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On Bessel's and Grüss Inequalities for Orthonormal Families in 2-Inner Product Spaces and Applications

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Abstract. A new counterpart of Bessel's inequality for orthonormal families in real or complex 2-inner product spaces is obtained. Applications for some Grüss inequality for determinantal integral inequalities are also provided.

1. Introduction

The concepts of 2-inner products and 2-inner product spaces have been intensively studied by many authors in the last three decades. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in [1]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let X be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $(\cdot, \cdot|\cdot)$ is a $\mathbb{K}$-valued function defined on $X \times X \times X$ satisfying the following conditions:

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($2I_1$) $(x, x|z) \geq 0$ and $(x, x|z) = 0$ if and only if x and z are linearly dependent,
($2I_2$) $(x, x|z) = (z, z|x)$,
($2I_3$) $(y, x|z) = \overline{(x, y|z)}$,
($2I_4$) $(\alpha x, y|z) = \alpha(x, y|z)$ for any scalar $\alpha \in \mathbb{K}$,
($2I_5$) $(x + x', y|z) = (x, y|z) + (x', y|z)$.

$(\cdot, \cdot|\cdot)$ is called a 2-*inner product* on X and $(X, (\cdot, \cdot|\cdot))$ is called a 2-*inner product space* (or 2-*pre-Hilbert space*). Some basic properties of 2-inner product spaces can be immediately obtained as follows [2]: (1) If $\mathbb{K} = \mathbb{R}$, then ($2I_3$) reduces to

$$(y, x|z) = (x, y|z).$$

(2) From ($2I_3$) and ($2I_4$), we have

$$(0, y|z) = 0, \quad (x, 0|z) = 0$$

and also

$$(x, \alpha y|z) = \bar{\alpha}(x, y|z). \tag{1.1}$$

(3) Using ($2I_2$)-($2I_5$), we have

$$(z, z|x \pm y) = (x \pm y, x \pm y|z) = (x, x|z) + (y, y|z) \pm 2\mathrm{Re}(x, y|z)$$

and

$$\mathrm{Re}(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)]. \tag{1.2}$$

In the real case $\mathbb{K} = \mathbb{R}$, (1.2) reduces to

$$(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)] \tag{1.3}$$

and, using this formula, it is easy to see that, for any $\alpha \in \mathbb{R}$,

$$(x, y|\alpha z) = \alpha^2(x, y|z). \tag{1.4}$$

In the complex case, using (1.1) and (1.2), we have

$$\mathrm{Im}(x, y|z) = \mathrm{Re}[-i(x, y|z)] = \frac{1}{4}[(z, z|x + iy) - (z, z|x - iy)],$$

which, in combination with (1.2), yields

$$(x, y|z) = \frac{1}{4}[(z, z|x + y) - (z, z|x - y)] + \frac{i}{4}[(z, z|x + iy) - (z, z|x - iy)]. \tag{1.5}$$

Using the above formula and (1.1), we have, for any $\alpha \in \mathbb{C}$,

$$(x, y|\alpha z) = |\alpha|^2(x, y|z). \tag{1.6}$$

However, for $\alpha \in \mathbb{R}$, (1.6) reduces to (1.4). Also, from (1.6) it follows that

$$(x, y|0) = 0.$$

(4) For any three given vectors $x, y, z \in X$, consider the vector $u = (y, y|z)x - (x, y|z)y$. By $(2I_1)$, we know that $(u, u|z) \geq 0$ with the equality if and only if u and z are linearly dependent. The inequality $(u, u|z) \geq 0$ can be rewritten as,

$$(y, y|z)[(x, x|z)(y, y|z) - |(x, y|z)|^2] \geq 0. \tag{1.7}$$

For $x = z$, (1.7) becomes

$$-(y, y|z)|(z, y|z)|^2 \geq 0,$$

which implies that

$$(z, y|z) = (y, z|z) = 0 \tag{1.8}$$

provided y and z are linearly independent. Obviously, when y and z are linearly dependent, (1.8) holds too. Thus (1.8) is true for any two vectors $y, z \in X$. Now, if y and z are linearly independent, then $(y, y|z) > 0$ and, from (1.7), it follows that

$$|(x, y|z)|^2 \leq (x, x|z)(y, y|z). \tag{1.9}$$

Using (1.8), it is easy to check that (1.9) is trivially fulfilled when y and z are linearly dependent. Therefore, the inequality (1.9) holds for any three vectors $x, y, z \in X$ and is strict unless the vectors $u = (y, y|z)x - (x, y|z)y$ and z are linearly dependent. In fact, we have the equality in (1.9) if and only if the three vectors x, y and z are linearly dependent. In any given 2-inner product space $(X, (\cdot, \cdot \mid \cdot))$, we can define a function $\|\cdot \mid \cdot\|$ on $X \times X$ by

$$\|x|z\| = \sqrt{(x, x|z)} \tag{1.10}$$

for all $x, z \in X$.

It is easy to see that this function satisfies the following conditions:
($2N_1$) $\|x|z\| \geq 0$ and $\|x|z\| = 0$ if and only if x and z are linearly dependent,
($2N_2$) $\|z|x\| = \|x|z\|$,
($2N_3$) $\|\alpha x|z\| = |\alpha|\|x|z\|$ for any scalar $\alpha \in \mathbb{K}$,
($2N_4$) $\|x + x'|z\| \leq \|x|z\| + \|x'|z\|$.
Any function $\|\cdot|\cdot\|$ defined on $X \times X$ and satisfying the conditions ($2N_1$)-($2N_4$) is called a 2-*norm* on X and $(X, \|\cdot \mid \cdot\|)$ is called a *linear* 2-*normed space* [5].

Whenever a 2-inner product space $(X, (\cdot, \cdot|\cdot))$ is given, we consider it as a linear 2-normed space $(X, \|\cdot \mid \cdot\|)$ with the 2-norm defined by (1.10).

Let $(X, (\cdot, \cdot|\cdot))$ be a 2-inner product space over the real or complex number field $\mathbb{K}$. If $(f_i)_{1\leq i\leq n}$ are linearly independent vectors in the 2-inner product space X, and, for a given $z \in X, (f_i, f_j|z) = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$ where δ_{ij} is the

Kronecker delta (we say that the family $(f_i)_{1\le i\le n}$ is $z-$*orthonormal*), then the following inequality is the corresponding *Bessel's inequality* (see for example [2]) for the $z-$*orthonormal family* $(f_i)_{1\le i\le n}$ in the 2-inner product space $(X,(\cdot,\cdot|\cdot))$:

$$\sum_{i=1}^{n}|(x,f_i|z)|^2\le\|x|z\|^2 \tag{1.11}$$

for any $x\in X$. For more details on this inequality, see the recent paper [2] and the references therein.

The following reverse of Bessel's inequality in 2-inner product spaces has been obtained in [3]:

Theorem 1. *Let* $\{e_i\}_{i\in I}$, F, ϕ_i,Φ_i, $i\in F$ *and* $x,z\in X$ *so that either*

(i) $\operatorname{Re}\left(\sum_{i\in F}\Phi_i e_i-x,x-\sum_{i\in F}\phi_i e_i|z\right)\ge 0$

or, equivalently,

(ii) $\left\|x-\sum_{i\in F}\dfrac{\phi_i+\Phi_i}{2}e_i|z\right\|\le\dfrac{1}{2}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\dfrac{1}{2}}$

holds. Then we have the inequality:

$$\begin{aligned}0&\le\|x|z\|^2-\sum_{i\in F}|(x,e_i|z)|^2\\&\le\frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2-\operatorname{Re}\left(\sum_{i\in F}\Phi_i e_i-x,x-\sum_{i\in F}\phi_i e_i|z\right)\\&\quad\left(\le\frac{1}{4}\sum_{i\in F}|\Phi_i-\phi_i|^2\right).\end{aligned}$$

The constant $\frac{1}{4}$ *is best possible.*

The main aim of the present paper is to establish a different reverse inequality for (1.11). Some companion results and applications for determinantal integral inequalities are also given.

2. Another reverse of Bessel's inequality

The following lemma holds.

Lemma 1. *Let* $\{e_i\}_{i\in I}$ *be a family of z-orthonormal vectors in* X, F *a finite part of* I, $\lambda_i\in\mathbb{K}$, $i\in F$, $r>0$ *and* $x\in X$. *If*

$$\left\|x-\sum_{i\in F}\lambda_i e_i|z\right\|\le r, \tag{2.1}$$

then we have the inequality

$$0 \le \|x|z\|^2 - \sum_{i\in F} |(x, e_i|z)|^2 \le r^2 - \sum_{i\in F} |\lambda_i - (x, e_i|z)|^2 . \tag{2.2}$$

Proof. Consider

$$\begin{aligned} I_1 &:= \left\| x - \sum_{i\in F} \lambda_i e_i | z \right\|^2 = \left(x - \sum_{i\in F} \lambda_i e_i, x - \sum_{j\in F} \lambda_j e_j | z \right) \\ &= \|x|z\|^2 - \sum_{i\in F} \lambda_i \overline{(x, e_i|z)} - \sum_{i\in F} \overline{\lambda_i} (x, e_i|z) + \sum_{i\in F} \sum_{j\in F} \lambda_i \overline{\lambda_j} (e_i, e_j|z) \\ &= \|x|z\|^2 - \sum_{i\in F} \lambda_i \overline{(x, e_i|z)} - \sum_{i\in F} \overline{\lambda_i} (x, e_i|z) + \sum_{i\in F} |\lambda_i|^2 \end{aligned}$$

and

$$\begin{aligned} I_2 &:= \sum_{i\in F} |\lambda_i - (x, e_i|z)|^2 = \sum_{i\in F} (\lambda_i - (x, e_i|z)) \left(\overline{\lambda_i} - \overline{(x, e_i|z)} \right) \\ &= \sum_{i\in F} \left[|\lambda_i|^2 + |(x, e_i|z)|^2 - \overline{\lambda_i} (x, e_i|z) - \lambda_i \overline{(x, e_i|z)} \right] \\ &= \sum_{i\in F} |\lambda_i|^2 + \sum_{i\in F} |(x, e_i|z)|^2 - \sum_{i\in F} \overline{\lambda_i} (x, e_i|z) - \sum_{i\in F} \lambda_i \overline{(x, e_i|z)}. \end{aligned}$$

If we subtract I_2 from I_1, we deduce an identity that is interesting in its own right:

$$\left\| x - \sum_{i\in F} \lambda_i e_i | z \right\|^2 - \sum_{i\in F} |\lambda_i - (x, e_i|z)|^2 = \|x|z\|^2 - \sum_{i\in F} |(x, e_i|z)|^2 , \tag{2.3}$$

from which we easily deduce (2.2). □

The following reverse of Bessel's inequality holds.

Theorem 2. *Let $\{e_i\}_{i\in I}$ be a family of z-orthonormal vectors in X, F a finite part of I, ϕ_i, Φ_i, $i \in I$ real or complex numbers. For $x \in X$, if either*

(i) $\operatorname{Re}\left(\sum_{i\in F} \Phi_i e_i - x, x - \sum_{i\in F} \phi_i e_i | z \right) \ge 0$,

or, equivalently,

(ii) $\left\| x - \sum_{i\in F} \frac{\phi_i + \Phi_i}{2} e_i | z \right\| \le \frac{1}{2} \left(\sum_{i\in F} |\Phi_i - \phi_i|^2 \right)^{\frac{1}{2}}$

holds, then the following reverse of Bessel's inequality

$$
\begin{aligned}
0 &\le \|x|z\|^2 - \sum_{i\in F} |(x, e_i|z)|^2 \\
&\le \frac{1}{4}\sum_{i\in F} |\Phi_i - \phi_i|^2 - \sum_{i\in F}\left|\frac{\phi_i + \Phi_i}{2} - (x, e_i|z)\right|^2 \\
&\left(\le \frac{1}{4}\sum_{i\in F} |\Phi_i - \phi_i|^2\right)
\end{aligned} \tag{2.4}
$$

is valid. The constant $\frac{1}{4}$ *is best possible.*

Proof. Firstly, we observe that, for $y, a, A \in X$, the following are equivalent

$$\mathrm{Re}\,(A - y, y - a|z) \ge 0 \tag{2.5}$$

and

$$\left\| y - \frac{a + A}{2}|z \right\| \le \frac{1}{2}\|A - a|z\|. \tag{2.6}$$

Now, for $a = \sum_{i\in F} \phi_i e_i$, $A = \sum_{i\in F} \Phi_i e_i$, we have

$$
\begin{aligned}
\|A - a|z\| &= \left\|\sum_{i\in F} (\Phi_i - \phi_i)\, e_i|z\right\| = \left(\left\|\sum_{i\in F} (\Phi_i - \phi_i)\, e_i|z\right\|^2\right)^{\frac{1}{2}} \\
&= \left(\sum_{i\in F} |\Phi_i - \phi_i|^2 \|e_i|z\|^2\right)^{\frac{1}{2}} = \left(\sum_{i\in F} |\Phi_i - \phi_i|^2\right)^{\frac{1}{2}},
\end{aligned}
$$

which gives, for $y = x$, the desired equivalence.

Now, if we apply Lemma 1 for $\lambda_i = \frac{\phi_i + \Phi_i}{2}$ and

$$r := \frac{1}{2}\left(\sum_{i\in F} |\Phi_i - \phi_i|^2\right)^{\frac{1}{2}},$$

then we deduce the first inequality in (2.4).

Let us prove that $\frac{1}{4}$ is best possible in the second inequality in (2.4).

Assume that there is a $c > 0$ such that

$$0 \le \|x|z\|^2 - \sum_{i\in F} |(x, e_i|z)|^2 \le c\sum_{i\in F} |\Phi_i - \phi_i|^2 - \sum_{i\in F}\left|\frac{\phi_i + \Phi_i}{2} - (x, e_i|z)\right|^2, \tag{2.7}$$

provided that ϕ_i, Φ_i, x and F satisfy (i) or (ii).

Now, let $F = \{1\}, e_1 = e, \|e|z\| = 1$ and $m \in X$ so that $\|m|z\| = 1$ and $(m, e|z) = 0$. For $\Phi_1 = \Phi, \phi_1 = \phi, \Phi \neq \phi$, define the vector

$$x := \frac{\Phi+\phi}{2}e + \frac{\Phi-\phi}{2}m.$$

A simple calculation shows that

$$(\Phi e - x, x - \phi e|z) = \left|\frac{\Phi-\phi}{2}\right|^2 (e - x, x - e|z) = 0$$

and thus the condition (i) of the theorem holds true for $F = \{1\}$.

Observe also that

$$\begin{aligned} \|x|z\|^2 &= \left\|\frac{\Phi+\phi}{2}e + \frac{\Phi-\phi}{2}m|z\right\|^2 \\ &= \left|\frac{\Phi+\phi}{2}\right|^2 + \left|\frac{\Phi-\phi}{2}\right|^2 \end{aligned}$$

and

$$(x, e|z) = \left(\frac{\Phi+\phi}{2}e + \frac{\Phi-\phi}{2}m, e|z\right) = \frac{\Phi+\phi}{2}.$$

Consequently, by (2.7), we deduce

$$\left|\frac{\Phi-\phi}{2}\right|^2 \leq c\,|\Phi-\phi|^2,$$

which gives $c \geq \frac{1}{4}$, and the proof is completed. □

Remark 1. If $F = \{1\}$, $e_1 = e$, $\|e|z\| = 1$ and, for $\phi, \Phi \in \mathbb{K}$ and $x \in X$, one has either

$$\operatorname{Re}(\Phi e - x, x - \phi e|z) \geq 0 \tag{2.8}$$

or, equivalently,

$$\left\|x - \frac{\phi+\Phi}{2}e|z\right\| \leq \frac{1}{2}|\Phi-\phi|, \tag{2.9}$$

then

$$\begin{aligned} 0 &\leq \|x|z\|^2 - |(x, e|z)|^2 \\ &\leq \frac{1}{4}|\Phi-\phi|^2 - \left|\frac{\phi+\Phi}{2} - (x, e|z)\right|^2 \left(\leq \frac{1}{4}|\Phi-\phi|^2\right). \end{aligned} \tag{2.10}$$

The constant $\frac{1}{4}$ is best possible.

3. A refinement of the Grüss inequality

The following result holds.

Theorem 3. *Let $\{e_i\}_{i\in I}$ be a family of z-orthonormal vectors in X, F a finite part of I, $\phi_i, \Phi_i,\ \gamma_i, \Gamma_i \in \mathbb{K},\ i \in F$ and $x, y \in X$. If either*

$$\operatorname{Re}\left(\sum_{i\in F}\Phi_i e_i - x, x - \sum_{i\in F}\phi_i e_i | z\right) \geq 0, \tag{3.1}$$
$$\operatorname{Re}\left(\sum_{i\in F}\Gamma_i e_i - y, y - \sum_{i\in F}\gamma_i e_i | z\right) \geq 0$$

or, equivalently,

$$\left\| x - \sum_{i\in F}\frac{\Phi_i + \phi_i}{2} e_i | z \right\| \leq \frac{1}{2}\left(\sum_{i\in F}\left|\Phi_i - \phi_i\right|^2\right)^{\frac{1}{2}}, \tag{3.2}$$
$$\left\| y - \sum_{i\in F}\frac{\Gamma_i + \gamma_i}{2} e_i | z \right\| \leq \frac{1}{2}\left(\sum_{i\in F}\left|\Gamma_i - \gamma_i\right|^2\right)^{\frac{1}{2}}$$

hold, then we have the inequalities

$$\begin{aligned} 0 &\leq \left|(x, y|z) - \sum_{i\in F}(x, e_i|z)\,(e_i, y|z)\right| \\ &\leq \frac{1}{4}\left(\sum_{i\in F}\left|\Phi_i - \phi_i\right|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{i\in F}\left|\Gamma_i - \gamma_i\right|^2\right)^{\frac{1}{2}} \\ &\qquad - \sum_{i\in F}\left|\frac{\Phi_i + \phi_i}{2} - (x, e_i|z)\right|\left|\frac{\Gamma_i + \gamma_i}{2} - (y, e_i|z)\right| \\ &\left(\leq \frac{1}{4}\left(\sum_{i\in F}\left|\Phi_i - \phi_i\right|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{i\in F}\left|\Gamma_i - \gamma_i\right|^2\right)^{\frac{1}{2}}\right). \end{aligned} \tag{3.3}$$

The constant $\frac{1}{4}$ is best possible.

Proof. Using Schwarz's inequality in the 2-inner product space $(X, (\cdot, \cdot|\cdot))$, one has

$$\left|\left(x - \sum_{i \in F} (x, e_i)\, e_i, y - \sum_{i \in F} (y, e_i)\, e_i | z\right)\right|^2 \leq \left\|x - \sum_{i \in F} \langle x, e_i \rangle\, e_i | z\right\|^2 \left\|y - \sum_{i \in F} \langle y, e_i \rangle\, e_i | z\right\|^2 \tag{3.4}$$

and, since a simple calculation shows that

$$\left(x - \sum_{i \in F} (x, e_i)\, e_i, y - \sum_{i \in F} (y, e_i)\, e_i | z\right) = (x, y|z) - \sum_{i \in F} (x, e_i|z)\,(e_i, y|z)$$

and

$$\left\|x - \sum_{i \in F} (x, e_i)\, e_i | z\right\|^2 = \|x|z\|^2 - \sum_{i \in F} |(x, e_i|z)|^2$$

for any $x, y \in X$, then, by (3.4) and the reverse of Bessel's inequality in Theorem 2, we have

$$\begin{aligned} &\left|(x, y|z) - \sum_{i \in F} (x, e_i|z)\,(e_i, y|z)\right|^2 \\ &\leq \left(\|x|z\|^2 - \sum_{i \in F} |(x, e_i|z)|^2\right)\left(\|y|z\|^2 - \sum_{i \in F} |(y, e_i|z)|^2\right) \\ &\leq \left[\frac{1}{4}\sum_{i \in F} |\Phi_i - \phi_i|^2 - \sum_{i \in F} \left|\frac{\Phi_i + \phi_i}{2} - (x, e_i|z)\right|^2\right] \\ &\qquad \times \left[\frac{1}{4}\sum_{i \in F} |\Gamma_i - \gamma_i|^2 - \sum_{i \in F} \left|\frac{\Gamma_i + \gamma_i}{2} - (y, e_i|z)\right|^2\right] \\ &:= K. \end{aligned} \tag{3.5}$$

Using Aczél's inequality for real numbers, i.e., we recall that

$$\left(a^2 - \sum_{i \in F} a_i^2\right)\left(b^2 - \sum_{i \in F} b_i^2\right) \leq \left(ab - \sum_{i \in F} a_i b_i\right)^2, \tag{3.6}$$

provided that $a, b, a_i, b_i > 0$, $i \in F$, (originally, Aczél proved it under more restrictive assumptions for a, b, a_i, b_i, i.e., either $a^2 - \sum_{i \in F} a_i^2$ or $b^2 - \sum_{i \in F} b_i^2$ are nonnegative, but those conditions are not necessary), we may state that

$$K \le \left[\frac{1}{4}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}}\cdot\left(\sum_{i\in F}|\Gamma_i-\gamma_i|^2\right)^{\frac{1}{2}} - \sum_{i\in F}\left|\frac{\Phi_i+\phi_i}{2}-(x,e_i|z)\right|\left|\frac{\Gamma_i+\gamma_i}{2}-(y,e_i|z)\right|\right]^2. \tag{3.7}$$

Using (3.5) and (3.7) we conclude that

$$\begin{aligned}&\left|(x,y|z)-\sum_{i\in F}(x,e_i|z)(e_i,y|z)\right|^2\\ &\le \left[\frac{1}{4}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}}\cdot\left(\sum_{i\in F}|\Gamma_i-\gamma_i|^2\right)^{\frac{1}{2}} - \sum_{i\in F}\left|\frac{\Phi_i+\phi_i}{2}-(x,e_i|z)\right|\left|\frac{\Gamma_i+\gamma_i}{2}-(y,e_i|z)\right|\right]^2.\end{aligned} \tag{3.8}$$

Taking the square root in (3.8) and taking into account that the quantity in the last square brackets is nonnegative (see for example (2.4)), we deduce the second inequality in (3.3).

The fact that $\frac{1}{4}$ is the best possible constant follows by Theorem 2 and we omit the details. □

The following corollary may be stated.

Corollary 1. *Let $e \in X$, $\|e|z\| = 1$, $\phi, \Phi, \gamma, \Gamma \in \mathbb{K}$ and $x, y \in X$ such that either*

$$\operatorname{Re}(\Phi e - x, x - \phi e|z) \ge 0, \quad \operatorname{Re}(\Gamma e - y, y - \gamma e|z) \ge 0 \tag{3.9}$$

or, equivalently,

$$\left\|x-\frac{\phi+\Phi}{2}e|z\right\| \le \frac{1}{2}|\Phi-\phi|, \quad \left\|y-\frac{\gamma+\Gamma}{2}e|z\right\| \le \frac{1}{2}|\Gamma-\gamma|. \tag{3.10}$$

Then we have the following refinement of Grüss' inequality

$$\begin{aligned}0 &\le |(x,y|z)-(x,e|z)(e,y|z)|\\ &\le \frac{1}{4}|\Phi-\phi||\Gamma-\gamma| - \left|\frac{\phi+\Phi}{2}-(x,e|z)\right|\left|\frac{\gamma+\Gamma}{2}-(y,e|z)\right|\\ &\left(\le \frac{1}{4}|\Phi-\phi||\Gamma-\gamma|\right).\end{aligned} \tag{3.11}$$

The constant $\frac{1}{4}$ *is best possible.*

4. Some companion inequalities

The following companion of the Grüss inequality also holds.

Theorem 4. *Let* $\{e_i\}_{i\in I}$ *be a family of orthonormal vectors in* X, F *a finite part of* I *and* $\phi_i, \Phi_i \in \mathbb{K}$, $i \in F$, $x, y \in X$ *and* $\lambda \in (0,1)$ *such that either*

$$\operatorname{Re}\left(\sum_{i\in F}\Phi_i e_i - (\lambda x + (1-\lambda)y), \lambda x + (1-\lambda)y - \sum_{i\in F}\phi_i e_i | z\right) \geq 0 \tag{4.1}$$

or, equivalently,

$$\left\|\lambda x + (1-\lambda)y - \sum_{i\in F}\frac{\Phi_i + \phi_i}{2}\cdot e_i | z\right\| \leq \frac{1}{2}\left(\sum_{i\in F}|\Phi_i - \phi_i|^2\right)^{\frac{1}{2}} \tag{4.2}$$

holds. Then we have the inequality

$$\begin{aligned}
&\operatorname{Re}\left[(x,y|z) - \sum_{i\in F}(x,e_i|z)(e_i,y|z)\right] \\
&\leq \frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}\sum_{i\in F}|\Phi_i - \phi_i|^2 \\
&\qquad - \frac{1}{4}\frac{1}{\lambda(1-\lambda)}\sum_{i\in F}\left|\frac{\Phi_i+\phi_i}{2} - (\lambda x + (1-\lambda)y, e_i|z)\right|^2 \\
&\left(\leq \frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}\sum_{i\in F}|\Phi_i - \phi_i|^2\right)
\end{aligned} \tag{4.3}$$

The constant $\frac{1}{16}$ *is the best possible constant in* (4.3) *in the sense that it cannot be replaced by a smaller constant.*

Proof. We know, for any $z, u, v \in X$, that one has

$$\operatorname{Re}(z,u|v) \leq \frac{1}{4}\|z+u|v\|^2.$$

Then, for any $a, b, z \in X$ and $\lambda \in (0,1)$, one has

$$\operatorname{Re}(a,b|z) \leq \frac{1}{4\lambda(1-\lambda)}\|\lambda a + (1-\lambda)b|z\|^2. \tag{4.4}$$

Since

$$(x,y|z) - \sum_{i\in F}(x,e_i|z)(e_i,y|z) = \left(x - \sum_{i\in F}(x,e_i)e_i, y - \sum_{i\in F}(y,e_i)e_i | z\right),$$

for any $x, y \in X$, then, by (4.4), we get

$$
\begin{aligned}
(4.5)\qquad & \operatorname{Re}\left[(x,y|z) - \sum_{i\in F}(x,e_i|z)(e_i,y|z)\right] \\
&= \operatorname{Re}\left[\left(x - \sum_{i\in F}\langle x,e_i\rangle e_i, y - \sum_{i\in F}\langle y,e_i\rangle e_i|z\right)\right] \\
&\le \frac{1}{4\lambda(1-\lambda)}\left\|\lambda\left(x - \sum_{i\in F}(x,e_i)e_i\right) + (1-\lambda)\left(y - \sum_{i\in F}(y,e_i)e_i\right)|z\right\|^2 \\
&= \frac{1}{4\lambda(1-\lambda)}\left\|\lambda x + (1-\lambda)y - \sum_{i\in F}(\lambda x + (1-\lambda)y, e_i)e_i|z\right\|^2 \\
&= \frac{1}{4\lambda(1-\lambda)}\left[\|\lambda x + (1-\lambda)y|z\|^2 - \sum_{i\in F}|(\lambda x + (1-\lambda)y, e_i|z)|^2\right].
\end{aligned}
$$

If we apply the reverse of Bessel's inequality in Theorem 2 for $\lambda x + (1-\lambda)y$, we may state that

$$
\begin{aligned}
(4.6)\qquad & \|\lambda x + (1-\lambda)y|z\|^2 - \sum_{i\in F}|(\lambda x + (1-\lambda)y, e_i|z)|^2 \\
&\le \frac{1}{4}\sum_{i\in F}|\Phi_i - \phi_i|^2 - \sum_{i\in F}\left|\frac{\Phi_i + \phi_i}{2} - (\lambda x + (1-\lambda)y, e_i|z)\right|^2 \\
&\le \frac{1}{4}\sum_{i\in F}|\Phi_i - \phi_i|^2.
\end{aligned}
$$

Now, by making use of (4.5) and (4.6), we deduce (4.3).

The fact that $\frac{1}{16}$ is the best possible constant in (4.3) follows by the fact that if in (4.1) we choose $x = y$, then it becomes (i) of Theorem 2, which implies for $\lambda = \frac{1}{2}$ the inequality (2.4), and so we have shown that $\frac{1}{4}$ is the best constant. □

Remark 2. In practical applications, we may use only the inequality between the first and the last term in (4.3).

Remark 3. If, in Theorem 4, we choose $\lambda = \frac{1}{2}$, then we get

$$\begin{aligned}(4.7)\qquad &\operatorname{Re}\left[(x, y|z) - \sum_{i\in F}(x, e_i|z)(e_i, y|z)\right] \\ &\le \frac{1}{4}\sum_{i\in F}|\Phi_i - \phi_i|^2 - \sum_{i\in F}\left|\frac{\Phi_i + \phi_i}{2} - \left(\frac{x+y}{2}, e_i|z\right)\right|^2 \\ &\left(\le \frac{1}{4}\sum_{i\in F}|\Phi_i - \phi_i|^2\right),\end{aligned}$$

provided

$$\operatorname{Re}\left(\sum_{i\in F}\Phi_i e_i - \frac{x+y}{2}, \frac{x+y}{2} - \sum_{i\in F}\phi_i e_i|z\right) \ge 0$$

or, equivalently,

$$\left\|\frac{x+y}{2} - \sum_{i\in F}\frac{\Phi_i + \phi_i}{2}\cdot e_i|z\right\| \le \frac{1}{2}\left(\sum_{i\in F}|\Phi_i - \phi_i|^2\right)^{\frac{1}{2}}.$$

Corollary 2. *With the assumptions of Theorem* 4 *and if*

$$(4.8)\qquad \operatorname{Re}\left(\sum_{i\in F}\Phi_i e_i - (\lambda x \pm (1-\lambda)y), \lambda x \pm (1-\lambda)y - \sum_{i\in F}\phi_i e_i|z\right) \ge 0$$

or, equivalently,

$$(4.9)\qquad \left\|\lambda x \pm (1-\lambda)y - \sum_{i\in F}\frac{\Phi_i + \phi_i}{2}\cdot e_i|z\right\| \le \frac{1}{2}\left(\sum_{i\in F}|\Phi_i - \phi_i|^2\right)^{\frac{1}{2}},$$

then we have the inequality

$$(4.10)\qquad \left|\operatorname{Re}\left[(x, y|z) - \sum_{i\in F}(x, e_i|z)(e_i, y|z)\right]\right| \le \frac{1}{16}\cdot\frac{1}{\lambda(1-\lambda)}\sum_{i\in F}|\Phi_i - \phi_i|^2.$$

The constant $\frac{1}{16}$ *is best possible in* (4.10).

Remark 4. If X is a real inner product space and $m_i, M_i \in \mathbb{R}$ with the property

$$(4.11)\qquad \left(\sum_{i\in F}M_i e_i - (\lambda x \pm (1-\lambda)y), \lambda x \pm (1-\lambda)y - \sum_{i\in F}m_i e_i|z\right) \ge 0$$

or, equivalently,

$$\left\|\lambda x \pm (1-\lambda) y - \sum_{i\in F} \frac{M_i + m_i}{2} \cdot e_i | z\right\| \leq \frac{1}{2}\left[\sum_{i\in F} (M_i - m_i)^2\right]^{\frac{1}{2}}, \tag{4.12}$$

then we have the Grüss type inequality

$$\left|(x, y|z) - \sum_{i\in F} (x, e_i|z)(e_i, y|z)\right| \leq \frac{1}{16} \cdot \frac{1}{\lambda(1-\lambda)} \sum_{i\in F} (M_i - m_i)^2 . \tag{4.13}$$

5. Applications for determinantal integral inequalities

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, Σ a $\sigma-$algebra of subsets of Ω and μ a countably additive and positive measure on Σ with values in $\mathbb{R} \cup \{\infty\}$.

Denote by $L^2_\rho(\Omega)$ the Hilbert space of all real-valued functions f defined on Ω that are $2-\rho-$integrable on Ω, i.e., $\int_\Omega \rho(s) |f(s)|^2 d\mu(s) < \infty$, where $\rho : \Omega \to [0, \infty)$ is a measurable function on Ω.

We can introduce the following 2-inner product on $L^2_\rho(\Omega)$ by formula

$$(f, g|h)_\rho := \frac{1}{2}\int_\Omega\int_\Omega \rho(s)\rho(t) \begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix} \begin{vmatrix} g(s) & g(t) \\ h(s) & h(t) \end{vmatrix} d\mu(s) d\mu(t), \tag{5.1}$$

where

$$\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix}$$

denotes the determinant of the matrix

$$\begin{bmatrix} f(s) & f(t) \\ h(s) & h(t) \end{bmatrix},$$

which generates the 2-norm on $L^2_\rho(\Omega)$ expressed by

$$\|f|h\|_\rho := \left(\frac{1}{2}\int_\Omega\int_\Omega \rho(s)\rho(t) \begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix}^2 d\mu(s) d\mu(t)\right)^{1/2} . \tag{5.2}$$

A simple calculation with integrals reveals that

$$(f, g|h)_\rho = \begin{vmatrix} \int_\Omega \rho f g d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho g h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix} \tag{5.3}$$

and

$$\|f|h\|_{\rho}=\begin{vmatrix}\int_{\Omega}\rho f^{2}d\mu & \int_{\Omega}\rho fhd\mu\\ \int_{\Omega}\rho fhd\mu & \int_{\Omega}\rho h^{2}d\mu\end{vmatrix}^{1/2}, \tag{5.4}$$

where, for simplicity, instead of $\int_{\Omega}\rho(s)f(s)g(s)d\mu(s)$, we have written $\int_{\Omega}\rho fgd\mu$.

We recall that the pair of functions $(q,p)\in L_{\rho}^{2}(\Omega)\times L_{\rho}^{2}(\Omega)$ is called *synchronous* if

$$(q(x)-q(y))(p(x)-p(y))\geq 0$$

for *a.e.* $x,y\in\Omega$.

We note that, if $\Omega=[a,b]$, then a sufficient condition for synchronicity is that the functions are both monotonic increasing or decreasing. This condition is not necessary.

Now, suppose that $h\in L_{\rho}^{2}(\Omega)$ is such that $h(x)\neq 0$ for $\mu-a.e.$ $x\in\Omega$. Then, by the definition of 2-inner product $(f,g|h)_{\rho}$, we have

$$\begin{aligned}(f,g|h)_{\rho} \\ =\ & \frac{1}{2}\int_{\Omega}\int_{\Omega}\rho(s)\rho(t)h^{2}(s)h^{2}(t)\left(\frac{f(s)}{h(s)}-\frac{f(t)}{h(t)}\right)\left(\frac{g(s)}{h(s)}-\frac{g(t)}{h(t)}\right)d\mu(s)d\mu(t)\end{aligned} \tag{5.5}$$

and thus a *sufficient condition* for the inequality

$$(f,g|h)_{\rho}\geq 0 \tag{5.6}$$

to hold is that the functions $\left(\frac{f}{h},\frac{g}{h}\right)$ are synchronous. It is obvious that this condition is not necessary. Using the representations (5.3), (5.4) and the inequalities for 2-inner products and 2-norms established in the previous sections, we have some interesting determinantal integral inequalities.

Proposition 1. *Let $h\in L_{\rho}^{2}(\Omega)$ be such that $h(x)\neq 0$ for $\mu-a.e.$ $x\in\Omega$ and $(f_i)_{i\in I}$ a family of functions in $L_{\rho}^{2}(\Omega)$ with the property that*

$$\begin{vmatrix}\int_{\Omega}\rho f_{i}f_{j}d\mu & \int_{\Omega}\rho f_{i}hd\mu\\ \int_{\Omega}\rho f_{j}hd\mu & \int_{\Omega}\rho h^{2}d\mu\end{vmatrix}=\delta_{i,j}$$

for any $i,j\in I$, where $\delta_{i,j}$ is the Kronecker delta.

If we assume that there exists the real numbers $M_i,m_i,i\in F$, where F is a given finite part of I, such that the functions

$$\sum_{i\in F}M_{i}\cdot\frac{f_{i}}{h}-\frac{f}{h},\frac{f}{h}-\sum_{i\in F}m_{i}\cdot\frac{f_{i}}{h}$$

are synchronous on Ω, *then we have the inequalities*

$$\begin{aligned} 0 &\leq \begin{vmatrix} \int_\Omega \rho f^2 d\mu & \int_\Omega \rho f h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix} - \sum_{i\in F} \begin{vmatrix} \int_\Omega \rho f_i f d\mu & \int_\Omega \rho f_i h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu \end{vmatrix}^2 \\ &\leq \frac{1}{4}\sum_{i\in F}(M_i - m_i)^2 \\ &\quad - \sum_{i\in F}\left| \frac{M_i + m_i}{2} - \det\begin{bmatrix} \int_\Omega \rho f_i f d\mu & \int_\Omega \rho f_i h d\mu \\ \int_\Omega \rho f h d\mu & \int_\Omega \rho h^2 d\mu \end{bmatrix} \right|^2 \\ &\quad \left(\leq \frac{1}{4}\sum_{i\in F}(M_i - m_i)^2 \right). \end{aligned}$$

The proof follows by Theorem 2 applied for the 2-inner product $(\cdot,\cdot|\cdot)_\rho$ and we omit the details.

Similar determinantal integral inequalities may be stated if one uses the other results for 2-inner products obtained above, but we do not present them here.

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