_1$ and $v_1v_2x_tx_t^+\cdots x_m^-y_m^{++}\cdots y_sv_1$. This cycle and C_1''' are weakly cycle complementary.

Hence, let $x_t^+ = x_m$, $y_m^+ = y_s$ and $V(x_t) = V(y_s)$. This is possible only if $|V(C_1)| = |V(C_2)| = 3$ and both cycles contain vertices from the same partite sets. Consequently, we deduce that C and C_1 are weakly complementary cycles, a contradiction.

Subcase 1.2.1.4. Assume that x_t is on the path along C_1 from x_m to x_s and that y_m is on the path along C_2 from y_t to y_s. Then D contains $v_1v_2x_tx_t^+\cdots x_m^-y_m^+y_m^{++}\cdots y_sv_1$ or $v_1v_2x_tx_t^+\cdots x_m^{--}y_m^+y_m^{++}\cdots y_sv_1$, say this cycle is C_1'. Furthermore, there is one of the cycles $v_3x_mx_m^+\cdots x_t^-y_s^+y_s^{++}\cdots y_mv_3$ and $v_3x_mx_m^+\cdots x_t^-y_s^{++}\cdots y_mv_3$. This cycle and C_1' are weakly complementary cycles, a contradiction.

Subcase 1.2.2. Assume that $p \geq 3$. If all partite sets appearing in $D_2\cup D_3\cup\cdots\cup D_{p-1}$ also appear in D_1 or D_p or $V(C)$, then we find the same weakly complementary cycles as in Subcase 1.2.1. If there are vertices of new partite sets in $D_2 \cup D_3 \cup \cdots \cup D_{p-1}$, then it is easy to see that these vertices can be inserted into the cycles of Subcase 1.2.1.

Case 2. Assume that there is exactly one vertex set D_i $(1 \leq i \leq p)$, which induces a non-trivial strong component of $D - V(C)$.

Subcase 2.1. Assume that $1 < i < p$. Let C_i be an arbitrary GodOel-cycle of D_i and let $v_j' \in D_j$ for all $j \in \{1,2,\cdots,p\}\setminus\{i\}$. If $p \geq 4$ or $V(v_1') \neq V(v_p')$, then D contains one of the cycles $v_1'v_2'\cdots v_{i-1}'v_{i+1}'v_{i+2}'\cdots v_p'v_1v_2v_3v_1'$, $v_1'v_2'\cdots v_{i-2}'v_{i+1}'v_{i+2}'\cdots v_p'v_1v_2v_3v_1'$ and $v_1'v_2'\cdots v_{i-1}'v_{i+2}'v_{i+3}'\cdots v_p'v_1v_2v_3v_1'$. This cycle and C_i are weakly complementary cycles, a contradiction. Hence, let $p = 3$ and $V(v_1') = V(v_3')$.

Suppose that $|D_1|, |D_3| \geq 2$ with $\{v_1', v_1''\} \subseteq D_1$ and $\{v_3', v_3''\} \subseteq D_3$. Let $x_i \in V(C_i)$ be arbitrary such that $x_i \notin V(v_1')$. Then D contains the cycle $C_1' = v_1'x_iv_3'v_1v_1'$. Furthermore there is one of the cycles $v_1''x_i^+x_i^{++}\cdots x_i^-v_3''v_2v_3v_1''$, $v_1''x_i^{++}\cdots x_i^-v_3''v_2v_3v_1''$, $v_1''x_i^+x_i^{++}\cdots x_i^{--}v_3''v_2v_3v_1''$ and $v_1''x_i^{++}\cdots x_i^{--}v_3''v_2v_3v_1''$. This cycle and C_1' are weakly complementary in D, a contradiction.

It follows that $|D_1| = 1$ or $|D_3| = 1$. Without loss of generality, let $D_3 = \{v_3'\}$. Suppose that $|D_1| \geq 2$ and $\{v_1', v_1''\} \subseteq D_1$. Because of $\kappa(D) \geq 3$ we conclude that there are vertices $x_j \in D_2$ and $v_m \in V(C)$ such that $x_j \to v_m$, say $x_j \to v_1$. Let $C_2 = C_2(x_j)$. If $x_j \notin V(v_1')$, then there is the cycle $C_1' = v_1'x_jv_1v_1'$. Furthermore D contains one of the cycles $v_1''x_j^+x_j^{++}\cdots x_j^-v_3'v_2v_3v_1''$, $v_1''x_j^{++}\cdots x_j^-v_3'v_2v_3v_1''$, $v_1''x_j^+x_j^{++}\cdots x_j^{--}v_3'v_2v_3v_1''$ and $v_1''x_j^{++}\cdots x_j^{--}v_3'v_2v_3v_1''$. This cycle and C_1' are weakly complementary cycles, a contradiction. If $x_j \in V(v_1')$, then D contains the weakly complementary cycles C and C_2, also a contradiction.

Hence, it remains to treat the case that $D_1 = \{v_1'\}$ and $D_3 = \{v_3'\}$. Let C_2 be a GodOel-cycle of $D[D_2]$. If there is a vertex $v_2' \in D_2$ such that $v_1' \in V(v_2')$, then C and C_2 are two weakly complementary cycles a contradiction. Consequently, let $v_1' \notin V(v_2')$ for all $v_2' \in D_2$. Suppose that there is a vertex $v_2' \in D_2 - V(C_2)$. Then D contains the weakly complementary cycles C_2 and $v_1v_1'v_2'v_3'v_2v_3v_1$, also a contradiction. Hence, let $D_2 = V(C_2)$. Since $\kappa(D) = 3$ there are vertices $x_j \in D_2$ and $v_m \in V(C)$ such that $x_j \to v_m$. If there is a vertex $x_p \in D_2$ with $x_p \neq x_j$ and $x_p \neq x_j^+$ such that $v_m^+ \to x_p$ or $v_m \to x_p$, then D contains the weakly complementary cycles $x_px_p^+\cdots x_jv_mv_m^+x_j$ and $v_1'x_j^+x_j^{++}\cdots x_p^-v_3'v_m^-v_1'$ or $x_px_p^+\cdots x_jv_mx_p$ and $v_1'x_j^+x_j^{++}\cdots x_p^-v_3'v_m^+v_m^-v_1'$, in both cases a contradiction. If $v_m^+ \to x_j$, then we find the two weakly complementary cycles $x_j v_m v_m^+ x_j$ and $v_1' x_j^+ x_j^{++} \cdots x_j^- v_3' v_m^+ v_m^- v_1'$, also a contradiction. Altogether, we see that $(D_2 - \{x_j^+\}) \rightsquigarrow \{v_m, v_m^+\}$. Since every vertex of $V(C)$ has an outer neighbor in D_2 we conclude that $\{v_m, v_m^+\} \to x_j^+$. Moreover, we obviously have $x_j \to v_m^+$ or $x_j^- \to v_m^+$ and $x_j \in V(v_m^+)$.

First, let $x_j \to v_m^+$. Analogously as above and noticing that $v_m^{++} = v_m^-$ we deduce that in this case $(D_2 - \{x_j^+\}) \rightsquigarrow v_m^- \to x_j^+$, and thus

$$\{v_m, v_m^+, v_m^-\} \to x_j^+ \to x_j^{++} \rightsquigarrow \{v_m, v_m^+, v_m^-\}.$$

Now $D - \{v_1', x_j^+\}$ is not strong, a contradiction to $\kappa(D) = 3$.

Second, let $x_j^- \to v_m^+$ and $x_j \in V(v_m^+)$. Analogously as above and noticing that $v_m^{++} = v_m^-$ we conclude that $(D_2 - \{x_j\}) \rightsquigarrow v_m^- \to x_j$, and thus D contains the weakly complementary cycles $x_j^- v_m^+ x_j^+ x_j^{++} \cdots x_j^-$ and $v_1' x_j v_3' v_m^- v_m v_1'$, a contradiction.

Subcase 2.2. Assume that $i = 1$. If there is an arc leading from $V(C)$ to D_2, say $v_3 \to v_2'$, then there are the weakly complementary cycles $v_1 v_2 v_3 v_2' v_3' \cdots v_p' v_1$ and the GodOel-cycle of D_1, a contradiction. Hence, let $D_2 \rightsquigarrow V(C)$. If there are vertices $v_m \in V(C)$ and $v_1' \in D_1$ such that $v_m \to v_1'$ and v_1' is not contained in a GodOel-cycle C_1 of $D[D_1]$, then C_1 and $v_m v_1' v_2' \cdots v_p' v_m^+ v_m^- v_m$ are two weakly complementary cycles of D, also a contradiction. Consequently, let all vertices $v_1' \in D_1$ that are outer neighbors of a vertex of $V(C)$ be on every GodOel-cycle of $D[D_1]$.

Subcase 2.2.1. Assume that $p \geq 3$. Furthermore, let $C' = x_1 x_2 \cdots x_l x_1$ be a GodOel-cycle of $D[D_1]$. Since $\kappa(D) = 3$ there are three pairwise non-incident arcs leading from $V(C)$ to D_1, say $v_1 \to x_s$, $v_2 \to x_t$ and $v_3 \to x_u$.

Subcase 2.2.1.1. Assume that x_u is on the path along C' from x_s to x_t. If $v_3 \notin V(v_2')$, then there is one of the cycles $v_1 x_s x_s^+ \cdots x_t^- v_2' v_3 v_1$ and $v_1 x_s x_s^+ \cdots x_t^{--} v_2' v_3 v_1$, and if $v_3 \in V(v_2')$, then D contains one of the cycles $v_1 x_s x_s^+ \cdots x_t^- v_2' v_1$ and $v_1 x_s x_s^+ \cdots x_t^{--} v_2' v_1$. In any case, let this cycle be called C''. If $x_s^- \notin V(v_p')$ or if $x_s^- \in V(v_p')$ and $p \geq 4$, then C'' and $v_2 x_t x_t^+ \cdots x_s^- v_3' v_4' \cdots v_p' v_2$ or $v_2 x_t x_t^+ \cdots x_s^- v_4' \cdots v_p' v_2$ are two weakly complementary cycles, a contradiction. If $p = 3$, $x_s^- \in V(v_3')$ and $x_s^- \neq x_t$, then C'' and $v_2 x_t x_t^+ \cdots x_s^{--} v_3' v_2$ are weakly complementary cycles of D, also a contradiction. Consequently, let $p = 3$, $x_s^- \in V(v_3')$ and $x_s^- = x_t$. If $v_2 \in V(v_2')$, then C and C' are two weakly complementary cycles of D, a contradiction. Hence, let $v_2 \notin V(v_2')$. But now we arrive at the weakly complementary cycles $v_1 x_s x_s^+ \cdots x_t^- v_3' v_3 v_1$ and $v_2 x_t v_2' v_2$, a contradiction.

Subcase 2.2.1.2. Assume that x_u is on the path along C' from x_t to x_s.

First, let $v_2' \in V(v_2)$. Then D contains one of the cycles $v_3 x_u x_u^+ \cdots x_t^- v_2' v_3$ and $v_3 x_u x_u^+ \cdots x_t^{--} v_2' v_3$. Let this cycle be called C''. If $p \geq 4$ or if $p = 3$ and $x_u^- \notin V(v_3')$, then C'' and $v_2 x_t x_t^+ \cdots x_u^- v_3' v_4' \cdots v_p' v_1 v_2$ or $v_2 x_t x_t^+ \cdots x_u^- v_4' \cdots v_p' v_1 v_2$ are two weakly complementary cycles, a contradiction. In the remaining case that $p = 3$ and $x_u^- \in V(v_3')$ the GodOel-cycle in $D[D_1]$ and the cycle C are two weakly complementary cycles of D, also a contradiction.

Second, let $v_2' \notin V(v_2)$. Then D contains one of the cycles $v_2x_tx_t^+\cdots x_s^-v_2'v_2$ and $v_2x_tx_t^+\cdots x_s^{--}v_2'v_2$. Let this cycle be called $\tilde{C}$. If $x_t^- \notin V(v_p')$ or if $x_t^- \in V(v_p')$ and $p \geq 4$, then $\tilde{C}$ and $v_1x_sx_s^+\cdots x_t^-v_3'v_4'\cdots v_p'v_3v_1$ or $v_1x_sx_s^+\cdots x_t^-v_4'\cdots v_p'v_3v_1$ are two weakly complementary cycles, a contradiction. If $p = 3$, $x_t^- \in V(v_3')$ and $x_t^- \neq x_s$, then $\tilde{C}$ and $v_1x_sx_s^+\cdots x_t^{--}v_3'v_3v_1$ are weakly complementary cycles, also a contradiction. Hence, let $p = 3$, $x_t^- \in V(v_3')$ and $x_t^- = x_s$. If $v_3 \in V(v_2')$ or $x_s \in V(v_2')$, then C and C' are two weakly complementary cycles, a contradiction. Otherwise if $v_3, x_s \notin V(v_2')$, then $v_2x_tx_t^+\cdots x_s^-v_3'v_2$ and $v_1x_sv_2'v_3v_1$ are weakly complementary cycles, again a contradiction.

Subcase 2.2.2. Assume that $p = 2$. If there are vertices $v_2' \in D_2$ and $v_1' \in D_1$ such that $v_2' \in V(v_1')$, then C and the GodOel-cycle of $D[D_1]$ are weakly complementary cycles, a contradiction. Consequently, we have $D_1 \to D_2 \to V(C)$. Let $C' = x_1x_2\cdots x_lx_1$ be the GodOel-cycle of $D[D_1]$. If there are vertices $v_i \in V(C)$ and $v_1' \in (D_1 - V(C'))$ such that $v_i \to v_1'$, then C' and $v_iv_1'v_2'v_i^+v_i^-v_i$ are weakly complementary cycles, a contradiction. Consequently, let $(D_1 - V(C')) \rightsquigarrow V(C)$. Because of $\kappa(D) = 3$ we observe that there are three pairwise non-incident arcs leading from $V(C)$ to $V(C')$, say $v_1 \to x_s$, $v_2 \to x_t$ and $v_3 \to x_u$. If $|D_2| \geq 2$ with $\{v_2', v_2''\} \subseteq D_2$, then D contains the weakly complementary cycles $v_1x_sx_s^+\cdots x_t^-v_2'v_3v_1$ and $v_2x_tx_t^+\cdots x_s^-v_2''v_2$, a contradiction. Hence, let $|D_2| = 1$.

Suppose that there are vertices v_i, x_j and x_s with $i \in \{1,2,3\}$, $j \in \{1,2,\cdots,l\}$ and $s \in \{1,2,\cdots,l\} \setminus \{j-2, j-1, j\}$ such that $v_i \to x_j$ and $x_{j-1} \to x_s$. Then D contains the weakly complementary cycles $x_{j-1}x_sx_{s+1}\cdots x_{j-1}$ and $v_ix_jx_{j+1}\cdots x_{s-1}v_2'v_i^+v_i^-v_i$, a contradiction. This leads to

Claim 1. If $l \geq 4$ and $x_j \in V(C')$ is the outer neighbor of a vertex of $V(C)$, then it follows that $x_s \rightsquigarrow x_{j-1}$ for all $s \in \{1,2,\cdots,l\} \setminus \{j-2, j-1, j\}$.

Claim 1 immediately implies the following claim.

Claim 2. If $l \geq 4$ and there are vertices $v_i, v_m \in V(C)$ and $x_j, x_p \in V(C')$ with $j \notin \{p+1, p, p-1\}$ such that $v_i \to x_j$ and $v_m \to x_p$, then it follows that $V(x_{j-1}) = V(x_{p-1})$.

Let $\{x_s, x_t, x_u\} = \{x_{p_1}, x_{p_2}, x_{p_3}\}$. If we pass the vertices of $V(C')$ along its orientation, then, without loss of generality, let this three vertices be appear in the order $x_{p_1}, x_{p_2}, x_{p_3}$ such that $p_1 - p_3 \geq \max\{p_3 - p_2, p_2 - p_1\}$ (all indices taken modulo l). Furthermore, let $\{v_i, v_j, v_m\} = \{v_1, v_2, v_3\}$ such that $v_i \to x_{p_1}$, $v_j \to x_{p_2}$ and $v_m \to x_{p_3}$. If $p_1 \geq p_3 + 3$ (modulo l), then Claim 2 implies that $V(x_{p_3-1}) = V(x_{p_1-1})$ and with Claim 1 we conclude that $x_{p_1-2} \to x_{p_3-1}$. Now we observe that D contains the weakly complementary cycles $x_{p_1-2}x_{p_3-1}x_{p_3}x_{p_3+1}\cdots x_{p_1-2}$ and $v_ix_{p_1}x_{p_1+1}\cdots x_{p_3-2}v_2'v_i^+v_i^-v_i$, a contradiction. Hence, let $p_1 \leq p_3 + 2$ (modulo l). This immediately yields that $l \leq 6$.

Subcase 2.2.2.1. Assume that $l = 6$. In this case $p_1 \leq p_3 + 2$ (modulo l) implies that, without loss of generality, $p_1 = 1$, $p_2 = 3$ and $p_3 = 5$. Applying Claim 2, we see that $V(x_2) = V(x_4) = V(x_6)$. Furthermore, with Claim 1 it follows that $x_1 \to x_4$.

Now it is obvious that D contains the weakly complementary cycles $x_1x_4x_5x_6x_1$ and $v_jx_3v_2'v_j^+v_j^-v_j$, a contradiction.

Subcase 2.2.2.2. Assume that $l = 5$. Then the fact that $p_1 \leq p_3 + 2$ (modulo l) implies that, without loss of generality, $p_1 = 1$, $p_2 = 3$ and $p_3 = 4$ or $p_1 = 1$, $p_2 = 2$ and $p_3 = 4$. If $p_1 = 1$, $p_2 = 3$ and $p_3 = 4$, then Claim 2 implies that $V(x_3) = V(x_5)$ and $V(x_2) = V(x_5)$, a contradiction. And if $p_1 = 1$, $p_2 = 2$ and $p_3 = 4$, then Claim 2 yields $V(x_3) = V(x_1)$ and $V(x_3) = V(x_5)$, also a contradiction.

Subcase 2.2.2.3. Assume that $l = 4$. Without loss of generality, we may suppose that $p_1 = 1$, $p_2 = 2$ and $p_3 = 3$. Claim 2 yields $V(x_2) = V(x_4)$ and with Claim 1 we conclude that $x_3 \to x_1$ or $x_3 \in V(x_1)$.

First, let $x_3 \to x_1$. Then we conclude that $x_4 \leadsto V(C)$ since otherwise if there is a vertex $v_l \in V(C)$ such that $v_l \to x_4$, then D contains the weakly complementary cycles $v_lx_4v_2'v_l^+v_l^-v_l$ and $x_1x_2x_3x_1$, a contradiction. Let $v_m \in V(C)$ such that $v_m \to x_2$. Then D contains the cycle $C'' := v_mx_2x_3x_4v_m$. If $v_m^- \to x_1$, then C'' and $v_m^-x_1v_2'v_m^+v_m^-$ are weakly complementary cycles, a contradiction. Hence, let $x_1 \leadsto v_m^-$, and thus $v_m^- \to x_3$ and $v_m^+ \to x_1$. If $x_4 \to v_m^-$, then $x_4v_m^-x_3x_4$ and $v_mv_m^+x_1v_2'v_m$ contain all partite sets of D, a contradiction. Consequently, we have $V(x_4) = V(v_m^-)$. But now D contains the weakly complementary cycles C'' and $v_m^+x_1v_2'v_m^+$, a contradiction.

Second, let $x_3 \in V(x_1)$. Suppose that there is a vertex $v_1' \in (D_1 - V(C'))$. Then v_1' is on a GodOel-cycle $\tilde{C}$ of $D[D_1]$. If $|V(\tilde{C})| \geq 6$, then with Subcase 2.2.2.1 we arrive at a contradiction. Hence, let $|V(\tilde{C})| = 4$. Since $(D_1 - V(\tilde{C})) \leadsto V(C)$ and $(D_1 - V(C')) \leadsto V(C)$, we conclude that $v_1' \in V(x_4)$ and $V(\tilde{C}) = (V(C') \cup \{v_1'\}) \setminus \{x_4\}$. So in any case, if $D_1 - V(C')$ is empty or not, we observe that $d^+_{D-V(C)}(x_2) = d^+_{D-V(C)}(x_4) = 2$, and thus x_2 as well as x_4 has an outer neighbor in $V(C)$. Without loss of generality, let $v_1 \to x_1$.

Assume that $v_2 \to x_3$ and $v_3 \to x_2$. If $x_3 \to v_1$, then the cycles $x_3v_1v_2x_3$ and $v_3x_2v_2'v_3$ contain vertices from all partite sets of D, a contradiction. Hence, let $v_1 \to x_3$. If $x_4 \to v_1$, then D contains the weakly complementary cycles $x_4v_1x_3x_4$ and $v_2v_3x_2v_2'v_2$, also a contradiction. Consequently, we have $v_1 \leadsto x_4$. If $x_4 \to v_3$ and $v_1 \notin V(x_4)$, then $x_4v_3v_1x_4$ and $v_2x_3v_2'v_2$ are weakly complementary cycles, a contradiction. If $v_1 \in V(x_4)$ and $x_1 \to v_2$, then $x_1v_2x_3x_4x_1$ and $v_3x_2v_2'v_3$ are weakly complementary cycles, a contradiction. Hence let $v_2 \to x_1$ in this case. If $x_4 \to v_3$ and $v_1 \in V(x_4)$, then the cycles $x_4v_3x_2x_3x_4$ and $v_2x_1v_2'v_2$ contain vertices from all partite sets, also a contradiction. Hence, in all cases we observe that $v_3 \to x_4$. Consequently, v_2 is an outer neighbor of x_4. But now D contains the weakly complementary cycles $x_4v_2v_3x_4$ and $v_1x_1v_2'v_1$, a contradiction.

Assume that $v_2 \to x_2$ and $v_3 \to x_3$. If $x_1 \to v_3$, then D contains the weakly complementary cycles $x_1v_3v_1x_1$ and $v_2x_2v_2'v_2$, a contradiction. Hence, let $v_3 \to x_1$. If $x_4 \to v_3$, then $x_4v_3x_3x_4$ and $v_1v_2x_2v_2'v_1$ are weakly complementary cycles of D, also a contradiction. Consequently, we deduce that $v_3 \leadsto x_4$. If $x_2 \to v_1$, then D contains the cycles $v_1v_2x_2v_1$ and $v_3x_1v_2'v_3$, a contradiction. Hence, let $v_1 \leadsto x_2$. If $v_1 \in V(x_2)$, then v_2 has to be an outer neighbor of x_4 and $x_4v_2x_2x_3x_4$ and

$v_3x_1v_2'v_3$ contain vertices from all partite sets of D, a contradiction. Hence, let $v_1 \notin V(x_2)$, and thus $v_1 \to x_2$. If $x_4 \to v_1$, then $x_4v_1x_2x_3x_4$ and $v_3x_1v_2'v_2v_3$ are weakly complementary cycles of D, a contradiction. Consequently, let $v_1 \to x_4$ and v_2 is an outer neighbor of x_4. This yields the weakly complementary cycles $x_4v_2v_3x_3x_4$ and $v_1x_2v_2'v_1$, a contradiction.

Subcase 2.2.2.4. Assume that $l = 3$.

First, let there be a vertex $v_1' \in (D_1 - V(C'))$. Then v_1' is on a GodOel-cycle $\tilde{C}$ of $D[D_1]$. If $|V(\tilde{C})| \geq 4$, then the previous subcases yield a contradiction. Hence, let $|V(\tilde{C})| = 3$, and thus $x_i \in (V(C') - V(\tilde{C}))$ for an $1 \leq i \leq 3$. If $v_j \to x_i$, then $v_jx_iv_2'v_j^+v_j^-v_j$ and $\tilde{C}$ are weakly complementary cycles, a contradiction.

Second, let $D_1 = V(C')$. In this case, D is a tournament of order seven and Theorem 2.2 implies that D is cycle complementary and thus weakly cycle complementary unless D is the tournament T_7 of Example 2.1. The tournament T_7 is 3-connected and not (weakly) cycle complementary.

Subcase 2.3. Assume that $i = p$. Observing the converse D^{-1} of D Subcase 2.2 yields a contradiction.

Case 3. Assume that D_i is a set of independent vertices for all $1 \leq i \leq p$. As seen above, it follows that $D_p \to V(C) \to D_1$, and obviously we have $D_i \to D_{i+1}$ for all $1 \leq i \leq p-1$. In the following, if we speak of a vertex v_i', then we mean that $v_i' \in D_i$.

First, we assume that $|D_1| \geq 2$ and $|D_p| \geq 2$ such that $\{v_1', v_1''\} \subseteq D_1$ and $\{v_p', v_p''\} \subseteq D_p$. If the vertices of D_1 and D_p belong to different partite sets, then D contains the weakly complementary cycles $v_1'v_p'v_1v_1'$ and $v_1''v_2'v_3' \cdots v_{p-1}'v_p''v_2v_3v_1''$, a contradiction. Hence, we have to analyze the case that $v_1' \in V(v_p')$. The fact that $\kappa(D) = 3$ implies that $|D_2 \cup D_3 \cup \cdots D_{p-1}| \geq 3$. If $p = 3$, then $v_1'v_2'v_3'v_1v_1'$ and $v_1''v_2''v_3''v_2v_3v_1''$ are weakly complementary cycles, if $p = 4$, then $v_1'v_2'v_4'v_1v_1'$ and $v_1''v_3'v_4''v_2v_3v_1''$ are weakly complementary cycles, and if $p \geq 5$, then $v_1'v_2'v_p'v_1v_1'$ and $v_1''v_3'v_4' \cdots v_{p-1}'v_p''v_2v_3v_1''$ or $v_1''v_4'v_5' \cdots v_{p-1}'v_p''v_2v_3v_1$ are weakly complementary cycles, in all cases a contradiction. Hence, it follows that $|D_1| = 1$ or $|D_p| = 1$. Without loss of generality, let $|D_p| = 1$. The fact that $\kappa(D) = 3$ yields that $|D_2 \cup D_3 \cup \cdots \cup D_{p-1}| \geq 2$.

Second, we assume that $|D_1| \geq 2$ such that $\{v_1', v_1''\} \subseteq D_1$. The fact that $\kappa(D) = 3$ implies that there are vertices $v_{p-1}' \in D_{p-1}$ and $v_j \in V(C)$, say $j = 1$, such that $v_{p-1}' \to v_1$. If the vertices of D_1 and D_p belong to different partite sets, then $v_1'v_p'v_3v_1'$ and $v_1''v_2'v_3' \cdots v_{p-1}'v_1v_2v_1''$ contain vertices from all partite sets of D, a contradiction. Hence, let $v_1' \in V(v_p')$. If $p \geq 4$, then $v_1'v_2'v_p'v_3v_1'$ and $v_1''v_3'v_4' \cdots v_{p-1}'v_1v_2v_1''$ or $v_1''v_4' \cdots v_{p-1}'v_1v_2v_1''$ are weakly complementary cycles of D and if $p = 3$, and thus $|D_2| \geq 2$, then $v_1'v_2''v_3'v_2v_3v_1'$ and $v_1''v_2'v_1v_1''$ are weakly complementary cycles, in all cases a contradiction.

Consequently, it remains to treat the case that $|D_1| = |D_p| = 1$. Suppose that $p = 3$. Since $\kappa(D) = 3$, we conclude that a vertex $v_2' \in D_2$ has at least two outer and two inner neighbors in $V(C)$, a contradiction. Hence, let $p \geq 4$.

Subcase 3.1. Assume that there are vertices $v_i \in V(C)$, $v'_k \in D_k$ and $v'_l \in D_l$ with $2 \le l \le k \le p-1$ such that $v'_k \to v_i \to v'_l$. In this case let k and l be chosen such that $k-l$ is minimal. Obviously, D contains the cycle $C' = v'_l v'_{l+1} \cdots v'_k v_i v'_l$. If $V(v'_{l-1}) \ne V(v'_{k+1})$, then C' and $v'_1 v'_2 \cdots v'_{l-1} v'_{k+1} v'_{k+2} \cdots v'_p v_i^+ v_i^- v'_1$ are weakly complementary cycles, a contradiction. Hence let $V(v'_{l-1}) = V(v'_{k+1})$. If $l-1 \ne 1$, then C' and $v'_1 v'_2 \cdots v'_{l-2} v'_{k+1} v'_{k+2} \cdots v'_p v_i^+ v_i^- v'_1$ are weakly complementary cycles, and if $k+1 \ne p$, then C' and $v'_1 v'_2 \cdots v'_{l-1} v'_{k+2} v'_{k+3} \cdots v'_p v_i^+ v_i^- v'_1$ contain vertices from all partite sets of D, in both cases a contradiction. Consequently, it remains to treat the case that $l = 2$, $k = p-1$ and $V(v'_1) = V(v'_p)$.

If $|D_m| \ge 2$ with $m \in \{2, p-1\}$ and $v''_m \in D_m - \{v'_m\}$, then C' and $v'_1 v''_m v'_p v_i^+ v_i^- v'_1$ are weakly complementary cycles, a contradiction. Hence let $D_2 = \{v'_2\}$ and $D_p = \{v'_p\}$. If $p \ge 6$ or $p = 5$ and $v'_3 \notin V(v_i)$, then we arrive at a contradiction to the minimality of $k-l$. If $p = 5$, $v'_3 \in V(v_i)$ and $v'_2 \notin V(v'_4)$, then D contains the weakly complementary cycles $v'_2 v'_4 v_i v'_2$ and $v'_1 v'_3 v'_5 v_i^+ v_i^- v'_1$, a contradiction. If $p = 5$, $v'_3 \in V(v_i)$ and $v'_2 \in V(v'_4)$, then the fact that $D - \{v'_4, v'_5\}$ is strongly connected implies that there is an arc leading from v'_3 to $\{v_i^+, v_i^-\}$. If $v'_3 \to v_i^+$, then $v'_1 v'_3 v_i^+ v_i^- v'_1$ and $v'_5 v_i v'_2 v'_5$ are weakly complementary cycles and if $v_i^+ \to v'_3 \to v_i^-$, then $v'_2 v'_3 v_i^- v_i v'_2$ and $v'_1 v'_4 v'_5 v_i^+ v'_1$ contain vertices from all partite sets of D, in both cases a contradiction. Finally, if $p = 4$, then we arrive at the contradiction that $D - \{v'_2, v'_3\}$ is not strong.

Subcase 3.2. Assume that there are vertices $v_i \in V(C)$, $v'_k \in D_k$ and $v'_l \in D_l$ with $2 \le k < l \le p-1$ such that $v'_k \to v_i \to v'_l$. In this case let k and l be chosen such that $l-k$ is minimal. If $v_i^+ \to v'_k$, then $v'_k v_i v_i^+ v'_k$ and $v'_1 v'_2 \cdots v'_{k-1} v'_{k+1} v'_{k+2} \cdots v'_p v_i^- v'_1$ or $v'_1 v'_2 \cdots v'_{k-1} v'_{k+2} \cdots v'_p v_i^- v'_1$ are two weakly complementary cycles of D, a contradiction. Hence, let $v'_k \rightsquigarrow v_i^+$. Analogously, if $v'_l \to v_i^-$, then we see that $v'_l v_i^- v_i v'_l$ and $v'_1 v'_2 \cdots v'_{l-1} v'_{l+1} v'_{l+2} \cdots v'_p v_i^+ v'_1$ or $v'_1 v'_2 \cdots v'_{l-2} v'_{l+1} v'_{l+2} \cdots v'_p v_i^+ v'_1$ are weakly complementary cycles, also a contradiction. Consequently, let $v_i^- \rightsquigarrow v'_l$. Since $l-k$ is minimal we conclude that $l-k = 1$ or $l-k = 2$ and $v_i \in V(v'_{k+1})$.

First let $l-k = 2$ and $v_i \in V(v'_{k+1})$. If $v'_{k+1} \to v_i^+$, then D contains the weakly complementary cycles $v'_1 v'_2 \cdots v'_{k+1} v_i^+ v_i^- v'_1$ and $v'_l v'_{l+1} \cdots v'_p v_i v'_l$, a contradiction. Hence, let $v_i^+ \to v'_{k+1}$. Analogously, we observe that $v'_{k+1} \to v_i^-$. If $v_i^- \in (V(v'_2) \cup V(v'_3) \cup \cdots \cup V(v'_{p-1}))$, then $v'_1 v'_2 \cdots v'_k v_i v'_1$ and $v'_{k+1} v'_{k+2} \cdots v'_p v_i^+ v'_{k+1}$ contain vertices from all partite sets of D, a contradiction. Consequently, let $v_i^- \notin (V(v'_2) \cup V(v'_3) \cup \cdots \cup V(v'_{p-1}))$, and thus $v_i^- \to v'_l$. If $v_i^- \to v'_k$, then Subcase 3.1 yields a contradiction. Hence let $v'_k \to v_i^-$. But now D contains the weakly complementary cycles $v'_1 v'_2 \cdots v'_k v_i^- v'_1$ and $v'_{k+1} v'_{k+2} \cdots v'_p v_i^+ v'_{k+1}$, a contradiction.

Second let $l = k+1$. If $v'_k \notin V(v_i^+)$ and thus, as seen above, $v'_k \to v_i^+$, then $v'_1 v'_2 \cdots v'_k v_i^+ v_i^- v'_1$ and $v'_{k+1} v'_{k+2} \cdots v'_p v_i v'_{k+1}$ contain vertices from all partite sets of D, a contradiction. Analogously, if $v'_{k+1} \notin V(v_i^-)$ and thus $v_i^- \to v'_{k+1}$, then $v'_1 v'_2 \cdots v'_k v_i v_i^+ v'_1$ and $v'_{k+1} v'_{k+2} \cdots v'_p v_i^- v'_{k+1}$ are weakly complementary cycles of D, also a contradiction. Consequently, let $v'_k \in V(v_i^+)$ and $v'_{k+1} \in V(v_i^-)$. If

$v_i^+ \to v'_{k+1}$, then D contains the cycles $v'_1v'_2\cdots v'_kv_iv'_1$ and $v'_{k+1}v'_{k+2}\cdots v'_pv_i^+v'_{k+1}$, a contradiction. If $v'_k \to v_i^-$, then $v'_1v'_2\cdots v'_kv_i^-v'_1$ and $v'_{k+1}v'_{k+2}\cdots v'_pv_iv'_{k+1}$ contain vertices from all partite sets of D, also a contradiction. Hence let $v'_{k+1} \to v_i^+$ and $v_i^- \to v'_k$. Now it follows that D contains the cycle $C' = v_i^-v'_kv'_{k+1}v_i^+v_i^-$. If $k \neq 2$ or $k \neq p-2$ or $k = 2 = p-2$ and $v'_1 \notin V(v'_p)$, then C' and one of the cycles $v'_1v'_2\cdots v'_{k-1}v'_{k+2}v'_{k+3}\cdots v'_pv_iv'_1$ and $v'_1v'_2\cdots v'_{k-2}v'_{k+2}v'_{k+3}\cdots v'_pv_iv'_1$ and $v'_1v'_2\cdots v'_{k-1}v'_{k+3}v'_{k+4}\cdots v'_pv_iv'_1$ are weakly complementary cycles of D, a contradiction. Hence let $k = 2$, $p = 4$ and $v'_1 \in V(v'_4)$. If $|D_m| \geq 2$ with $m \in \{2,3\}$, then C' and $v'_1v''_mv'_4v_iv'_1$ contain vertices from all partite sets of D, a contradiction. Consequently we have $|D_2| = |D_3| = 1$. Now it is obvious that $D - \{v'_2, v'_3\}$ is not strong, a contradiction to $\kappa(D) = 3$.

Subcase 3.3. Assume that $v_m \rightsquigarrow \bigcup_{j=2}^{p-1} D_j$ or $\bigcup_{j=2}^{p-1} D_j \rightsquigarrow v_m$ for each $m \in \{1,2,3\}$. Since $D - \{v'_1\}$ as well as $D - \{v'_p\}$ are strong, we deduce that there are vertices $v_i, v_j \in V(C)$ with $i \neq j$ such that $v_i \to v'_2$ and $v'_{p-1} \to v_j$, and thus $(D_2 \cup D_3 \cup \cdots \cup D_{p-1}) \rightsquigarrow v_j$ and $v_i \rightsquigarrow (D_2 \cup D_3 \cup \cdots D_{p-1})$. Let $v_m \in V(C) - \{v_i, v_j\}$.

Suppose that $v_j \to v_i$. Then D contains the cycle $C' = v'_2v'_3\cdots v'_{p-1}v_jv_iv'_2$. If $v'_1 \notin V(v'_p)$, then D contains the weakly complementary cycles C' and $v'_1v'_pv_mv'_1$, a contradiction. Hence let $v'_1 \in V(v'_p)$. If $|D_l| \geq 2$ with $l \in \{2, p-1\}$, then C' and $v'_1v''_lv'_pv_mv'_1$ are weakly complementary cycles of D, also a contradiction. Consequently, let $|D_2| = |D_{p-1}| = 1$. If $p \geq 5$, then $v'_1v'_2v'_pv_mv'_1$ and $v_iv'_3v'_4\cdots v'_{p-1}v_jv_i$ or $v_iv'_4v'_5\cdots v'_{p-1}v_jv_i$ contain vertices from all partite sets of D, a contradiction. And if $p = 4$, then $D - \{v'_2, v'_3\}$ is not strong, a contradiction to $\kappa(D) = 3$.

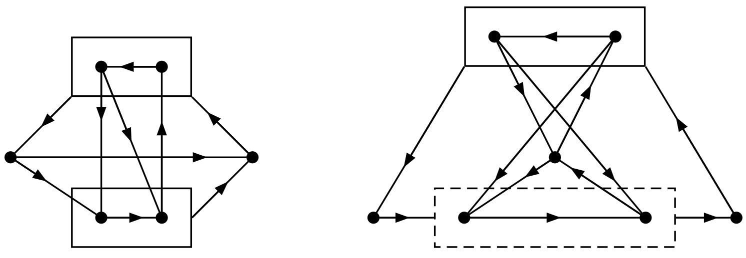
Consequently, it remains to consider the case that $v_i \to v_j \to v_m \to v_i$. If $v_j \notin V(v'_2)$ and $v_i \notin V(v'_{p-1})$, then D contains the weakly complementary cycles $v'_1v'_2v_jv_mv'_1$ and $v'_3v'_4\cdots v'_pv_iv'_3$ or $v'_4v'_5\cdots v'_pv_iv'_4$, a contradiction. If $v_j \in V(v'_2)$ and $v_i \in V(v'_{p-1})$, then it is straightforward to see that $v'_1v'_{p-1}v_jv_mv'_1$ and $v'_2v'_3\cdots v'_{p-2}v'_pv_iv'_2$ or $v'_2v'_3\cdots v'_{p-3}v'_pv_iv'_2$ are two cycles that contain vertices from all partite sets of D, also a contradiction. Hence let $v_j \in V(v'_2)$ and $v_i \to v'_{p-1}$. If $v'_2 \to v_m$, then D contains the weakly complementary cycles $v_iv'_2v_mv_i$ and $v'_1v'_3v'_4\cdots v'_pv_jv'_1$ or $v'_1v'_4v'_5\cdots v'_pv_jv'_1$, a contradiction. Consequently, let $v_m \to v'_2$. Now, we have two vertices $v_j, v_m \in V(C)$ such that $v_m \to v'_2$, $v'_{p-1} \to v_j$ and $v_j \to v_m$. As above we arrive at a contradiction. Finally if $v'_2 \to v_j$ and $v_i \in V(v'_{p-1})$, then observing the converse D^{-1} of D, we also arrive at a contradiction. This completes the proof of this theorem. □

A direct consequence of this theorem is the following result.

Corollary 3.2. *Every 3-strong c-partite tournament with $c \geq 3$ and at least 8 vertices is weakly cycle complementary.*

The following example presents two 2-strong c-partite tournaments with $c \geq 3$ and at least 6 vertices that even do not contain two vertex disjoint cycles, and thus no weakly complementary cycles.

Example 3.3. The 4-partite tournament D_1 and the 3-partite tournament D_2 of Figure 3 are 2-strong and do not have any two vertex disjoint cycles. Thus they do not have two vertex disjoint cycles with vertices from all partite sets, which means that D_1 and D_2 are not weakly cycle complementary.



D_1 D_2

Figure 3: The 2-strong multipartite tournaments D_1 and D_2 without a pair of vertex disjoint cycles.

Since the authors found some more 2-strong multipartite tournaments with this property it may become difficult to characterize all 2-strong multipartite tournaments that are weakly cycle complementary. Nevertheless it would be interesting to solve this problem.

Problem 3.4. Characterize all 2-strong c-partite tournaments D with $c \geq 3$ and $|V(D)| \geq 6$ that are weakly cycle complementary.

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