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A Note on Subnormal and Hyponormal Derivations

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Abstract. In this note we prove that if A and B^* are subnormal operators and X is a bounded linear operator such that $AX - XB$ is a Hilbert-Schmidt operator, then $f(A)X - Xf(B)$ is also a Hilbert-Schmidt operator and

$$||f(A)X - Xf(B)||_2 \leq L\,||AX - XB||_2,$$

for f belonging to a certain class of functions. Furthermore, we investigate the similar problem in the case that S, T are hyponormal operators and $X \in \mathcal{L}(\mathcal{H})$ is such that $SX - XT$ belongs to a norm ideal $(J, ||\cdot||_J)$ and prove that $f(S)X - Xf(T) \in J$ and

$$||f(S)X - Xf(T)||_J \leq C\,||SX - XT||_J,$$

for f in a certain class of functions.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$ and by $\mathcal{C}_2(\mathcal{H})$ the Hilbert-Schmidt class. For $T \in \mathcal{L}(\mathcal{H})$, $\sigma(T)$ denotes the spectrum of T, and for a compact subset $\Sigma \subset C$, $Lip\,(\Sigma)$ denotes the set of Lipschitz functions on Σ. Furthermore, $Rat(\Sigma)$ denotes the algebra of rational functions with poles off Σ, and $R(\Sigma)$ denotes the the closure of $Rat(\Sigma)$ in the supremum norm over Σ.

For operators $A, B \in \mathcal{L}(\mathcal{H})$, the mapping $\Delta_{A,B}(X) = AX - XB$ is called a (generalized) derivation. If A, B are normal (subnormal or co-subnormal, hyponormal or co-hyponormal) operators, then $\Delta_{A,B}$ will be called a normal (subnormal, hyponormal) derivation, respectively.

Next, we recall some theorems that involve normal derivations, and then we extend some of these theorems to the case in which A, B^* are subnormal operators and to the case in which $A = S,\ B = T$ are hyponormal operators.

In [7], a generalization of Fuglede-Putnam theorem for normal operators was proved. For further results concerning normal derivations, the reader can see [8] and [9].

Theorem A ([7]). *If $A,\, B \in \mathcal{L}(\mathcal{H})$ are normal operators and $X \in \mathcal{L}(\mathcal{H})$ satisfies*

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$AX - XB \in \mathcal{C}_2(\mathcal{H})$, then $A^*X - XB^* \in \mathcal{C}_2(\mathcal{H})$ and

$$||AX - XB||_2 = ||A^*X - XB^*||_2.$$

In [3], Furuta extended the above result to subnormal operators.

Theorem B ([3]). *If A, $B^* \in \mathcal{L}(\mathcal{H})$ are subnormal operators and $X \in \mathcal{L}(\mathcal{H})$ satisfies $AX - XB \in \mathcal{C}_2(\mathcal{H})$, then $A^*X - XB^* \in \mathcal{C}_2(\mathcal{H})$ and*

$$||AX - XB||_2 \geq ||A^*X - XB^*||_2.$$

In his paper [4] Kittaneh proved the following theorem using a famous result of Voiculescu [6] according to which every normal operator can be written as the sum of a diagonal operator and a Hilbert-Schmidt operator of arbitrarily small Hilbert-Schmidt norm.

Theorem C ([4]). *Let A, $B \in \mathcal{L}(\mathcal{H})$ be normal operators and $X \in \mathcal{L}(\mathcal{H})$ such that $AX - XB \in \mathcal{C}_2(\mathcal{H})$, and let $f \in \mathrm{Lip}\,(\sigma(A) \cup \sigma(B))$. Then $f(A)X - Xf(B)$ is also a Hilbert-Schmidt operator and*

$$||f(A)X - Xf(B)||_2 \leq L\,||AX - XB||_2,$$

where L is the Lipschitz constant of the function f.

2. Subnormal derivations

In this section we investigate the validity of this inequality in the case that A, B^* are subnormal operators, but with a drawback concerning the extent of the class of functions in which f can run.

The following lemma is elementary and can be easily established making use of the *minimal normal extension* of a subnormal operator. Its proof is left for the reader.

Lemma 1. *If S_1, $S_2 \in \mathcal{L}(\mathcal{H})$ are subnormal operators, then there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and normal operators N_1, $N_2 \in \mathcal{L}(\mathcal{K})$ that are extensions of S_1, S_2, respectively, and $\sigma(N_i) \subseteq \sigma(S_i)$, $i = 1, 2$.*

For a subnormal operator $S \in \mathcal{L}(\mathcal{H})$ and a function $f \in R(\sigma(S))$, one can associate an operator $f(S) \in \mathcal{L}(\mathcal{H})$ as follows. Let $r_n \in Rat(\sigma(S))$, $n \in N$, such that

$$||f - r_n||_{\sigma(S),\infty} \to 0, \quad \text{as } n \to \infty,$$

and let $N_S \in \mathcal{L}(\mathcal{K})$, where $\mathcal{K} \supset \mathcal{H}$, be the minimal normal extension of S. Since $\sigma(N_S) \subseteq \sigma(S)$, we have

$$r_n(N_S) = \begin{pmatrix} r_n(S) & S'_{12} \\ 0 & S'_{22} \end{pmatrix},$$

and $r_n(N_S) \to f(N_S)$ in the operator norm of $\mathcal{L}(\mathcal{K})$. Therefore $r_n(S)$ converges in the operator norm of $\mathcal{L}(\mathcal{H})$ to an operator that will be denoted by $f(S)$. It is obvious that this operator does not depend on the sequence $\{r_n\}$. In a similar way, for $f \in R(\sigma(T))$, one can define $f(T)$, when $T^* \in \mathcal{L}(\mathcal{H})$ is a subnormal operator.

Theorem 2. *Let A, $B^* \in \mathcal{L}(\mathcal{H})$ be subnormal operators and $X \in \mathcal{L}(\mathcal{H})$ such that $AX - XB \in \mathcal{C}_2(\mathcal{H})$, and let $\Sigma = \sigma(A) \cup \sigma(B)$ and $f \in Lip\,(\Sigma) \cap R(\Sigma)$. Then $f(A)X - Xf(B)$ is also a Hilbert-Schmidt operator and*

$$||f(A)X - Xf(B)||_2 \le L\,||AX - XB||_2,$$

where L is the Lipschitz constant of the function f.

Proof. For subnormal operators A, $B^* \in \mathcal{L}(\mathcal{H})$, according to previous lemma, there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and there are some normal operators N_A, $N_{B^*} \in \mathcal{L}(\mathcal{K})$ such that relative to the decomposition of $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^{\perp}$, we have

$$N_A = \begin{pmatrix} A & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad N_{B^*} = \begin{pmatrix} B^* & B_{12} \\ 0 & B_{22} \end{pmatrix},$$

and $\sigma(N_A) \subseteq \sigma(A)$, $\sigma(N_{B^*}) \subseteq \sigma(B^*)$.

If we put $\tilde{X} = X \oplus 0$ on $\mathcal{H} \oplus \mathcal{H}^{\perp}$, then we have $N_A\,\tilde{X} - \tilde{X}\,N_{B^*}^* = (AX - XB) \oplus 0$, and therefore $N_A\,\tilde{X} - \tilde{X}\,N_{B^*}^* \in \mathcal{C}_2(\mathcal{K})$.

For $r \in Rat(\Sigma)$, where $\Sigma = \sigma(A) \cup \sigma(B)$, a simple calculation shows that

$$r(N_A) = \begin{pmatrix} r(A) & A'_{12} \\ 0 & A'_{22} \end{pmatrix} \text{ and } r(N_{B^*}^*) = \begin{pmatrix} r(B) & 0 \\ B'_{21} & B'_{22} \end{pmatrix}. \tag{1}$$

Thus, if $f \in Lip\,(\Sigma) \cap R(\Sigma)$, using a limiting argument, one can see that $f(N_A)$ and $f(N_{B^*}^*)$ have similar matrix representation as in (1), but with f replacing r. According to Theorem C,

$$f(N_A)\tilde{X} - \tilde{X}f(N_{B^*}^*) \in \mathcal{C}_2(\mathcal{K})$$

and

$$||f(N_A)\tilde{X} - \tilde{X}f(N_{B^*}^*)||_2 \le L\,||N_A\tilde{X} - \tilde{X}N_{B^*}^*||_2.$$

Since $f(N_A)\tilde{X} - \tilde{X}f(N_{B^*}^*) = (f(A)X - Xf(B)) \oplus 0$, the proof is finished. $\square$

Corollary 3. *Let A, $B^* \in \mathcal{L}(\mathcal{H})$ be subnormal operators and $X \in \mathcal{L}(\mathcal{H})$ such that $AX - XB \in \mathcal{C}_2(\mathcal{H})$, and let $\Sigma = \sigma(A) \cup \sigma(B)$ and $f \in Lip\,(\Sigma) \cap R(\Sigma)$. Then*

$$||f(A)^*X - Xf(B)^*||_2 \le ||f(A)X - Xf(B)||_2,$$

and thus

$$||f(A)^*X - Xf(B)^*||_2 \le L||AX - XB||_2,$$

where L is the Lipschitz constant of the function f.

Proof. The first inequality is a consequence of Theorem B after observing that $f(A)$ and $f(B)^*$ are subnormal operators. The second inequality follows from Theorem 2. □

3. Hyponormal derivations

In this section we approach the same problem, but in the case in which $A = S$, $B = T$ are hyponormal operators and the Hilbert-Schmidt class replaced with an arbitrary norm ideal.

For a hyponormal operator $T \in \mathcal{L}(\mathcal{H})$, the analytic functional calculus can be extended to a class $A^\alpha(\sigma(T))$ of "pseudo-analytic" functions on $\sigma(T)$ that satisfy a certain growth condition at the boundary.

The extension of the analytic functional calculus for a hyponormal operator was introduced by Dynkin (cf. [1], [2]) and it also can be found in [5].

We briefly review the definition and the main tools that are necessary. Let Σ be a perfect compact set of the complex plane and let α be a positive non-integer with k its integer part, $[\alpha]$. The class $A^\alpha(\Sigma)$ is defined as the set of $(k+1)$-tuples of continuous functions on Σ, $(f_0, \cdots, f_k) : \Sigma \to C^{k+1}$ that are related by

$$f_j(z) = f_j(z_0) + \frac{f_{j+1}(z_0)}{1!}(z - z_0) + \cdots + \frac{f_k(z_0)}{(k-j)!}(z - z_0)^{k-j} + R_j(z_0, z),$$

and

$$|R_j(z_0, z)| \le C_j\, |z - z_0|^{\alpha - j}, \tag{2}$$

for $j = 0, \cdots, k$ and $z, z_0 \in \Sigma$. Since Σ is a perfect set,

$$f_j(z) = \lim_{z \to z_0} \frac{f_{j-1}(z) - f_{j-1}(z_0)}{z - z_0}, \quad j = 0, \cdots, k-1,$$

and thus the $(k+1)$-tuple depends only on f_0. The space $A^\alpha(\Sigma)$ endowed with the maximum of the smallest constants that satisfy (2) plus the supremum norm on Σ of f_0 becomes a unital Banach algebra and is a closed subalgebra of $Lip\,(\alpha, \Sigma)$, the algebra of Lipschitz functions of order α.

Theorem D ([2]). *Let Σ be a perfect compact set, $f \in C(\Sigma)$, and α a positive non-integer. The following are equivalent:*

(*a*) $f \in A^\alpha(\Sigma)$;

(*b*) *f has an extension $F \in C^1(C \setminus \Sigma)$ with $|\overline{\partial} F(z)| \le C \cdot \mathrm{dist}(z, \Sigma)^{\alpha-1}$, $z \notin \Sigma$;*

(*c*) *There exists $\phi \in C_0(C)$ such that*

$$f(z) = \int \frac{\phi(w)}{w - z}\, d\mu(w), \quad z \in \Sigma,$$

and $|\phi(w)| \le C_0 \cdot ||f||_{A^\alpha(\Sigma)} \cdot \mathrm{dist}(w, \Sigma)^{\alpha-1}$, $w \in C$, where μ is planar Lebesgue measure and C_0 is a constant that does not depend on f.

If $T \in \mathcal{L}(\mathcal{H})$ is a hyponormal operator, then $||T|| = ||T||_\sigma$, where $||T||_\sigma$ denotes the spectral radius of T, that is $\underset{z\in\sigma(T)}{\rightarrow}$ sup$|z|$. It is well known that if $z \notin \sigma(T)$, then $(z-T)^{-1}$ is also hyponormal and thus

$$(3) \qquad ||(z-T)^{-1}|| = \frac{1}{\operatorname{dist}(z, \sigma(T))}.$$

Thus, for a hyponormal operator T whose spectrum $\sigma(T)$ is a perfect set and for a function $f \in A^\alpha(\sigma(T))$ with $\alpha > 2$, one can associate an operator defined by

$$\int \phi(w)(w-T)^{-1}\, d\mu(w),$$

that will be denoted by $f(T)$. The above integral does not depend on ϕ, that is the definition of $f(T)$ is not ambiguous, and the mapping $\phi \mapsto f(T)$ acting from $A^\alpha(\sigma(T))$ into $\mathcal{L}(\mathcal{H})$ is a continuous, unital morphism of Banach algebras, and which extends the Riesz-Dunford calculus.

Let $(J, ||\cdot||_J)$ be a norm ideal, that is, a proper two-sided ideal J of $\mathcal{L}(\mathcal{H})$ with a norm $||\cdot||_J$ that satisfies: $(J, ||\cdot||_J)$ is a Banach space and $||AXB||_J \leq ||A||\,||B||\,||X||_J$, for all $X \in J$ and any A, $B \in \mathcal{L}(\mathcal{H})$. In particular, the Shatten-von Neumann p-classes, $\mathcal{C}_p(\mathcal{H})$, for $p \geq 1$, are instances of norm ideals.

Theorem 4. *Let $(J, ||\cdot||_J)$ be a norm ideal, let $S, T \in \mathcal{L}(\mathcal{H})$ be hyponormal operators for which both $\sigma(S)$ and $\sigma(T)$ are perfect sets, let f belong to $A^\alpha(\Sigma)$ with $\alpha > 3$ and $\Sigma = \sigma(S) \cup \sigma(T)$, and let $X \in \mathcal{L}(\mathcal{H})$ such that $SX - XT \in J$. Then $f(S)X - Xf(T) \in J$ and*

$$||f(S)X - Xf(T)||_J \leq C_1 \cdot ||f||_{A^\alpha(\Sigma)} \cdot ||SX - XT||_J,$$

where C_1 is a constant that depends on Σ but it does not depend on f.

Proof. For $f \in A^\alpha(\Sigma)$, according to Theorem D, there exists $\phi \in C_0(C)$ such that

$$f(z) = \int \frac{\phi(w)}{w-z}\, d\mu(w), \quad z \in \Sigma,$$

and

$$|\phi(w)| \leq C_0 \cdot ||f||_{A^\alpha(\Sigma)} \cdot \operatorname{dist}(w, \Sigma)^{\alpha-1}, \quad w \in C.$$

Therefore

$$(4) \qquad f(S)X - Xf(T) = \int \phi(w)[(w-S)^{-1}X - X(w-T)^{-1}]\, d\mu(w).$$

The domain of integration is supp(ϕ), which is a compact set that has in common with Σ only possibly boundary points ofΣ. For $w \in \operatorname{supp}(\phi) \cap (C \setminus \Sigma)$,

$$\begin{aligned} & (w-S)^{-1}X - X(w-T)^{-1} \\ = \ & (w-S)^{-1}[X(w-T) - (w-S)X](w-T)^{-1} \\ = \ & (w-S)^{-1}[SX - XT](w-T)^{-1} \in J, \end{aligned}$$

and, according to (3),

$$\begin{aligned} & ||(w-S)^{-1}X - X(w-T)^{-1}||_J \\ \leq\ & \mathrm{dist}(w,\sigma(S))^{-1}\cdot \mathrm{dist}(w,\sigma(T))^{-1}\cdot ||SX-XT||_J \\ \leq\ & C'\cdot \mathrm{dist}(w,\Sigma)^{-2}\cdot ||SX-XT||_J, \end{aligned}$$

where C' is a constant that depends on Σ. Therefore the integrant in (4) belongs to the norm ideal J and

$$||\phi(w)(w-S)^{-1}X - X(w-T)^{-1}||_J \leq C_0\cdot C'\cdot ||f||_{A^\alpha(\Sigma)}\cdot dist(w,\Sigma)^{\alpha-3}||SX-XT||_J,$$

for $w\in \mathrm{supp}(\phi)\cap(C\setminus\Sigma)$. After integration one obtains the desired conclusion of the theorem. □

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