Crosscap Numbers of Two-component Links

GENGYU ZHANG
Department of Mathematics, Tokyo Institute of Technology, Tokyo 152-8551, Japan
e-mail : zhang@math.titech.ac.jp

ABSTRACT. We define the crosscap number of a 2-component link as the minimum of the first Betti numbers of connected, non-orientable surfaces bounding the link. We discuss some properties of the crosscap numbers of 2-component links.

1. Introduction

Throughout this paper we work in the piecewise linear category, and knots and links we work with are embedded in the 3-sphere S^3. The crosscap number of a knot K was introduced by Clark [1] in 1978. It is defined to be the minimum of the first Betti numbers of non-orientable surfaces bounding K. Various notations for the crosscap number of a knot have been used in the past research on it, see for example [1], [6], [10], [3], [4], and in this paper we denote it by $\gamma(K)$.

Clark observed in [1] that the inequality $\gamma(K) \leq 2g(K) + 1$ holds and raised the question whether some knots exist for which the equality holds. Murakami and Yasuhara [6] brought a concrete calculation for the knot 7_4 which is the first example known to satisfy the equality above. It has been shown in [3] that there exist numerous knots for which the equality holds.

Given a knot, generally it is hard to determine the crosscap number for it. Clark gave a necessary and sufficient condition for the crosscap number 1 knots, which says that a knot has crosscap number 1 if and only if it is a $(2, n)$-cable knot. Recently the crosscap numbers for several families of knots, such as the torus knots in [10], the 2-bridge knots in [3], and the pretzel knots in [4], have been determined.

In this paper we define the crosscap number for two-component links and discuss some properties of it. By following the technique used in [6], we calculate the crosscap number of the two-component link 6_3^2 as an example. Here we use the notation of Rolfsen [9] to denote a link in his link table.

---

2. Definitions

The crosscap number of a knot was first introduced by Clark [1] in 1978.

Definition 2.1([1]). The *crosscap number* $\gamma(K)$ of a knot K is the minimal number of the first Betti numbers of all the connected, non-orientable surfaces bounding K. The crosscap number of the unknot U is defined to be 1.

Note that in this paper we define the crosscap number of the unknot to be 1, instead of 0 defined by Clark [1].

Clark also gave an upper bound for the crosscap number of a knot in terms of its genus.

Proposition 2.2([1]). Let K be a knot, and $g(K)$ denote the genus of K. Then

$$(1) \qquad\qquad \gamma(K) \leq 2g(K) + 1.$$

Beginning with the knot 7_4 proved by Murakami and Yasuhara in [6], it has been shown in [3] that numerous knots are suited for the equality in (1).

The crosscap number of a two-component link is defined similarly to that of a knot.

Definition 2.3. The *crosscap number* $\gamma(L)$ of a two-component link L is the minimum of the first Betti numbers of connected, non-orientable surfaces bounding the link, i.e. we have

$$\gamma(L) := \min\{\beta_1(F)|F \text{ is a connected non-orientable surface bounding } L\}.$$

It is not hard to see that for a 2-component link L, its crosscap number is at least 2, i.e. $\gamma(L) \geq 2$. This is because the projective plane $\mathbf{R}P^2$ is a closed non-orientable surface with minimum first Betti number, and the first Betti number of the surface obtained by cutting two disks off $\mathbf{R}P^2$ is 2.

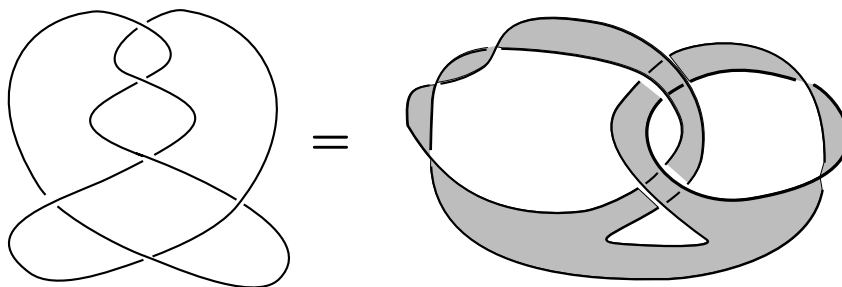Let 6_2^2 be the two-component link illustrated in Figure 1.

Figure 1: Non-orientable surface bounding 6_2^2 with $\beta_1 = 2$

Example 2.4. We have $\gamma(6_2^2) = 2$ as shown in the picture.

3. Behavior of crosscap numbers under split union

In this section we will see how the crosscap number of a split union of two knots can be evaluated by the crosscap numbers of the knots.

Definition 3.1. A link $L = L_1 \cup L_2$ is called *splittable* if there exists some 2-sphere S^2 embedded in S^3 bounding two 3-balls B_1 and B_2 with $B_1 \cup B_2 = S^3$ such that L_i lies in the interior of B_i ($i = 1, 2$). A *split union* of two knots K_1 and K_2, denoted by $K_1 \circ K_2$, is the splittable link with the 2-sphere bounding two 3-balls such that K_i lies in the interior of B_i ($i = 1, 2$).

Firstly, we consider the crosscap number of the split union of any two knots. In actuality, for the split union of any two knots, its crosscap number can be known in terms of genera and crosscap numbers of the knots. Namely, we have the following equality.

Theorem 3.2. *Let K_1 and K_2 be any two knots. We have*

$$\gamma(K_1 \circ K_2) = \min\{\gamma(K_1) + 2g(K_2) + 1, \gamma(K_2) + 2g(K_1) + 1, \gamma(K_1) + \gamma(K_2) + 1\}.$$

Proof. Let G' be a connected, non-orientable surface which gives the minimum first Betti number for the split union $K_1 \circ K_2$. There exists a 2-sphere S separating K_1 and K_2 with $S \cap G' \neq \emptyset$. Take an innermost cirle α of $S \cap G'$ in S bounding no disk in G'. (We can always make it because we can remove the circle α without changing the first Betti number by doing surgery on the surface G' if it bounds a disk in G'.)

If α is a non-separating curve in G', then a surgery along α on the surface G' gives us a new connected surface bounding $K_1 \circ K_2$ with first Betti number smaller than the previous one by two, either orientable or non-orientable. By adding a half twisted band, we can have a non-orientable surface with smaller first Betti number than that of G' from where a contradiction produces.

Hence we may assume that α is a separating curve which separates the surface G' into two surfaces, say G'_1 and G'_2. Each of them bounds K_1 or K_2 separately; otherwise we can get a connected, non-orientable surface bounding $K_1 \circ K_2$ with smaller first Betti number by doing surgery along α on G', which contradicts the minimality of the crosscap number. Then at least one of the two surfaces G'_1 and G'_2 is non-orientable. In all there are three possibilities: G'_1 orientable and G'_2 non-orientable, G'_1 non-orientable and G'_2 orientable, or both non-orientable. Therefore we get the inequality $\min\{\gamma(K_1) + 2g(K_2) + 1, \gamma(K_2) + 2g(K_1) + 1, \gamma(K_1) + \gamma(K_2) + 1\} \leq \gamma(K_1 \circ K_2)$.

On the other hand, we may assume that the knot K_i bounds an orientable surface S_i with genus $g(K_i)$ and a non-orientable surface G_i with first Betti number $\gamma(K_i)$ ($i = 1, 2$). Then three non-orientable surfaces bounding $K_1 \circ K_2$ will be produced if we connect the surfaces S_1 and G_2, G_1 and S_2, and G_1 and G_2 by

tubes. The first Betti numbers of these three surfaces are $\gamma(K_1) + 2g(K_2) + 1$, $\gamma(K_2) + 2g(K_1) + 1$ and $\gamma(K_1) + \gamma(K_2) + 1$ respectively. This gives us an upper bound of the crosscap number of the split union, i.e. $\gamma(K_1 \circ K_2) \leq \min\{\gamma(K_1) + 2g(K_2) + 1, \gamma(K_2) + 2g(K_1) + 1, \gamma(K_1) + \gamma(K_2) + 1\}$.

Then we have the proof of the equality.   □

Corollary 3.3. *Let K_1 and K_2 be any two knots. Then the following inequalities hold:*

$$
\begin{aligned}
\gamma(K_1 \circ K_2) &\leq \gamma(K_1) + \gamma(K_2) + 1, \\
\gamma(K_1 \circ K_2) &\leq \gamma(K_1) + 2g(K_2) + 1, \\
\gamma(K_1 \circ K_2) &\leq \gamma(K_2) + 2g(K_1) + 1.
\end{aligned}
\tag{2}
$$

Using Clark's inequality (1), we have

Corollary 3.4. *Let K_1 and K_2 be any two knots. Then $\gamma(K_1 \circ K_2) = \gamma(K_1) + \gamma(K_2) + 1$ if and only if $\gamma(K_i) < 2g(K_i) + 1$, $i = 1$ and 2.*

This corollary is equivalent to the following.

Corollary 3.5. $\gamma(K_1 \circ K_2) = \gamma(K_1) + \gamma(K_2)$ *if and only if $\gamma(K_i) = 2g(K_i) + 1$, $i = 1$ or 2.*

Note that if $\gamma(K_1) < 2g(K_1) + 1$ and $\gamma(K_2) = 2g(K_2) + 1$, then $\gamma(K_1 \circ K_2) = \gamma(K_1) + 2g(K_2) + 1$; if $\gamma(K_1) = 2g(K_1) + 1$ and $\gamma(K_2) < 2g(K_2) + 1$, then $\gamma(K_1 \circ K_2) = \gamma(K_2) + 2g(K_1) + 1$. In fact, the crosscap number $\gamma(K_1 \circ K_2)$ is exactly equal to $\gamma(K_1) + \gamma(K_2)$ in both cases.

If we apply the argument above to the case when K_2 is the unknot U, we have the following corollary.

Corollary 3.6. *Let U denote the unknot and K be any knot. Then we have*

$$\gamma(K \circ U) = \gamma(K) + 1.$$

Next, by applying the homology theory, we discuss a little more the examples for which the equality in (2) holds.

Proposition 3.7. *Let $D(L)$ denote the double branched cover of S^3 branched along the two-component link L. Then the minimum number of generators for $H_1(D(L); \mathbf{Z})$ has the crosscap number $\gamma(L)$ as an upper bound.*

Proof. Let F denote a non-orientable surface which has the minimum first Betti number, $\gamma(L)$, bounding L. Then corresponding to this surface, there is a $\gamma(L) \times \gamma(L)$ Goeritz matrix built in the way of Gordon and Litherland [2, §2]. Then this Goeritz matrix becomes a relation matrix for $H_1(D(L); \mathbf{Z})$, see Appendix A, from which the result follows.   □
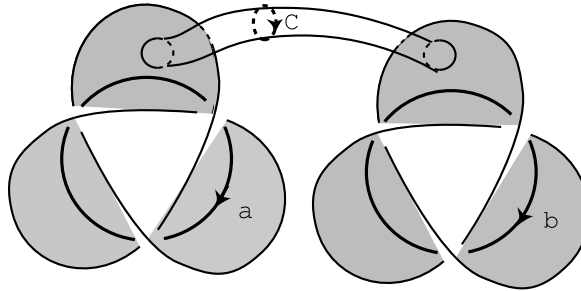
Figure 2: Tubular connection

Remark 3.8. It is known that $g(3_1) = 1$ and $\gamma(3_1) = 1$. Thus $\gamma(3_1) < 2g(3_1) + 1$, and by Corollary we have $\gamma(3_1 \circ 3_1) = 3 = \gamma(3_1) + \gamma(3_1) + 1$. Now we prove this in terms of homology theory.

By connecting the two non-orientable surfaces bounding trefoils with a tube, a non-orientable surface F bounding $3_1 \circ 3_1$ can be built. see Figure 2. The Georitz matrix corresponding to this surface with indicated generators $\{a, b, c\}$ is as follows:

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the first homology group of the double cover of S^3 branched over $3_1 \circ 3_1$, can be known as $H_1(D(3_1 \circ 3_1); \mathbf{Z}) = \mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}$. By the fundamental theorem of abelian groups, $H_1(D(3_1 \circ 3_1); \mathbf{Z})$ cannot be presented by a 2×2 matrix.

So the crosscap number of the split union of two trefoil knots cannot be 2 by Proposition 3.7. Then we have $\gamma(3_1 \circ 3_1) = 3 = \gamma(3_1) + \gamma(3_1) + 1$.

4. Upper bounds of crosscap numbers of two-component links

Denote by $\overrightarrow{L}$ and $\overleftarrow{L}$ the two different relative orientations for a 2-component link L. Let $g(\overrightarrow{L})$ and $g(\overleftarrow{L})$ denote the genera of L under these two different orientations.

Theorem 4.1. *With the notations above, we have*

(3) $$\gamma(L) \le 2\min(g(\overrightarrow{L}), g(\overleftarrow{L})) + 2.$$

Proof. Denote $\min(g(\overrightarrow{L}), g(\overleftarrow{L}))$ by g. Then there exists an orientable Seifert surface F, whose standard form is illustrated in Figure 3, with genus g bounding the 2-component link L, so the first Betti number of this surface becomes $2g + 1$. By adding a half twist to the surface, we obtain a non-orientable surface from F as illustrated in Figure 4, whose first Betti number is $2g + 2$. By the definition of the crosscap number, we know that the inequality holds. $\square$
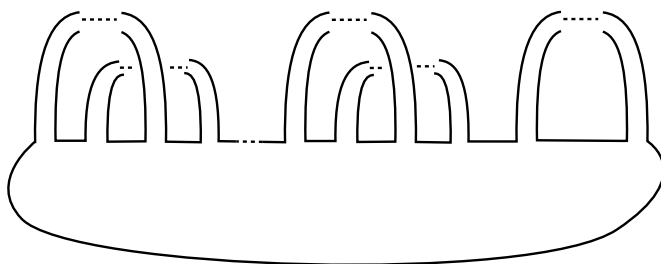
Figure 3: Standard form of a Seifert surface bounding a 2-component link
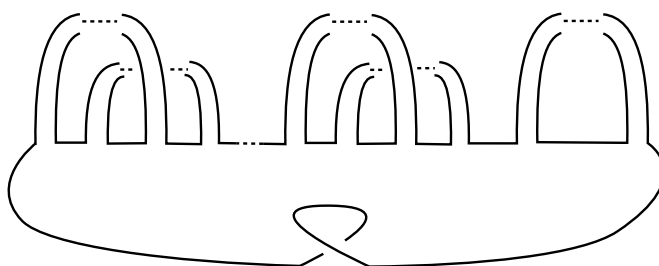

Figure 4: Surface after adding a half-twisted band

Does there exist any 2-component link for which the equality in (3) holds?

There exists an infinite sequence of 2-component links $T(2, 2n)$ ($n \in \mathbf{Z}$) for which the equality in (3) holds, where $T(p, q)$ denotes a torus knot or link. Take an example of torus link $T(2, 10)$ as illustrated in Figure 5. It bounds a genus 0 orientable Seifert surface, which gives us an upper bound 2 for the crosscap number of the link by using the inequality (3). Therefore we have the fact that the crosscap number of the torus link $T(2, 2n)$ is 2, for which the equality in (3) holds.
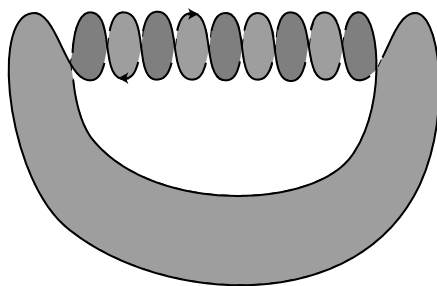

Figure 5: Torus link $T(2, 10)$

Let $n(L)$ denote the minimum crossing number of a link L. We can give another

upper bound for the crosscap number of a two-component link in terms of $n(L)$. For a knot, Murakami and Yasuhara in [6] proved the following proposition.

Proposition 4.2([6]). *Let $n(K)$ denote the crossing number of a knot K. Then*

(4)
$$\gamma(K) \le \left\lfloor \frac{n(K)}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer that does not exceed x.

For a two-component link, we have the following result.

Proposition 4.3. *Let L be a two-component link, excluding the unlink. Then we have*
$$\gamma(L) \le \left\lfloor \frac{n(L)}{2} \right\rfloor + 1,$$

where $\lfloor x \rfloor$ denotes the greatest integer that does not exceed x.

Note that for the Hopf link, the equality in Proposition 4.3 holds.

Proof. If L is a splittable link $K_1 \circ K_2$, then from Proposition 4.2 we have $\gamma(K_i) \le \left\lfloor \frac{n(K_i)}{2} \right\rfloor$ for $i = 1, 2$, and from Corollary 3.3 we have $\gamma(L) \le \gamma(K_1) + \gamma(K_2) + 1$. Then the inequality $\gamma(L) \le \left\lfloor \frac{n(K_1)}{2} \right\rfloor + \left\lfloor \frac{n(K_2)}{2} \right\rfloor + 1$ follows. Hence the theorem holds for splittable links due to the fact that $\left\lfloor \frac{n(K_1)}{2} \right\rfloor + \left\lfloor \frac{n(K_2)}{2} \right\rfloor \le \left\lfloor \frac{n(L)}{2} \right\rfloor$.

Now let D be a link diagram of non-splittable link L with the minimum crossing number $n(L)$. Then we have $n(L) + 2$ regions of S^2 divided by the link diagram. Color these regions black and white in a checkerboard way. Since L is a non-splittable link, all the regions of the same color can be connected to each other by half-twisted bands at the crossings. If both the white and black surfaces can be orientable, then $\gamma(L)$ should be less than or equal to the first Betti number of these surfaces plus one, where the "1" indicates the added first Betti number by adding a half-twisted band in that case. Denote the numbers of black and white regions by $n(b)$ and $n(w)$ respectively. Note that the number of the edges is twice that of vertices. Then by using a relation between Euler characteristic and the first Betti number together with Euler's formula, we have

$$\gamma(L) \le 2 + n(L) - \max\{n(b), n(w)\}.$$

It is not hard to know that $\max\{n(b), n(w)\} \ge \frac{1}{2}(n(L) + 2)$ in the case that $n(L)$ is even, and $\max\{n(b), n(w)\} \ge \frac{1}{2}(n(L) + 3)$ in the case that $n(L)$ is odd. Therefore the result follows.   □

5. An example of calculation

In this section, we will calculate the crosscap number of the two-component link 6_3^2 as an example.

Lemma 5.1. *Let L be a two-component link $K_1 \cup K_2$. Assume that it bounds a connected, non-orientable surface F with the first Betti number 2. Then we can choose a generator system for $H_1(F; \mathbf{Z})$ such that the Goeritz matrix $G_F(L)$ corresponding to this system is of the following form:*

$$
(5) \qquad G_F(L) = \begin{pmatrix} 2n+1 & 2k \\ 2k & 2m \end{pmatrix},
$$
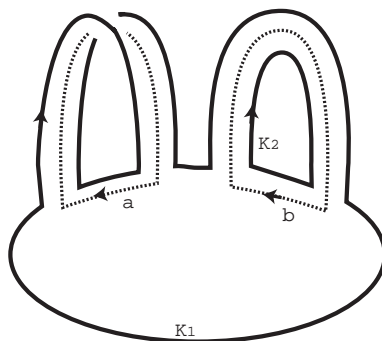
where k, m, n are integers.
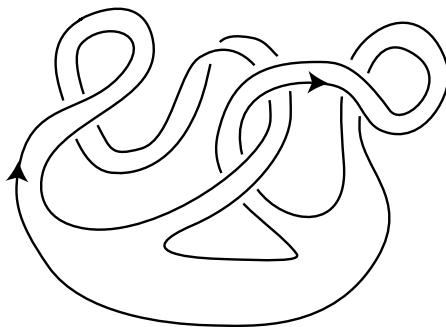

Figure 6: A crosscap number two surface

Figure 7: An example of a non-orientable surface with $\beta_1 = 2$

Proof. We may assume that F is a disk with a non-orientable band and an orientable one as indicated in Figure 6, where each band may be knotted and linked with the

other; see for example Figure 7. Choose a generator system $\{a, b\}$ as in Figure 6 and orient the two components K_1 and K_2 so that the two boundaries of each band have the same orientations as that of the 1-cycle passing through it.

Then the Goeritz matrix of the surface F, refer to [2, §2], corresponding to this generator system is

$$\begin{pmatrix} lk(a, \tau a) & lk(a, \tau b) \\ lk(b, \tau a) & lk(b, \tau b) \end{pmatrix} = \begin{pmatrix} 2\delta(A, A) + 1 & 2\delta(A, B) \\ 2\delta(B, A) & 2\delta(B, B) \end{pmatrix},$$

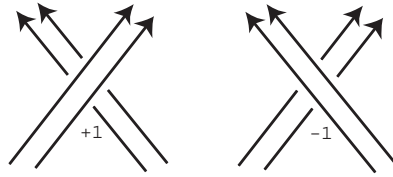where $\tau(x)$ denotes the orientation double cover of a cycle x.

Figure 8: Crossing types of bands

Here A and B are the bands which a and b pass through respectively, and $\delta(X, Y)$ is the sum of signs of crossings of bands X and Y with signs determined as in Figure 8. Note that $lk(a, \tau b) = lk(b, \tau a) = 2lk(a, b)$, and that both of them are even. This finishes the proof. □

Lemma 5.2. *With the orientations as above in the proof of Lemma , the linking number between K_1 and K_2 is $m+2k$ and the modified Euler number, see Appendix , of the surface F is (-2) times the sum of all the entries in the Goeritz matrix $G_F(L)$. Namely,*

$$lk(K_1, K_2) = m + 2k, \quad \overline{e}(F) = -2(2n + 1 + 2k + 2k + 2m).$$

Proof. According to the proof of the above lemma, we have $m = \delta(B, B)$ and $k = lk(a, b)$. It is obvious that the calculation of the linking number between K_1 and K_2 includes these two parts, which gives us $lk(K_1, K_2) = \delta(B, B) + 2lk(a, b) = m + 2k$.

We also have the fact that the modified normal Euler number of F with the orientations above is $-\{4\delta(A, A) + 2 + 4\delta(A, B) + 4\delta(B, A) + 4\delta(B, B)\}$. Then since $m = \delta(B, B)$, $k = \delta(A, B)$ and $n = \delta(A, A)$, we obtain the second equality. □

We will prove the following conclusion.

Theorem 5.3. *Let 6_3^2 denote the two-component link as illustrated in Figure 9, forgetting the orientations. We have $\gamma(6_3^2) = 3$.*

Proof. It is clear that the surface bounding the two-compnent 6_3^2 illustrated in Figure 10 is a non-orientable one with first Betti number 3.
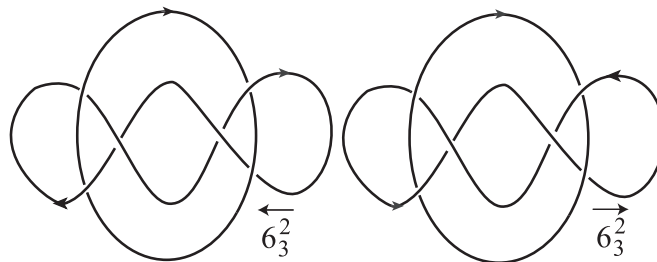
Figure 9: 2-component link 6_3^2 with two orientations

Now note that the crosscap number of a two-component link is at least 2, hence we may assume that $\gamma(6_3^2) = 2$. Then there exists a non-orientable surface F bounding the link with $H_1(F; \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}$. The Goeritz matrix $G_F(6_3^2)$ associated with F should be a 2 by 2 matrix, determined by a generator system of $H_1(F; \mathbf{Z})$. A different choice of basis of $H_1(F; \mathbf{Z})$ gives another matrix J such that $J = P^T G P$ where P is an integral unimodular matrix. The integral congruent class which $G_F(6_3^2)$ belongs to does not change. Since the link 6_3^2 is the two-bridge link $S(12, 5)$,
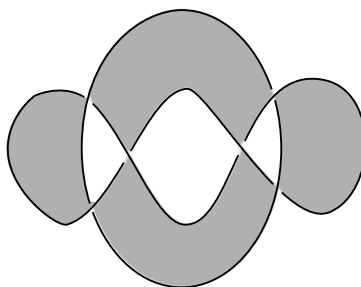
Figure 10: Non-orientable surface with $\beta_1 = 3$

the double branched cover $D(6_3^2)$ of S^3 branched over 6_3^2 is the lens space $L(12, 5)$ with $H_1(D(6_3^2); \mathbf{Z}) = \mathbf{Z}/12\mathbf{Z}$ and the linking form $\lambda(g, g) = \pm\dfrac{5}{12}$ for some properly chosen generator of $H_1(D(6_3^2); \mathbf{Z})$. The determinant of the Goeritz matrix $G_F(6_3^2)$ is known to be ± 12 since the absolute value of the determinant of a Goeritz matrix of a link is equal to the order of $H_1(D(L); \mathbf{Z})$.

By applying an elementary theorem of integral binary quadratic forms (see, for example [8]), we enumerate all the congruent classes of 2 by 2 integral matrices with discriminant ± 48, 4 times of the matrix determinant.

Then the result is as follows:

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 12 \end{pmatrix}, X_2 = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}, X_3 = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}, X_4 = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix},$$

$$X_5 = \begin{pmatrix} 1 & 0 \\ 0 & -12 \end{pmatrix}, X_6 = \begin{pmatrix} -1 & 0 \\ 0 & 12 \end{pmatrix}, X_7 = \begin{pmatrix} 2 & 0 \\ 0 & -6 \end{pmatrix},$$

$$X_8 = \begin{pmatrix} -2 & 0 \\ 0 & 6 \end{pmatrix}, X_9 = \begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}, X_{10} = \begin{pmatrix} -3 & 0 \\ 0 & 4 \end{pmatrix}.$$

It is obvious that the matrices X_2, X_4, X_7 and X_8 cannot present the cyclic group $\mathbf{Z}/12\mathbf{Z}$, and therefore cannot be the relation matrices of $H_1(D(6_3^2); \mathbf{Z})$. On the other hand, only the matrix X_3 presents the linking form $\pm\dfrac{5}{12}$ for the link 6_3^2, see Appendix B.

If X_3 were a Georitz matrix for the link 6_3^2, then there should exist some integral unimodular matrix Q such that $Q^T G_F(L)Q = X_3$. The inverse matrix of Q is also integral unimodular and we denote it by P. Changing the basis of $H_1(D(6_3^2); \mathbf{Z})$ by using $P := \begin{pmatrix} r & u \\ s & v \end{pmatrix}$, the Goeritz matrix $G_F(L)$ is of the form:

$$P^T \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} P = \begin{pmatrix} 3r^2 + 4s^2 & 3ru + 4sv \\ 3ru + 4sv & 3u^2 + 4v^2 \end{pmatrix}.$$

Recall that the formula by Gordon and Litherland [2], also see Appendix A, relates the signature of a link with the signature of the Goeritz matrix, $\sigma(\overline{L}) = \sigma(G_F) + \dfrac{1}{2}\overline{e}(F)$. The signature of a link is defined as the signature of the symmetrized Seifert matrix, the difference between the number of positive eigenvalues and negative ones of the matrix.

For some appropriate Seifert surfaces, the corresponding Seifert matrices of the two-component link under two different orientations, $\overleftarrow{6_3^2}$ and $\overrightarrow{6_3^2}$, see Figure 9, are as follows:

$$V(\overleftarrow{6_3^2}) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, V(\overrightarrow{6_3^2}) = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Then we obtain the signatures of $\overleftarrow{6_3^2}$ and $\overrightarrow{6_3^2}$ as 3 and -1 respectively. We will see that for either orientation, there exists no solution for which the formula by Gordon and Litherland holds.

According to Lemma 5.2, we see that the modified normal Euler number $\overline{e}(F)$ is $-2[3(r+u)^2 + 4(s+v)^2]$. Now since $\sigma(X_3) = 2$ we have

(6) $$\sigma(\overleftarrow{6_3^2}) = 3 = 2 - [3(r+u)^2 + 4(s+v)^2],$$

or

(7) $$\sigma(\overrightarrow{6_3^2}) = -1 = 2 - [3(r+u)^2 + 4(s+v)^2].$$

Since a negative number cannot be equal to the sum of perfect squares, it is not hard to see that there exists no integral solution of r, s, u, v to the equality (6).

For $\overrightarrow{6_3^2}$, we establish another equality in terms of the linking number by Lemma 5.2. We have

$$(8) \qquad\qquad lk(\overrightarrow{6_3^2}) = 2 = \frac{3u^2 + 4v^2}{2} + 3ru + 4sv.$$

Then we combine the equalities (7) and (8) to get the following system of equations:

$$\begin{cases} 3(r+u)^2 + 4(s+v)^2 = 3, \\ 3u^2 + 4v^2 + 6ru + 8sv - 4 = 0. \end{cases}$$

Because the values of r, u, s and v are taken in $\mathbf{Z}$, from the first equation we have $r + u = \pm 1$ and $s + v = 0$. Transform the second equation to the following:

$$3(u+r)^2 - 3r^2 + 4(s+v)^2 - 4s^2 - 4 = 0.$$

Putting the values of $u + r$ and $s + v$ into the transformed equation, we have $3 - 3r^2 - 4s^2 - 4 = 0$, i.e., $3r^2 + 4s^2 = -1$, for which there exists no integral solution of r and s.

So it turns out that there exist no integral solutions for this system of equations, which contradicts our assumption. Namely, there exists no connected, non-orientable surface with first Betti number 2. Therefore, we have $\gamma(6_3^2) = 3$.   □

Appendix A. A formula of signature

The definition of the signature of knots and links was developed by Trotter [11] and Murasugi [7]. For an oriented link in S^3 with Seifert matrix V, define the *signature* of L to be $\sigma(L) = \sigma(V + V^T)$ where $\sigma(M)$ is the difference between the number of the positive eigenvalues and that of the negative eigenvalues of a symmetric matrix M. Note that the signature of a link is up to the relative orientations for each component. See, for example [9] or [5].

Gordon and Litherland [2] has shown how to define a quadratic form related with Goeritz matrix by using any spanning surface, and related the signature of this form to the signature of a link. We recall their formula here for readers' information.

Lemma A.1([2]). *Let F be any surface bounding an unoriented link L, and let $\overline{L}$ denote the link L together with some orientation on each component of it. Then*

the signature $\sigma(\overline{L})$ can be calculated out of the Goeritz matrix G_F and the modified normal Euler number $\overline{e}(F)$, namely we have

(1)
$$\sigma(\overline{L}) = \sigma(G_F) + \frac{1}{2}\overline{e}(F).$$

Definition A.2([2]). Let $L = K_1 \cup K_2$ be a 2-component link with a connected surface F bounding it. We define $\overline{e}(F)$ to be $-\sum_{i,j=1}^{2} lk(K_i, K_j')$ where K_i' is the intersection of F and the boundary of the regular neighborhood of K_i in S^3, i.e. $K_i' = F \cap \partial N(K_i)$, with orientation parallel to that of K_i. We name it the *modified normal Euler number.*

Appendix B. Linking form of a link

Let M denote a closed, oriented 3-dimensional manifold. Denote the torsion part of $H_1(M; \mathbf{Z})$ by $\mathrm{Tor}H_1(M; \mathbf{Z})$. A chain complex of M is as follows:

$$\cdots \longrightarrow C_2(M) \xrightarrow{\partial_2} C_1(M) \xrightarrow{\partial_1} 0,$$

where $C_i(M)$ is the abelian group generated by i-simplices of M, and ∂_i is the boundary homomorphism ($i = 1, 2, \cdots$). Suppose that $x, y \in \mathrm{Tor}H_1(M; \mathbf{Z})$ are represented by 1-cycles a and b respectively. There exists $n \in \mathbf{Z}$ such that $[nb]$ is homologous to zero in $H_1(M; \mathbf{Z})$ and nb forms the boundary of some 2-chain, say Δ in $C_2(M)$, i.e. $\partial_2(\Delta) = nb$. Define a bilinear form $\lambda : \mathrm{Tor}H_1(M; \mathbf{Z}) \times \mathrm{Tor}H_1(M; \mathbf{Z}) \longrightarrow \mathbf{Q}/\mathbf{Z}$ as follows:

$$(x, y) \xmapsto{\lambda} \mathrm{Int}(a, \Delta)/n,$$

where Int denotes the intersection number between a 1-cycle and a 2-chain. This bilinear form λ is called the *linking form* on the 3-manifold M.

Then the linking form defined on the double branched cover $D(L)$ of L

$$\lambda : \mathrm{Tor}H_1(D(L); \mathbf{Z}) \times \mathrm{Tor}H_1(D(L); \mathbf{Z}) \longrightarrow \mathbf{Q}/\mathbf{Z}$$

is called *the linking form of the link L.* A Goeritz matrix G of the link L is a relation matrix for $H_1(D(L); \mathbf{Z})$, and the first homology group of $D(L)$ and the linking form on $D(L)$, $(H_1(D(L); \mathbf{Z}), \lambda)$, can be calculated out of the Goeritz matrix. Precisely speaking, $H_1(D(L); \mathbf{Z}) = \mathbf{Z}^n / \mathrm{Im}(G)$ where n is the size of the Goeritz matrix G, and the linking form is given by $\pm G^{-1}$ if $H_1(D(L); \mathbf{Z})$ is finite , i.e. $\lambda(g_i, g_j) = \pm(G^{-1})_{ij}$ (mod 1) for the generators g_i and g_j of $H_1(D(L); \mathbf{Z})$, where $\{g_1, g_2, \cdots, g_n\}$ is a generator system of $H_1(M; \mathbf{Z})$ corresponding to the presentation of $\mathbf{Z}^n / \mathrm{Im}(G)$. Here the sign $\pm$ depends on the orientation.

References

[1] B. E. Clark, *Crosscaps and knots*, Internat. J. Math. Sci., **1(1)**(1978), 113-223, MR 0478131 (57 #17620).

[2] C. McA. Gordon and R. A. Litherland, *On the signature of a link*, Invent. Math., **47(1)**(1978), 53 - 69, MR 0500905 (58 #18407).

[3] M. Hirasawa and M. Teragaito, *Crosscap numbers of 2-bridge knots*, Topology, **45(3)**(2006), 513 - 530.

[4] K. Ichihara and S. Mizushima, *Crosscap numbers of pretzel knots*, arXiv : math.GT/0608480.

[5] W. B. R. Lickorish, *An Introduction to Knot Theory*, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997, MR 1472978 (98f:57015).

[6] H. Murakami and A. Yasuhara, *Crosscap number of a knot*, Pacific J. Math., **171(1)**(1995), 261 - 273, MR 1362987 (96k:57006).

[7] K. Murasugi, *On the signature of links*, Topology, **9**(1970), 283 - 298, MR 0261585 (41 #6198).

[8] I. Niven, H. S. Zuckerman, and H. L. Montgomery, *An Introduction to the Theory of Numbers*, John Wiley & Sons Inc., New York, 1991, MR 1083765 (91i:11001).

[9] D. Rolfsen, *Knots and Links*, Mathematics Lecture Series, vol. 7, Publish or Perish Inc., Houston, TX, 1990, MR 95c:57018.

[10] M. Teragaito, *Crosscap numbers of torus knots*, Topology Appl., **138(1-3)**(2004), 219 - 238, MR 2035482 (2005a:57007).

[11] H. F. Trotter, *Homology of group systems with applications to knot theory*, Ann. of Math., **76(2)**(1962), 464 - 498, MR 0143201 (26 #761).