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Conformally Flat Totally Umbilical Submanifolds in Some Semi-Riemannian Manifolds

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Abstract. We prove that totally umbilical submanifold M of an extended quasi-recurren manifold is also extended quasi-recurrent. If, moreover, M is conformally flat then, locally, M is isometric to the manifold with known metric. Some curvature properties of such submanifold are investigated. Making use of these results we shall prove the existence of totally umbilical submanifold being pseudosymmetric in the sense of Ryszard Deszcz and satisfying some other curvature conditions.

1. Introduction

Let (N, g) be a Riemannian or semi-Riemannian manifold of dimension n. A manifold N is said to be extended quasi-recurrent if there exist a 1-forms a and b such that the Riemann curvature tensor R satisfies

$$(1)\qquad \begin{aligned} \nabla_W R(X,Y,U,V) &= 2a(W)R(X,Y,U,V) + a(X)R(W,Y,U,V) \\ &\quad + a(Y)R(X,W,U,V) + a(U)R(X,Y,W,V) \\ &\quad + a(V)R(X,Y,U,W) + 2b(W)G(X,Y,U,V) \\ &\quad + b(X)G(W,Y,U,V) + b(Y)G(X,W,U,V) \\ &\quad + b(U)G(X,Y,W,V) + b(V)G(X,Y,U,W) \end{aligned}$$

on the set $U_R = \{x \in N,\ R(x) \neq 0\}$, where $G(X,Y,U,V) = g(Y,U)g(X,V) - g(Y,V)g(X,U)$. Manifolds satisfying condition of the type (1) were introduced by Prvanović ([16]). In ([9]) a local structure theorem for conformally flat manifolds satisfying (1) was proved. In the earlier papers ([6]) and ([7]) the condition (1) with $b = 0$ were considered and some theorems on totally umbilical submanifolds were proved. In the present paper we deal with totally umbilical submanifolds in manifolds satisfying (1) with $b \neq 0$ in general. We shall prove that the Weyl conformal curvature tensor C of such submanifold satisfies $L\ C = 0$, L being a function

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depending on the mean curvature vector and the 1-form a. Moreover, the submanifold is also extended quasi-recurrent and, locally, is isometric to the manifold with known metric. If, in addition, the submanifold is conformally flat, then the rank of its Ricci tensor is equal or less then one and 1-forms induced from a and b are closed. By the use of these results we shall give an example of manifold with non-recurrent curvature tensor that the tensor itself as well as the Ricci tensor satisfy some curvature conditions.

In the subsequent paper ([10]) we shall indicate some classes of manifolds that on totally umbilical submanifold relation $L\,C = 0$ holds, C being the Weyl conformal curvature tensor of the submanifold.

Throughout the paper all manifolds under consideration are assumed to be smooth Hausdorff connected and their metrics are not assumed to be definite.

2. Preliminaries

Let $(N, \overline{g})$ be covered by a system of coordinate neighbourhoods $\{U; x^r\}$. We denote by g_{ij}, Γ^k_{ij}, R_{hijk}, R_{ij}, r the components of the metric tensor $\overline{g}$, the Christoffel symbols, the curvature tensor R, the Ricci tensor S and the scalar curvature of $(N, \overline{g})$ respectively. Here and in the sequel the indices h, i, j, k, l, m, r, s, t, u, v run over the range $1, 2, \cdots, n$. Let (M, g) be an m-dimensional manifold covered by a system of coordinate neighbourhoods $\{V; y^a\}$ immersed in manifold (N, g) and let $x^r = x^r(y^a)$ be its local expression. Then the local components g_{ab} of the induced metric tensor of (M, g) are related to g_{rs} by $g_{ab} = g_{rs}B^r_aB^s_b$, where $B^r_a = \dfrac{\partial x^r}{\partial y^a}$. In what follows we shall adopt the convention

$$B^{rs}_{ab} = B^r_aB^s_b, \quad B^{rst}_{abc} = B^r_aB^s_bB^t_c, \quad B^{rstu}_{abcd} = B^r_aB^s_bB^t_cB^u_d.$$

We denote by Γ^c_{ab}, K_{abcd}, K_{bc}, K the components of the Christoffel symbols, the curvature tensor, the Ricci tensor and the scalar curvature of (M, g) with respect to g_{ab}. Then the components C_{abcd} of the Weyl conformal curvature tensor are given by

$$\begin{aligned} C_{abcd} &= K_{abcd} - \frac{1}{m-2}(g_{bc}K_{ad} - g_{bd}K_{ac} + g_{ad}K_{bc} - g_{ac}K_{bd}) \\ &\quad + \frac{K}{(m-1)(m-2)}G_{abcd}. \end{aligned}$$

Here and in the sequel the indices a, b, c, d, e, f run over the range $1, 2, \cdots, m, (m < n)$.

The van der Waerden-Bertolotti covariant derivative of B^r_a is given by

$$B^r_{a;b} = \nabla_b B^r_a = \partial_b B^r_a + \Gamma^r_{st}B^{st}_{ba} - B^r_c\Gamma^c_{ba}, \tag{2}$$

where the semicolon denotes covariant differentiation with respect to the metric of the submanifold.

The vector field H^r defined by

$$H^r = \frac{1}{m} g^{ef} \nabla_e B_f^r$$

is called the mean curvature vector of (M, g). Using (2) and the equation

$$\Gamma_{bc}^a = (\partial_c B_b^r + \Gamma_{st}^r B_{cb}^{st}) B_d^u g^{da} g_{ru},$$

we obtain on (M, g) the relation

$$g_{rs} H^r B_a^s = 0. \tag{3}$$

Let N_x^r, x, y, $z = m+1, \cdots, n$, be pairwise orthogonal unit vectors, normal to M. Then

$$g_{rs} N_x^r N_x^s = e_x, \quad g_{rs} N_x^r N_y^s = 0,\ x \neq y, \quad g_{rs} N_x^r B_a^s = 0$$

and

$$g^{rs} = B_{ab}^{rs} g^{ab} + \sum_x e_x N_x^r N_x^s,$$

where e_x is the indicator of the vector N_x^r.

The Schouten curvature tensor H_{ab}^r of M is defined by $H_{ab}^r = \nabla_b B_a^r$. Then the second fundamental form H_{abx} is related to H_{ab}^i by $H_{ab}^i = \sum_x e_x H_{abx} N_x^i$. If

$$H_{ab}^r = g_{ab} H^r, \tag{4}$$

then M is called a totally umbilical submanifold of N. Then $H_{abx} = g_{ab} H_x$, where $H_y = H^r N_y^s g_{rs}$, and

$$H^r = \sum_x e_x H_x N_x^r.$$

Furthermore, on a totally umbilical submanifold the Gauss, Codazzi and Weingarten equations take the form ([12], [13], [14])

$$K_{abcd} = R_{rstu} B_{abcd}^{rstu} + H(g_{bc} g_{ad} - g_{bd} g_{ac}), \tag{5}$$

$$R_{rstu} B_{abc}^{rst} N_x^u = A_{ax} g_{bc} - A_{bx} g_{ac}$$

and

$$N_{z;a}^s = -H_z B_a^s + \sum_y e_y L_{azy} N_y^s$$

respectively, where

$$H = g_{rs} H^r H^s, \quad A_{ax} = \partial_a H_x + \sum_y e_y L_{ayx} H_y,$$

$$L_{azy} = g_{rs} N_y^r N_{z;a}^s.$$

Moreover, we have ([13], [14])

$$R_{rstu}H^r B^{stu}_{bcd} = \frac{1}{2}(g_{bc}H_d - g_{bd}H_c), \quad H_c = H_{;c}, \tag{6}$$

$$\begin{aligned} K_{abcd;e} &= R_{rstu,v}B^{rstuv}_{abcde} + H_e(g_{bc}g_{ad} - g_{bd}g_{ac}) \\ &+\frac{1}{2}\left[H_a(g_{bc}g_{ed} - g_{bd}g_{ec}) + H_b(g_{ec}g_{ad} - g_{ed}g_{ac})\right. \\ &\left.+ H_c(g_{be}g_{ad} - g_{bd}g_{ae}) + H_d(g_{bc}g_{ae} - g_{be}g_{ac})\right] \end{aligned} \tag{7}$$

and

$$H^r_{;a} = -HB^r_a + \sum_y e_y A_{ay} N^r_y.$$

Finally, letting

$$E_{bc} = R_{hijk}H^h B^{ij}_{bc} H^k, \qquad A_{bc} = \sum_x e_x A_{bx} A_{cx}, \qquad H_{ae} = H_{;ae},$$

from the results of ([14], p. 108) it follows that

$$\begin{aligned} HK_{abce} &= R_{rstu,v}B^{rst}_{abc}H^u B^v_e + g_{ae}E_{bc} - g_{be}E_{ac} + A_{ae}g_{bc} - A_{be}g_{ac} \\ &+H^2(g_{ae}g_{bc} - g_{be}g_{ac}) - \frac{1}{2}(H_{ae}g_{bc} - H_{be}g_{ac}). \end{aligned} \tag{8}$$

In the sequel we shall often need the following lemmas:

Lemma 1 ([11]), Lemma 1). *Let M be a semi-Riemannian manifold of dimension $n \geq 3$. If B is a tensor field of type $(0,4)$ on M with components B_{hijk} satisfying*

$$B_{hijk} = -B_{ihjk} = B_{jkhi}, \qquad B_{hijk} + B_{hjki} + B_{hkij} = 0, \tag{9}$$

$$B_{hijk,lm} - B_{hijk,ml} = 0,$$

and P, Q are 1-forms on M, such that their components P_l, Q_l satisfy

$$P_r B^r_{ijk} = g_{ij}Q_k - g_{ik}Q_j,$$

then

$$Q_l\left[B_{hijk} - \frac{S}{n(n-1)}(g_{ij}g_{hk} - g_{ik}g_{hj})\right] = 0,$$

where $S = B_{pqrs}g^{qr}g^{ps}$.

If the Riemann curvature tensor of the manifold M satisfies (9), then M is called semi-symmetric.

Lemma 2. *Let N, $\dim N = n \geq 3$, be a conformally flat manifold. Then on N the following well-known relations hold:*

$$R_{hijk} = \frac{1}{n-2}\left(g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik}\right) + \frac{r}{(n-1)(n-2)}(g_{ij}g_{hk} - g_{ik}g_{hj}), \tag{10}$$

$$R_{ij,k} - R_{ik,j} - \frac{1}{2(n-1)}\left(g_{ij}r_{,k} - g_{ik}r_{,j}\right) = 0, \tag{11}$$

where $R_{ij} = S(\partial_i, \partial_j)$, $r = TrS$.

Lemma 3([15]). (a) *Let (A_j), (B_j) be two sequences of numbers linearly independent as the vectors of the space R^n. If T_{ij}, S_{ij} are numbers satisfying the conditions*

$$T_{ij}A_k + T_{jk}A_i + T_{ki}A_j + S_{ij}B_k + S_{jk}B_i + S_{ki}B_j = 0,$$
$$T_{ij} = T_{ji}, \qquad S_{ij} = S_{ji},$$

then there exist numbers D_j such that

$$T_{ij} = -B_iD_j - B_jD_i, \qquad S_{ij} = A_iD_j + A_jD_i.$$

(b) *Let T_{ij}, A_j be numbers satisfying*

$$T_{ij}A_k + T_{jk}A_i + T_{ki}A_j = 0, \qquad T_{ij} = T_{ji}.$$

Then either each T_{ij} is zero or each A_j is zero.

3. Main results

In the local coordinates (1) takes the form

$$R_{hijk,l} = 2a_lR_{hijk} + a_hR_{lijk} + a_iR_{hljk} + a_jR_{hilk} + a_kR_{hijl} + 2b_lG_{hijk} + b_hG_{lijk} + b_iG_{hljk} + b_jG_{hilk} + b_kG_{hijl}, \tag{12}$$

where $G_{hijk} = g_{ij}g_{hk} - g_{hj}g_{ik}$.

Theorem 4. *Let M $(\dim M \geq 3)$ be a totally umbilical submanifold of the manifold N and suppose that on N condition* (1) *is satisfied. Then*

$$(g_{rs}H^rH^s - a_rH^r)C_{abcd} = 0 \tag{13}$$

holds on M, where C_{abcd} are components of the Weyl conformal curvature tensor of the submanifold M.

Proof. Transvecting (12) with $H^hB^{ijkl}_{bcde}$ we can follow step by step the proof of ([7], Theorem 1) to obtain (13). See also ([8], Theorem 5). □

Theorem 5. *Let M $(\dim M \geq 3)$ be a conformally flat totally umbilical submanifold of the manifold N and suppose that on N condition* (1) *is satisfied. If $a_e(x) \neq 0$, $x \in M$, then*

$$rank\,[K_{ad}] \leq 1. \tag{14}$$

More precisely,

$$K_{ad} = h a_a a_d \tag{15}$$

and

$$a_f K^f_d = K a_d \tag{16}$$

on some neighbourhood, h being a function.

Proof. Let $a_e = a_r B^r_e$, $b_e = b_r B^r_e$, $H_e = \partial_e H = H_{;e}$, $c_e = b_e - H a_e + \frac{1}{2} H_e$, $g_{ad} = g_{rs} B^{rs}_{ad}$. Transvecting (12) with B^{hijkl}_{abcde} and making use of (5) and (7) we get

$$\begin{aligned} K_{abcd;e} &= 2a_e K_{abcd} + a_a K_{ebcd} + a_b K_{aecd} + a_c K_{abed} + a_d K_{abce} \\ &\quad + 2c_e G_{abcd} + c_a G_{ebcd} + c_b G_{aecd} + c_c G_{abed} + c_d G_{abce}. \end{aligned} \tag{17}$$

Contracting (17) with g^{bc} we obtain

$$\begin{aligned} K_{ad;e} &= 2a_e K_{ad} + a_a K_{ed} + a_d K_{ae} + a_f K^f_{ade} + a_f K^f_{dae} \\ &\quad + 2mc_e g_{ad} + (m-2)(c_a g_{ed} + c_d g_{ae}) \end{aligned} \tag{18}$$

whence

$$K_{;e} = 2K a_e + 4 a_f K^f_e + 2(m-1)(m+2) c_e. \tag{19}$$

Now, suppose that M is conformally flat. Put $U_{ade} = a_f K^f_{ade} + a_f K^f_{dae}$. Apply Lemma 2 to (17), then substitute (18) to obtain

$$\begin{aligned} & g_{bc} U_{ade} - g_{bd} U_{ace} + g_{ad} U_{bce} - g_{ac} U_{bde} \\ = & -\frac{K}{m-1} (2a_e G_{abcd} + a_a G_{ebcd} + a_b G_{aecd} + a_c G_{abed} + a_d G_{abce}) \\ & + \left[\frac{K_{;e}}{m-1} - 2(m+2) c_e \right] G_{abcd} \end{aligned}$$

which, by contractions, yields

$$\begin{aligned} (m-2) U_{ade} + g_{ad} U_{bce} g^{bc} &= -\frac{K}{m-1} [2m a_e g_{ad} + (m-2)(a_a g_{ed} + a_d g_{ae})] \\ &\quad + [K_{;e} - 2(m-1)(m+2) c_e]\, g_{ad} \end{aligned}$$

and

$$U_{ade}g^{ad} = -\frac{m+2}{m-1}Ka_e - m(m+2)c_e + \frac{m}{2(m-1)}K_{;e}.$$

The last two relations applied to (18) give

$$\begin{aligned}(20)\qquad & K_{ad;e} - \frac{1}{2(m-1)}K_{;e}g_{ad} \\ = \quad & 2a_eK_{ad} + a_aK_{ed} + a_dK_{ae} + (m-2)(c_eg_{ad} + c_ag_{ed} + c_dg_{ae}) \\ & -\frac{K}{m-1}(a_eg_{ad} + a_ag_{ed} + a_dg_{ae}).\end{aligned}$$

Alternating (20) in (d, e) and applying (11) we obtain

$$a_eK_{ad} - a_dK_{ae} = 0,$$

whence (14) - (16) result. Thus the theorem is proved. □

From (17) we have

Theorem 6. *A totally umbilical submanifold of an extended quasi-recurrent manifold is also extended quasi-recurrent.*

Now, from [9], Theorem 10, we get

Theorem 7. *Let M be an $m-$dimensional totally umbilical submanifold of an $n-$dimensional extended quasi-recurrent manifold with nowhere vanishing curvature tensor K and induced 1-form a. If M is conformally flat then is locally isometric to $m-$dimensional manifold with metric g given by*

$$(21)\qquad g_{ab}dx^adx^b = (dx^1)^2 + p^2f_{pq}dx^pdx^q,$$

where p is a function depending on x^1 only, such that

$$\left(E + p'^2\right)^2 + \left(p''\right)^2 \neq 0, \qquad pp'' \neq E + p'^2,$$
$$E = [(m-1)(m-2)]^{-1}\overline{K}$$

and $f_{pq}dx^pdx^q$ is a metric of $(m-1)-$dimensional manifold of constant curvature $\overline{K}$.

Theorem 8. *Let M $(\dim M \geq 3)$ be a conformally flat totally umbilical submanifold of the manifold N and suppose that on N condition (1) is satisfied. If M is semi-symmetric, then*

$$a_e\left[K_{abcd} - \frac{K}{m(m-1)}G_{abcd}\right] = 0.$$

Proof. We can suppose $a_e \neq 0$. Since the curvature tensor of M is of the form (10), by transvection with a^a and the use of (15) and (16) we obtain

$$a_f K^f_{bcd} = \frac{K}{m(m-1)}(g_{bc}a_d - g_{bd}a_c).$$

This, together with Lemma 1, completes the proof. □

Lemma 9. *Let M $(\dim M \geq 3)$ be a conformally flat totally umbilical submanifold of the manifold N and suppose that on N condition* (1) *is satisfied. If $a_e(x) \neq 0$, $x \in M$, then on some neighbourhood*

$$K_{;e} = 6Ka_e + 2(m-1)(m+2)c_e, \tag{22}$$

$$\begin{aligned} K_{ad:e} &= 4a_eK_{ad} + 2mc_e + (m-2)(c_a g_{ed} + c_d g_{ae}) \\ &\quad + \frac{K}{m-1}(2a_e g_{ad} - a_a g_{ed} - a_d g_{ae}), \end{aligned} \tag{23}$$

$$\begin{aligned} (m-2)K_{abcd;e} &= 4a_e(g_{bc}K_{ad} - g_{bd}K_{ac} + g_{ad}K_{bc} - g_{ac}K_{bd}) \\ &\quad + 2z_e G_{abcd} + z_a G_{ebcd} + z_b G_{aecd} + z_c G_{abed} \\ &\quad + z_d G_{abce}, \end{aligned} \tag{24}$$

where

$$z_e = (m-2)c_e - \frac{1}{m-1}Ka_e.$$

Proof. The first equation results from (19) and (16). The second one we get from (20), (15) and the former one. Finally, since the Riemann curvature tensor K of M satisfies (10), by covariant differentiation and the use of (22) and (23) we get (24). □

Theorem 10. *Under assumptions of Theorem 5, let $a_e(x) \neq 0$, $x \in M$. Then, on some neighbourhood of x, $da = 0$.*

Proof. That $a_{e,f} - a_{f,e} = 0$ at points that $a(x) \neq 0$ and $c(x) = 0$ it follows from Theorem 6 and [7], Proposition 3. Suppose now $a(x) \neq 0$ and $c(x) \neq 0$. Let $U_{ad} = 4K_{ad} - h(a_{a,d} + a_{d,a}) - \frac{1}{2}(h_a a_d + h_d a_a)$. Symmetrizing (23) in (a, d, e) and applying (15) we get

$$a_e U_{ad} + a_a U_{de} + a_d U_{ea} + 4(m-1)(c_e g_{ad} + c_a g_{de} + c_d g_{ea}) = 0.$$

If a_e and c_e were linearly independent, then, by Lemma 3(a), we would have $rank\,[g_{ab}] \leq 2$, a contradiction to the assumption. Thus, on some neighbourhood of x, there exist a function f, such that $c_e = fa_e$ and Lemma 3(b) gives $T_{ad} = U_{ad} + 4(m-1)fg_{ad} = 0$.

On the other hand, substituting (15) into (23) we have

$$\begin{aligned} (25) \qquad & h_e a_a a_d + h a_{a,e} a_d + h a_a a_{d,e} \\ = \; & 4h a_a a_d a_e + 2\left(mf + \frac{K}{m-1}\right) a_e g_{ad} \\ & + \left((m-2)f - \frac{K}{m-1}\right)(a_a g_{ed} + a_d g_{ae}). \end{aligned}$$

Put $A = a^f a_f$, $V_a = \frac{1}{2}\left[h^f a_f - 4hA - 2\left((m-2)f - \frac{K}{m-1}\right)\right] a_a + h a^f a_{a,f}$. Transvecting (25) with a^e we find

$$(26) \qquad 2\left(mf + \frac{K}{m-1}\right) A g_{ad} = V_a a_d + V_d a_a.$$

If $\left(mf + \frac{K}{m-1}\right) A \neq 0$, we would have $rank\,[g_{ab}] \leq 2$.

Transvecting (25) with a^a we get

$$\begin{aligned} (27) \qquad & A h_e a_d + \frac{1}{2} h A_{;e} a_d + h A a_{d,e} \\ = \; & 4hA a_d a_e + 2\left(mf + \frac{K}{m-1}\right) a_d a_e + \left((m-2)f - \frac{K}{m-1}\right)(A g_{ed} + a_d a_e). \end{aligned}$$

If $\left(mf + \frac{K}{m-1}\right) \neq 0$ at a point x, then on some neighbourhood $A = 0$. Consequently, from (15) and (16), we have $K = hA = 0$ and from (27) it follows, that $f = 0$, a contradiction. Thus

$$f = -\frac{K}{m(m-1)}.$$

Consequently, by the use of Lemma 9, we have

$$(28) \qquad K_e = \frac{4(m-1)}{m} K a_e.$$

But $K(x) = 0$ would yield $c_e(x) = 0$, a contradiction. This completes the proof. □

Remark 11. Generalising the well known notion of the Ricci-recurrent manifold, i.e. the manifold that the Ricci tensor S satisfies

$$\nabla S = a \otimes S$$

at all points that $S \neq 0$, some authors (see for example [2], [4], [5]) considered the condition

$$\nabla_{X_1} S(X_2, X_3) = \sum_{\sigma \in S_3} \overset{\sigma}{a}(X_{\sigma(1)}) S(X_{\sigma(2)}, X_{\sigma(3)}) + \sum_{\sigma \in S_3} \overset{\sigma}{c}(X_{\sigma(1)}) g(X_{\sigma(2)}, X_{\sigma(3)})$$

with 1-forms $\overset{\sigma}{a}$, $\overset{\sigma}{c}$ not all necessary different from zero, S_3 being the group of permutations. Lemma 9 together with (28) yield

$$K_{ad;e} = 4a_e K_{ad} - \frac{2}{m} K(a_a g_{ed} + a_d g_{ae}). \tag{29}$$

Since the scalar curvature in the metric (21) is

$$K = (m-2)\left[(m-1)p^2(E + p'^2) + pp''\right]$$

and $K \neq 0$ in general, Theorem 7 proves the existence of manifolds satisfying (29). Consequently, we have an example of conformally flat manifold that the covariant derivative of the curvature tensor satisfies

$$\begin{aligned} K_{abcd;e} &= 4ha_e(g_{bc}a_a a_d - g_{bd}a_a a_c + g_{ad}a_b a_c - g_{ac}a_b a_d) \\ &\quad + 2c_e g_{abcd} + c_a g_{ebcd} + c_b g_{aecd} + c_c g_{abed} + c_d g_{abce}, \end{aligned} \tag{30}$$

where $c_e = -\dfrac{2}{m(m-1)} K a_e$.

Differentiating covariantly (15) and combining the result with (29) and (28) we find

$$a_{a;e} = 2a_a a_e - \frac{2}{m}\frac{K}{h} g_{ae} - \frac{h_{;e}}{2h} a_a,$$

whence, in virtue of Theorem 10, we get $h_{;a} = \alpha a_a$, α being a function on some neighbourhood of $x \in M$, $a_a(x) \neq 0$. Then straightforward calculations yields

$$K_{ad;ef} - K_{ad;fe} = L\left(g_{ae}K_{fd} - g_{af}K_{ed} + g_{de}K_{af} - g_{df}K_{ae}\right)$$

which is equivalent to the Ricci-pseudosymmetric condition on the set $U_K = \{x \in M,\ K_{ae}(x) \neq 0\}$. Since M is conformally flat it must be pseudosymmetric.

Thus we conclude with

Theorem 12. *There exist totally umbilical conformally flat submanifolds being simultaneously pseudosymmetric in the sense of Ryszrd Deszcz. Moreover their Ricci and Riemann curvature tensors satisfy* (29) *and* (30) *respectively.*

We strongly emphasize the fact the notion of pseudosymmetry in the sense of Ryszard Deszcz and the one of Chaki have nothing in common.

Proposition 13. *Let M $(\dim M \geq 3)$ be a totally umbilical submanifold of the manifold N satisfying* (12). *If C does not vanish on a dense subset then*

$$a_c H_b - a_b H_c = 0,$$

and

$$R_{rstu} H^r B^{st}_{bc} H^u = \frac{1}{2} a_b H_c + \frac{F_{ae} g^{ae}}{m} g_{bc}$$

holds on M.

Proof. We can suppose $C(x) \neq 0$, $x \in M$. Then, by (13),

$$H = g_{rs}H^r H^s = a_r H^r. \tag{31}$$

From the second Bianchi identity we have

$$R_{rstu,v}B^{rst}_{abc}H^u B^v_e - R_{rsvu,t}B^{rsv}_{abe}H^u B^t_c = -R_{rsvt,u}B^{rsvt}_{abec}H^u.$$

Applying (8) twice to the left hand side and (12) to the right one, by the use of (31), (3), (6) and (5), we get

$$g_{ae}N_{bc} - g_{be}N_{ac} + g_{bc}N_{ae} - g_{ac}N_{be} = 0, \tag{32}$$

where

$$N_{bc} = E_{bc} + A_{bc} - \frac{1}{2}H_{bc} + \frac{1}{2}a_b H_c + \frac{1}{2}a_c H_b - b_u H^u g_{bc}.$$

(32) is the Kulkarni-Nomizu product of the metric tensor g and symmetric $(0,2)-$ tensor N. It is clear that if $rank\,[g_{bc}] > 1$ and (32) holds, then $N_{bc}g^{bc} = 0$. Hence we have

$$N_{bc} = 0. \tag{33}$$

On the other hand, transvecting (12) with $B^{hij}_{abc}H^k B^l_e$, by the use of (31), (3), (6) and (5), we obtain

$$\begin{aligned} R_{hijk,l}B^{hij}_{abc}H^k B^l_e \;\; = \;\; & a_e(g_{cb}H_a - g_{ca}H_b) + \frac{1}{2}\left[a_a(g_{cb}H_e - g_{ce}H_b)\right. \\ & + a_b(g_{ce}H_a - g_{ca}H_e) + a_c(g_{eb}H_a - g_{ea}H_b)\left.\right] \\ & + HK_{abce} + (b_u H^u - H^2)g_{abce}. \end{aligned}$$

Substituting into (8), by the use of (33), we get

$$\begin{aligned} & g_{ae}E_{bc} - g_{be}E_{ac} + g_{ac}E_{be} - g_{bc}E_{ae} + \\ & \frac{1}{2}\left[a_c(g_{eb}H_a - g_{ea}H_b) + a_e(g_{cb}H_a - g_{ca}H_b) + g_{ce}(H_a a_b - H_b a_a)\right] = 0. \end{aligned}$$

Contracting the last equation with g^{ae} and alternating the result in (b,c) we find

$$(m-2)(a_c H_b - a_b H_c) = 0,$$

whence

$$g_{ae}F_{bc} - g_{be}F_{ac} + g_{ac}F_{be} - g_{bc}F_{ae} = 0, \tag{34}$$

where $F_{bc} = F_{cb} = E_{bc} - \frac{1}{2}a_b H_c$. Symmetrising (34) in pairs (a,b), (e,c) we obtain $g_{ae}F_{bc} - g_{bc}F_{ae} = 0$. It follows, that $mF_{bc} = g_{bc}F_{ae}g^{ae}$. This completes the proof. □

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