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The Real Rank of CCR C^*-Algebras

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Abstract. We estimate the real rank of CCR C^*-algebras under some assumptions. As applications we determine the real rank of the reduced group C^*-algebras of non-compact connected, semi-simple and reductive Lie groups and that of the group C^*-algebras of connected nilpotent Lie groups.

1. Introduction

The real rank for C^*-algebras was introduced by Brown and Pedersen [3]. By definition, we say that a unital C^*-algebra $\mathfrak{A}$ has the real rank $n = \mathrm{RR}(\mathfrak{A})$ if n is the smallest non-negative integer such that for any $\varepsilon > 0$ given, any self-adjoint element $(a_j)_{j=1}^{n+1} \in \mathfrak{A}^{n+1}$ with $a_j = a_j^*$ is approximated by a self-adjoint element $(b_j)_{j=1}^{n+1} \in \mathfrak{A}^{n+1}$ with $b_j = b_j^*$ such that $\|a_j - b_j\| < \varepsilon$ $(1 \leq j \leq n+1)$ and $\sum_{j=1}^{n+1} b_j^2$ is invertible in $\mathfrak{A}$. For a non-unital C^*-algebra, its real rank is defined by that of its unitization by C. By definition, $\mathrm{RR}(\mathfrak{A}) \in \{0, 1, 2, \cdots, \infty\}$.

On the other hand, CCR C^*-algebras are very well known in the C^*-algebra theory such as the representation theory and structure theory of C^*-algebras (or group C^*-algebras). Recall that a C^*-algebra $\mathfrak{A}$ is CCR (or liminary) if for any irreducible representation π of $\mathfrak{A}$, the image $\pi(\mathfrak{A})$ is either isomorphic to a matrix algebra $M_n(C)$ over C or to K the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space (Dixmier [4, Section 4.2] and Pedersen [11, Section 6.1]).

However, it seems that the real rank of CCR C^*-algebras has been unknown. It is in part because it is difficult in general to compute the real rank of extensions of C^*-algebras. Thus as the first step we impose an assumption on CCR C^*-algebras that they have no finite dimensional irreducible representations. Then we can show below that those CCR C^*-algebras have real rank less than or equal to one by using their composition series and some results on the real rank for extensions of C^*-algebras (Nagisa, Osaka and Phillips [9]) and for tensor products of C^*-algebras with K (Beggs and Evans [2]).

As applications, we show that the real rank of the reduced group C^*-algebras of

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non-compact connected semi-simple Lie groups is one, and that of the reduced group C^*-algebras of non-compact connected reductive Lie groups is also one by using [13] in part for the structure of these group C^*-algebras. These results should be new and interesting. Compare them with the stable rank of those group C^*-algebras [13] (and see below). See also Kaniuth [7].

Moreover, following the same methods for the real rank of those CCR C^*-algebras and using [2] for the real rank of tensor products of commutative C^*-algebras with $M_n(C)$ we show that the real rank of CCR C^*-algebras which have no infinite dimensional irreducible representations can be estimated in terms of homogeneous subquotients with continuous trace.

Finally, we estimate the real rank of a CCR C^*-algebra that is decomposed into a closed ideal and a quotient that have no finite and no infinite dimensional irreducible representations respectively. As the final application, we show that the real rank of the group C^*-algebras of connected nilpotent Lie groups is equal to the dimension of the spaces of their 1-dimensional representations. For the stable rank of these group C^*-algebras, see [17] (and see below).

2. The real rank of CCR C^*-algebras

Theorem 2.1. *Let $\mathfrak{A}$ be a CCR C^*-algebra. Suppose that $\mathfrak{A}$ has no finite dimensional irreducible representations. Then* $\mathrm{RR}(\mathfrak{A}) \leq 1$.

Proof. Since $\mathfrak{A}$ is of type I, it has a composition series $(\mathfrak{I}_j)$ of essential closed ideals such that the union $\cup_j \mathfrak{I}_j$ is dense in $\mathfrak{A}$ and the subquotients $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ have continuous trace and each $\mathfrak{I}_j$ is essential in $\mathfrak{I}_{j+1}$ ([11, Theorem 6.2.11], [4, Theorem 4.5.5]). Thus, the subquotients $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ for $j \geq 1$ with $\mathfrak{I}_0 = \{0\}$ have Hausdorff spectrums, and by the assumption they are regarded as continuous field C^*-algebras $\Gamma_0(X_j, \{K\})$ on their spectrums X_j with fibers the constant K consisting of (certain) continuous operator fields on X_j vanishing at infinity ([4, Theorem 10.5.4]). Since the continuous field C^*-algebras $\Gamma_0(X_j, \{K\})$ are locally trivial ([4, Theorem 10.8.8]), they are inductive limits of the tensor products $C_0(U_{jk}) \otimes K$ for (certain) increasing open subsets U_{jk} of X_j with the unions $\cup_k U_{jk} = X_j$, where $C_0(U_{jk})$ are the C^*-algebras of continuous functions on U_{jk} vaninshing at infinity.

As the first step, we have the following exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{I}_1 & \longrightarrow & \mathfrak{I}_2 & \longrightarrow & \mathfrak{I}_2/\mathfrak{I}_1 & \longrightarrow & 0 \\ & & \| & & \iota\downarrow & & \tau\downarrow & & \\ 0 & \longrightarrow & \mathfrak{I}_1 & \longrightarrow & M(\mathfrak{I}_1) & \xrightarrow{q} & M(\mathfrak{I}_1)/\mathfrak{I}_1 & \longrightarrow & 0 \end{array}$$

where $M(\mathfrak{I}_1)$ is the multiplier algebra of $\mathfrak{I}_1$, and $\mathfrak{I}_2$ is isomorphic to the pull back $M(\mathfrak{I}_1) \oplus_{q,\tau} \mathfrak{I}_2/\mathfrak{I}_1$ defined by $\{(x, y) \in M(\mathfrak{I}_1) \oplus \mathfrak{I}_2/\mathfrak{I}_1 \mid q(x) = \tau(y)\}$, where q is the canonical quotient map and τ is the Busby invariant associated with the extension, and ι is the canonical inclusion (cf. [19]). Since $\mathfrak{I}_1 \cong \Gamma_0(X_1, \{K\})$, we have $M(\mathfrak{I}_1) \cong \Gamma^b(X_1, \{B\})$ the C^*-algebra of a bounded continuous field on X_1 with fibers the constant B the C^*-algebra of all bounded operators with the strict topology (cf. [1]). However, since $\mathfrak{I}_2$ is CCR and ι is injective from that $\mathfrak{I}_1$ is essential in $\mathfrak{I}_2$, we in fact that $\mathfrak{I}_2$ is embedded in $\Gamma^b(X_1, \{K\})$. Therefore, we have

$$\mathfrak{I}_2 \cong \Gamma^b(X_1, \{K\}) \oplus_{q,\tau} \mathfrak{I}_2/\mathfrak{I}_1.$$

Furthermore, we note that $\Gamma^b(X_1, \{K\})$ is stable, that is,

$$\Gamma^b(X_1, \{K\}) \cong \Gamma^b(X_1, \{K\}) \otimes K,$$

which follows from $\Gamma^b(X_1, \{K\}) \otimes M_n(C) \cong \Gamma^b(X_1, \{K \otimes M_n(C)\}) \cong \Gamma^b(X_1, \{K\})$ for any $n \geq 1$, and we can replace $M_n(C)$ with K. Note also that continuous fields with fibers K are always locally trivial (see [4, Chapter 10]), and it is known that an inductive limit of stable C^*-algebras is also stable [6].

By using [9, Proposition 1.6] and [2, Proposition 3.3],

$$\begin{aligned} \mathrm{RR}(\mathfrak{I}_2) &\leq \max\{\mathrm{RR}(\Gamma^b(X_1, \{K\}) \otimes K), \mathrm{RR}(\mathfrak{I}_2/\mathfrak{I}_1)\} \\ &\leq \max\{1, \mathrm{RR}(\mathfrak{I}_2/\mathfrak{I}_1)\} = 1 \end{aligned}$$

since $\mathfrak{I}_2/\mathfrak{I}_1 \cong \varinjlim C_0(U_{2k}) \otimes K$ so that

$$\mathrm{RR}(\varinjlim C_0(U_{2k}) \otimes K) \leq \sup_k \mathrm{RR}(C_0(U_{2k}) \otimes K)) \leq 1.$$

As the second step, we have the following commutative diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{I}_1 & \longrightarrow & \mathfrak{I}_3 & \longrightarrow & \mathfrak{I}_3/\mathfrak{I}_1 & \longrightarrow & 0 \\ & & \Vert & & {\scriptstyle \iota_1}\downarrow & & {\scriptstyle \tau_1}\downarrow & & \\ 0 & \longrightarrow & \mathfrak{I}_1 & \longrightarrow & M(\mathfrak{I}_1) & \xrightarrow{q_1} & M(\mathfrak{I}_1)/\mathfrak{I}_1 & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{I}_2/\mathfrak{I}_1 & \longrightarrow & \mathfrak{I}_3/\mathfrak{I}_1 & \longrightarrow & \mathfrak{I}_3/\mathfrak{I}_2 & \longrightarrow & 0 \\ & & \Vert & & {\scriptstyle \iota_2}\downarrow & & {\scriptstyle \tau_2}\downarrow & & \\ 0 & \longrightarrow & \mathfrak{I}_2/\mathfrak{I}_1 & \longrightarrow & M(\mathfrak{I}_2/\mathfrak{I}_1) & \xrightarrow{q_2} & M(\mathfrak{I}_2/\mathfrak{I}_1)/(\mathfrak{I}_2/\mathfrak{I}_1) & \longrightarrow & 0. \end{array}$$

Note that $\mathfrak{I}_2/\mathfrak{I}_1$ is essential in $\mathfrak{I}_3/\mathfrak{I}_1$ since $\mathfrak{I}_2$ is essential in $\mathfrak{I}_3$. By the same reasoning as in the first case, we have

$$\begin{aligned} \mathfrak{I}_3/\mathfrak{I}_1 &\cong M(\mathfrak{I}_2/\mathfrak{I}_1) \oplus_{q_2,\tau_2} \mathfrak{I}_3/\mathfrak{I}_2 \\ &\cong \Gamma^b(X_2, \{K\}) \oplus_{q_2,\tau_2} \mathfrak{I}_3/\mathfrak{I}_2 \cong (\Gamma^b(X_2, \{K\}) \otimes K) \oplus_{q_2,\tau_2} \mathfrak{I}_3/\mathfrak{I}_2. \end{aligned}$$

By using [9, Proposition 1.6] and [2, Proposition 3.3],

$$\begin{aligned} \mathrm{RR}(\mathfrak{I}_3/\mathfrak{I}_1) &\leq \max\{\mathrm{RR}(\Gamma^b(X_2, \{K\}) \otimes K), \mathrm{RR}(\mathfrak{I}_3/\mathfrak{I}_2)\} \\ &\leq \max\{1, \mathrm{RR}(\mathfrak{I}_3/\mathfrak{I}_2)\} = 1 \end{aligned}$$

since $\mathfrak{I}_3/\mathfrak{I}_2 \cong \varinjlim C_0(U_{3k}) \otimes K$ so that

$$\mathrm{RR}(\varinjlim C_0(U_{3k}) \otimes K) \le \sup_k \mathrm{RR}(C_0(U_{3k}) \otimes K)) \le 1.$$

Furthermore, we have

$$\begin{aligned} \mathfrak{I}_3 &\cong M(\mathfrak{I}_1) \oplus_{q_1,\tau_1} \mathfrak{I}_3/\mathfrak{I}_1 \\ &\cong \Gamma^b(X_1, \{K\}) \oplus_{q_1,\tau_1} \mathfrak{I}_3/\mathfrak{I}_1 \cong (\Gamma^b(X_1, \{K\}) \otimes K) \oplus_{q_1,\tau_1} \mathfrak{I}_3/\mathfrak{I}_1. \end{aligned}$$

Therefore, by [9, Proposition 1.6] and [2, Proposition 3.3] we obtain

$$\begin{aligned} \mathrm{RR}(\mathfrak{I}_3) &\le \max\{\mathrm{RR}(\Gamma^b(X_1, \{K\}) \otimes K), \mathrm{RR}(\mathfrak{I}_3/\mathfrak{I}_1)\} \\ &\le \max\{1, \mathrm{RR}(\mathfrak{I}_3/\mathfrak{I}_1)\} = 1. \end{aligned}$$

For the general step, we use the following diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{D}_j & \longrightarrow & \mathfrak{I}_n/\mathfrak{I}_{j-1} & \longrightarrow & \mathfrak{I}_n/\mathfrak{I}_j & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \iota_j} & & \downarrow{\scriptstyle \tau_j} & & \\ 0 & \longrightarrow & \mathfrak{D}_j & \longrightarrow & M(\mathfrak{D}_j) & \xrightarrow{q_j} & M(\mathfrak{D}_j)/(\mathfrak{D}_j) & \longrightarrow & 0 \end{array}$$

for $1 \le j \le n-1$, where $\mathfrak{D}_j = \mathfrak{I}_j/\mathfrak{I}_{j-1}$ and $\mathfrak{I}_0 = \{0\}$. By the same reasoning as in the first case, we have

$$\begin{aligned} \mathfrak{I}_n/\mathfrak{I}_{j-1} &\cong M(\mathfrak{D}_j) \oplus_{q_j,\tau_j} \mathfrak{I}_n/\mathfrak{I}_j \\ &\cong \Gamma^b(X_j, \{K\}) \oplus_{q_j,\tau_j} \mathfrak{I}_n/\mathfrak{I}_j \cong (\Gamma^b(X_j, \{K\}) \otimes K) \oplus_{q_j,\tau_j} \mathfrak{I}_n/\mathfrak{I}_j. \end{aligned}$$

By [9, Proposition 1.6] and [2, Proposition 3.3] we obtain

$$\begin{aligned} \mathrm{RR}(\mathfrak{I}_n/\mathfrak{I}_{j-1}) &\le \max\{\mathrm{RR}(\Gamma^b(X_j, \{K\}) \otimes K), \mathrm{RR}(\mathfrak{I}_n/\mathfrak{I}_j)\} \\ &\le \max\{1, \mathrm{RR}(\mathfrak{I}_n/\mathfrak{I}_j)\}. \end{aligned}$$

Using this inequality repeatedly for j varying, we have $\mathrm{RR}(\mathfrak{I}_n) \le 1$.

Since the union $\cup_j \mathfrak{I}_j$ is dense in $\mathfrak{A}$, we obtain $\mathrm{RR}(\mathfrak{A}) \le \sup_j \mathrm{RR}(\mathfrak{I}_j) \le 1$ as desired. □

Remark. It is shown in Takai and the author [18, Proposition 3.1] that

$$\mathrm{sr}(\mathfrak{A}) \le 2$$

for $\mathfrak{A}$ a separable C^*-algebra of type I which have no finite dimensional irreducible representations, where $\mathrm{sr}(\cdot)$ means the stable rank of C^*-algebras (see [12]). By definition, $\mathrm{sr}(\mathfrak{A}) \in \{1, 2, \cdots, \infty\}$. However, their method is much different from the method in the proof above.

As an important application, we have

Corollary 2.2. *Let G be a non-amenable CCR locally compact group and $C_r^*(G)$ its reduced group C^*-algebra. Then*

$$\mathrm{RR}(C_r^*(G)) \leq 1.$$

Proof. It is known that if G is a non-amenable locally compact group, then $C_r^*(G)$ has no finite dimensional irreducible representations (cf. [4, Chapter 18 and 18.9.5]). □

Remark. It is shown in [13, Proposition 2.3] that

$$\mathrm{sr}(C_r^*(G)) \leq 2$$

for G a non-amenable locally compact group of type I.

In particular, we obtain

Theorem 2.3. *Let G be a non-compact connected real semi-simple Lie group and $C_r^*(G)$ its reduced group C^*-algebra. Then*

$$\mathrm{RR}(C_r^*(G)) = 1.$$

Proof. It is known that G is CCR (cf. [4, Chapter 17 and 17.4.6]), and G is non-amenable since it is non-compact. Thus, $\mathrm{RR}(C_r^*(G)) \leq 1$ by Corollary 2.2. When G has real rank one as a Lie group via Iwasawa decomposition, it follows from the structure of $C_r^*(G)$ given by [13, Lemma 2.2] that $C_r^*(G)$ has a closed ideal of the form $C_0(R) \otimes K$. Since $C_0(R) \otimes K$ has no projections we have $\mathrm{RR}(C_0(R) \otimes K) \geq 1$. By [2, Proposition 3.3] we have $\mathrm{RR}(C_0(R) \otimes K) \leq 1$. Therefore, by [5, Theorem 1.4] we have $\mathrm{RR}(C_r^*(G)) \geq \mathrm{RR}(C_0(R) \otimes K) = 1$. When G has real rank more than one as a Lie group, it follows from the structure of $C_r^*(G)$ given by [13, Lemma 2.1] that $C_r^*(G)$ has a closed ideal of the form $C_0(X) \otimes K$ for X a certain non-compact connected locally compact Hausdorff space. Hence, by the same argument as above we deduce $\mathrm{RR}(C_r^*(G)) \geq 1$. □

Remark. It is shown in [13, Theorem 2.5] that

$$\mathrm{sr}(C_r^*(G)) = \min\{2, \mathrm{rr}(G)\}$$

for G a non-compact connected real semi-simple Lie group, where $\mathrm{rr}(G)$ is the real rank of G.

Furthermore, we obtain

Theorem 2.4. *Let G be a non-compact connected real reductive Lie group and $C_r^*(G)$ its reduced group C^*-algebra. Then*

$$\mathrm{RR}(C_r^*(G)) = 1.$$

Proof. Since G is reductive, non-compact and connected, there exists a non-compact connected semi-simple quotient Lie group S of G. In fact, $S \cong G/R$ for R the radical of G (i.e, the largest connected solvable Lie group in G). Then $C_r^*(S)$ is a quotient of $C_r^*(G)$ since R is amenable. Thus, we obtain $\mathrm{RR}(C_r^*(G)) \geq \mathrm{RR}(C_r^*(S)) = 1$ by [5, Theorem 1.4] and our Theorem 2.3. Since G is CCR and non-amenable, we have $\mathrm{RR}(C_r^*(G)) \leq 1$ by Corollary 2.2. □

Remark. It is shown in [13, Theorem 3.1] ([15, Theorem 3.1]) that

$$\mathrm{sr}(C_r^*(G)) = \min\{2, \max\{\mathrm{rr}([G,G]), \dim Z^\wedge + 1\}\}$$

for G a non-compact connected real reductive Lie group, where $[G,G]$ is the commutator group of G, and Z is the center of G and $Z^\wedge$ is its dual group.

As another consequence of Theorem 2.1 we have

Corollary 2.5. *Let $\mathfrak{A}$ be an inductive limit of CCR C^*-algebras which have no finite dimensional irreducible representations. Then*

$$\mathrm{RR}(\mathfrak{A}) \leq 1.$$

Proof. This follows from the property of the real rank for inductive limits of C^*-algebras (cf. [3, Proposition 3.1] and use its generalization). □

On the other hand, we now recall that a C^*-algebra $\mathfrak{A}$ is n-homogeneous if for any irreducible representation π of $\mathfrak{A}$, its image $\pi(\mathfrak{A})$ of $\mathfrak{A}$ is isomorphic to $M_n(C)$, and $\mathfrak{A}$ is ∞-homogeneous if $\pi(\mathfrak{A}) \cong K$ for any irreducible representation π of $\mathfrak{A}$ (cf. [4]). Following the methods as in Theorem 2.1 we can obtain

Theorem 2.6. *Let $\mathfrak{A}$ be a CCR C^*-algebra. Suppose that $\mathfrak{A}$ has no infinite dimensional irreducible representations. Then we have*

$$\mathrm{RR}(\mathfrak{A}) = \sup_j \lceil \dim X_j/(2n_j - 1) \rceil$$

for a composition series $\{\mathfrak{I}_j\}$ of $\mathfrak{A}$ such that subquotients $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ are isomorphic to $C_0(X_j) \otimes M_{n_j}(C)$ for some n_j, where $\lceil x \rceil$ means the least integer $\geq x$.

Proof. We have treated the case for ∞-homogeneous C^*-algebras in Theorem 2.1. Using the notations as in the proof of Theorem 2.1 we deal with the case for subquotients with continuous trace to be n-homogeneous C^*-algebras. In fact, since $\mathfrak{A}$ is CCR, by the assumption we have a composition series $\{\mathfrak{I}_j\}$ of $\mathfrak{A}$ with such subquotients $\mathfrak{I}_j/\mathfrak{I}_{j-1}$. Thus, we replace the subquotients $\mathfrak{I}_j/\mathfrak{I}_{j-1} = \Gamma_0(X_j, \{K\})$ in the proof of Theorem 2.1 with $\Gamma_0(X_j, \{M_{n_j}(C)\})$ for some n_j. Furthermore, since each $\Gamma_0(X_j, \{M_{n_j}(C)\})$ is locally trivial, we may assume and replace that $\mathfrak{I}_j/\mathfrak{I}_{j-1} = C_0(X_j) \otimes M_{n_j}(C)$ (if necessary by transfinite induction for the subquotients). Then we have $M(\mathfrak{I}_1) \cong C^b(X_1) \otimes M_{n_1}(C) \cong C(\beta X_1) \otimes M_{n_1}(C)$, where

$C^b(X_1)$ is the C^*-algebra of bounded continuous functions on X_1, and βX_1 is the Stone-Čech compactification of X_1 (cf. [1]). Thus, using [9, Proposition 1.6] and [2, Corollary 3.2] we obtain

$$\begin{aligned} \mathrm{RR}(\mathfrak{I}_2) &\leq \max\{\mathrm{RR}(M(\mathfrak{I}_1)), \mathrm{RR}(\mathfrak{I}_2/\mathfrak{I}_1)\} \\ &= \max\{\mathrm{RR}(C(\beta X_1)\otimes M_{n_1}(C)), \mathrm{RR}(C_0(X_2)\otimes M_{n_2}(C))\} \\ &= \max\{\lceil \dim \beta X_1/(2n_1-1)\rceil, \lceil \dim X_2/(2n_2-1)\rceil\} \end{aligned}$$

and repeating this process above we obtain

$$\mathrm{RR}(\mathfrak{A}) \leq \sup_j \lceil \dim X_j/(2n_j-1)\rceil.$$

On the other hand, by [5] we have

$$\mathrm{RR}(\mathfrak{A}) \geq \mathrm{RR}(\mathfrak{I}_j) \geq \mathrm{RR}(\mathfrak{I}_j/\mathfrak{I}_{j-1}) = \lceil \dim X_j/(2n_j-1)\rceil.$$

Therefore, we obtain the conclusion as desired. □

Remark. For any C^*-algebra $\mathfrak{A}$, we have

$$\mathrm{RR}(\mathfrak{A}) \geq \sup_{1\leq n<\infty} \lceil \dim X_n/(2n-1)\rceil,$$

where X_n means the subspace of the spectrum of $\mathfrak{A}$ consisting of n-dimensional irreducible representations of $\mathfrak{A}$. Note that some or all X_n may be empty. Moreover, by the same way we have

$$\mathrm{sr}(\mathfrak{A}) \geq \sup_{1\leq n<\infty} (1 + \lceil [\dim X_n/2]/n\rceil),$$

where $[x]$ means the maximum integer $\leq x$. For a C^*-algebra $\mathfrak{A}$ of type I, it is shown by [16, Corollary 2.8] that

$$\mathrm{sr}(\mathfrak{A}) \leq \max\{2, \sup_{1\leq n<\infty} (1 + \lceil [(1+\dim X_n)/2]/n\rceil)\}.$$

As a general result containing Theorem 2.1 and Theorem 2.6 in part,

Theorem 2.7. *Let $\mathfrak{A}$ be a CCR C^*-algebra. Suppose that $\mathfrak{A}$ is decomposed into the exact sequence*:

$$0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{D} \to 0,$$

where $\mathfrak{I}$ is a CCR C^*-algebra that has no finite dimensional irreducible representations and $\mathfrak{D}$ is a CCR C^*-algebra that has no infinite dimensional irreducible representations. Then

$$\mathrm{RR}(\mathfrak{D}) \leq \mathrm{RR}(\mathfrak{A}) \leq \max\{1, \mathrm{RR}(\mathfrak{D})\}.$$

Namely, in other words,

$$\sup_j \lceil \dim X_j/(2j-1) \rceil \le \mathrm{RR}(\mathfrak{A}) \le \max\{1, \sup_j \lceil \dim X_j/(2j-1) \rceil\},$$

where X_j are the subspaces of the spectrum of $\mathfrak{A}$ consisting of j-dimensional irreducible representations.

Remark. Note that quotients and closed ideals of CCR C^*-algebras are CCR. But extensions of CCR C^*-algebras by CCR C^*-algebras are not always CCR (cf. [4]). In fact, such extensions may not have Hausdorff spectrums.

To prove this theorem we need the following lemma :

Lemma 2.8. *Let $\mathfrak{A}$ be a CCR C^*-algebra. Suppose that $\mathfrak{A}$ is decomposed into the exact sequence*:

$$0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{D} \to 0,$$

where $\mathfrak{I}$ is an essential closed ideal of $\mathfrak{A}$ and is a continuous trace C^*-algebra that has no finite dimensional irreducible representations and $\mathfrak{D}$ is a CCR C^*-algebra. Then

$$\mathrm{RR}(\mathfrak{A}) \le \max\{1, \mathrm{RR}(\mathfrak{D})\}.$$

Proof. We use the first step of the argument in the proof of Theorem 2.1. Namely, replace $\mathfrak{I}_1$ and $\mathfrak{I}_2/\mathfrak{I}_1$ with $\mathfrak{I}$ and $\mathfrak{D}$ respectively. □

Remark. If a C^*-algebra $\mathfrak{A}$ has an ∞-homogeneous closed ideal $\mathfrak{I}$ with continuous trace, then it is shown by [10] that

$$\mathrm{sr}(\mathfrak{A}) \le \max\{2, \mathrm{sr}(\mathfrak{A}/\mathfrak{I})\}.$$

Proof of Theorem 2.7. Since $\mathfrak{A}$ is CCR (and of type I), there exists an essential composition series $\{\mathfrak{I}_j\}$ of $\mathfrak{A}$ such that subquotients $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ are of continuous trace and each $\mathfrak{I}_j$ is essential in $\mathfrak{I}_{j+1}$. For each j, we consider the following exact sequence:

$$0 \to \mathfrak{I} \cap \mathfrak{I}_j \to \mathfrak{I}_j \to \mathfrak{D}_j \to 0,$$

where the quotient $\mathfrak{D}_j$ corresponds to the subspace $\mathfrak{D}^\wedge \cap (\mathfrak{I}_j^\wedge \setminus (\mathfrak{I} \cap \mathfrak{I}_j)^\wedge)$, where $\mathfrak{B}^\wedge$ for a C^*-algebra $\mathfrak{B}$ means the spectrum of $\mathfrak{B}$ consisting of its irreducible representations up to unitary equivalence. Now set $\mathfrak{K}_k = \mathfrak{I} \cap \mathfrak{I}_k$ for $1 \le k \le j$ and $\mathfrak{K}_0 = \{0\}$. Then we consider the following exact sequences:

$$0 \to \mathfrak{K}_k/\mathfrak{K}_{k-1} \to \mathfrak{I}_j/\mathfrak{K}_{k-1} \to \mathfrak{I}_j/\mathfrak{K}_k \to 0$$

for $1 \le k \le j$. Note that $\mathfrak{I}_j/\mathfrak{K}_{k-1}$ are CCR and $\mathfrak{K}_k/\mathfrak{K}_{k-1}$ are ∞-homogeneous continuous trace C^*-algebras. Hence, using the argument for the general step in the proof of Theorem 2.1 and Lemma 2.8 repeatedly we obtain

$$\mathrm{RR}(\mathfrak{I}_j/\mathfrak{K}_{k-1}) \le \max\{1, \mathrm{RR}(\mathfrak{I}_j/\mathfrak{K}_k)\}$$

for $1 \leq k \leq j$. This implies that

$$\mathrm{RR}(\mathfrak{I}_j) \leq \max\{1, \mathrm{RR}(\mathfrak{D}_j)\} \leq \max\{1, \mathrm{RR}(\mathfrak{D})\}.$$

Note that $\mathfrak{D}_j$ are subquotients of $\mathfrak{D}$. Since the union of $\mathfrak{I}_j$ is dense in $\mathfrak{A}$, we have

$$\mathrm{RR}(\mathfrak{A}) \leq \sup \mathrm{RR}(\mathfrak{I}_j) \leq \max\{1, \mathrm{RR}(\mathfrak{D})\}$$

as desired. □

As an important application, we obtain

Theorem 2.9. *Let G be a connected nilpotent Lie group and $C^*(G)$ its group C^*-algebra. Then*

$$\mathrm{RR}(C^*(G)) = \dim G_1^\wedge,$$

where $G_1^\wedge$ is the space of all 1-dimensional representations of G.

Proof. Since $C^*(G)$ (or G) is CCR and we have the following exact sequence:

$$0 \to \mathfrak{I} \to C^*(G) \to C_0(G_1^\wedge) \to 0,$$

where $\mathfrak{I}$ is a CCR C^*-algebra that has no finite dimensional irreducible representations (note that since G is a solvable Lie group, the Lie's theorem tells us that G (or $C^*(G)$) has no finite dimensional irreducible representations except 1-dimensional ones), we can use Theorem 2.7 so that

$$\mathrm{RR}(C_0(G_1^\wedge)) \leq \mathrm{RR}(C^*(G)) \leq \max\{1, \mathrm{RR}(C_0(G_1^\wedge))\}.$$

By [3, Proposition 1.1] we have $\mathrm{RR}(C_0(G_1^\wedge)) = \dim G_1^\wedge$. Moreover, [14] implies that $\dim G_1^\wedge = 0$ if and only if G is isomorphic to the k-torus T^k. In this case we have $\mathrm{RR}(C^*(G)) = 0$ since $C^*(T^k) \cong C_0(Z^k)$ by the Fourier transform and $C_0(Z^k) \cong \oplus_{Z^k} C$ the c_0-direct sum of C. Furthermore, by [14] we have $\dim G_1^\wedge = 1$ if and only if G is isomorphic to either $R \times T^k$ or R. In this case we have $\mathrm{RR}(C^*(G)) = 1$ since $C^*(R \times T^k) \cong C_0(R \times Z^k)$. If $\dim G_1^\wedge \geq 2$, then the inequalities obtained above imply the conclusion. □

Remark. Note that the space $G_1^\wedge$ is homeomorphic to the dual group $(G/[G,G])^\wedge$ of the quotient connected abelian Lie group $G/[G,G]$ so that $G/[G,G] \cong R^s \times T^t$ and hence $G_1^\wedge \approx R^s \times Z^t$ for some $s, t \geq 0$. For G a connected nilpotent Lie group, it is shown by [14] that

$$\mathrm{sr}(C^*(G)) = [\dim G_1^\wedge/2] + 1.$$

This generalizes the case for G a simply connected nilpotent Lie group by [17].

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